On multidimensional topological solitons in gauged sigma models with spontaneously broken $Z(2)$ symmetry

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ABSTRACT. New gauged sigma models with broken $Z(2)$ symmetry in $(D+1)$-dimensional space-time are proposed, which admit search for topological solitons by using "hedgehog-like" ansatzes. Three-dimensional particle-like solutions of these models can be considered as classical prototypes of massive quantum particles.

RÉSUMÉ. On propose des sigma modèles de jauge nouveaux à symétrie $Z(2)$ brisée, en dimension d'espace temps $(D+1)$; ces modèles admettent des solitons topologiques comme solutions, qui peuvent être considérés comme des prototypes classiques de particules massives quantiques.

1. Sigma models take one of the central places in the modern mathematical physics, which is due to their universality: they appear in various branches of fundamental science: particle and nuclear physics, superfluid He^3 phases, high-temperature superconductivity, microphysics of magnets and ferroelectrics. Classical sigma models describe evolution in time of N-component unit isovector field $s_a(\mathbf{x},t)$ in $(D+1)$-dimensional space-time; field manifolds of these models are unit spheres S^{N-1}. The most frequently investigated cases correspond to $D=2,3$ and $N=2,3,4$.

Starting from the pioneer papers by Skyrme [1], there exists an increasing interest in the investigation of localized particle-like solutions (solitons) to those sigma models, which can be characterized by existence of the topological indices Q_t ("topological charges") [2-5]. In such sigma models the localized distributions of unit isovector $s_a(\mathbf{x})$ are divided into classes with different Q_t; solitons with the nonzero topological charges are referred to as "topological solitons". It should be noted that the existence of the solitons is not yet guaranteed within the sigma models admitting presence of nontrivial topological charges, though for nonzero topcharges additional possibilities arise for solitons to exist. For the wide class of nonlinear one-field models with positive definite Hamiltonian density the particle-like localized solutions with finite energy cannot be found for $D \geq 2$ due to the Derrick's theorem [6], whose proof is based on scaling transformations. The known ways to overcome this serious obstacle are:

1) to add to a Hamiltonian density of a model additional higher-order derivative stabilizing terms (first it was proposed in [1]),

2) to consider time-dependent localized solutions instead of static ones,

3) to study models comprising several basic fields.

Below we shall discuss the gauged sigma models which describe interaction of the anisotropic unit isovector fields with the gauge fields (two-field models). Their non-gauged counterparts are the sigma models whose Lagranians possess both some global continuous symmetry and the spontaneously broken discrete $Z(2)$ symmetry. We have already started investigation of D-dimensional $(D \geq 1)$ soliton solutions within the two-field models with spontaneously broken $Z(2)$ symmetry in [7], where interaction of the "easy-axis" anisotropic 3-component unit isovector field with the vector field was considered and 2D topological solitons were found.

2. First let us consider the 3-component unit isovector field $s_a(\mathbf{x})$ with the easy-axis anisotropy defined by the Lagrangian density

$$\mathcal{L} = (\partial_\mu s_a)^2 - V(\mathbf{s}), \quad s_a s_a = 1, \quad V(\mathbf{s}) = 1 - s_3^2, \quad (1)$$

$$\mu = 0, 1, ..., D, \quad a = 1, 2, 3.$$

(it was called the A3-field in [7]; "A3" stands for "anisotropic 3-component"). It is important to note that the Lagrangian (1) possesses $U(1) \times Z(2)$ internal symmetry; its vacuum manifold comprises two points on the S^2 sphere: $s_3 = 1$ and $s_3 = -1$ and possesses discrete $Z(2)$ symmetry [7]. At $D = 1$ the model (1), which can be viewed as generalization of the sine-Gordon equation [8], possesses kink and antikink solutions [8-10], which break the $Z(2)$ symmetry of the vacuum manifold [7]. Futhermore, the $Z(2)$ symmetry is also broken [7] on $2D$ nonstationary topological solitons [11] of the model (1).

The Lagrangian density (1) can be derived when describing in continuous approximation easy-axis Heisenberg antiferromagnets [10] and ferroelectrics [12] with the easy-axis anisotropy; thus, the pattern of the symmetry breaking under discussion is realized in condensed matter physics. It seems interesting to investigate possible implications of this pattern (which differs from that used in the Standard Model) for high-energy physics, especially having in mind intensive use of sigma models in modern particle physics. Consideration of anisotropic unit isovector fields, realizing such an "alternative" mechanism of symmetry breaking, as constituents of particle physics models, can be grounded in two (possibly interrelated) ways. First, one can conjecture that some of these scalar (with respect to Lorentz transformations) fields belong to the set of basic quantum fields. Second, these fields may arise as coherent states [13] of more fundamental (e.g., fermionic) fields. Moreover, these two viewpoints may prove dual with respect to each other (cf. the equivalence [14] of the quantum sine-Gordon model and the zero charge sector of the massive Thirring model). Anyhow, it would be a highly attractive picture if similar patterns of spontaneous symmetry breaking take place in condensed matter and high-energy physics.

Next consider minimal interaction of the A3-field with the Maxwell field $A_\mu(x)$, described by the gauge-invariant Lagrangian ("the A3M model"):

$$\mathcal{L} = \bar{\mathcal{D}}_\mu s_- \mathcal{D}^\mu s_+ + \partial_\mu s_3 \partial^\mu s_3 - V(s_a) - \frac{1}{4}F_{\mu\nu}^2, \quad (2)$$

$$\bar{\mathcal{D}}_\mu = \partial_\mu + ieA_\mu, \quad \mathcal{D}_\mu = \partial_\mu - ieA_\mu,$$

$$s_+ = s_1 + is_2, \quad s_- = s_1 - is_2,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad V(s_a) = \beta(1 - s_3^2),$$

where β, e are coupling constants and $\mu, \nu = 0, 1, ..., D$. Lagrangian (2) can be rewritten in the form

$$\mathcal{L} = (\partial_\mu s_a)^2 - V(s_a) - \frac{1}{4}F_{\mu\nu}^2 + 2eA_\mu(s_2\partial^\mu s_1 - s_1\partial^\mu s_2) + e^2(s_1^2 + s_2^2)A_\mu A^\mu. \quad (3)$$

We shall start the investigation of the topological solitons of the A3M model (2) for $D = 2$ and look for stationary solitons using the "hedgehog-like" ansatz for the A3-field

$$\begin{aligned} s_1 &= \cos m\chi \sin\theta(R), \quad s_2 = \sin m\chi \sin\theta(R), \\ s_3 &= \cos\theta(R), \end{aligned} \quad (4)$$

$$\sin\chi = \frac{y}{R}, \quad \cos\chi = \frac{x}{R}, \quad R^2 = x^2 + y^2,$$

where m is an integer number, and the standard 2D ansatz for the vector field A_μ, describing localized distributions of a stationary magnetic field:

$$\begin{aligned} A_0 = 0, \quad A_1 &= A_x = -ma(R)\tfrac{y}{R^2}, \\ A_2 &= A_y = ma(R)\tfrac{x}{R^2}. \end{aligned} \quad (5)$$

Making rescaling

$$a = \alpha e^{-1}, \quad R = re^{-1}, \quad (6)$$

we get for stationary Hamiltonian density $\mathcal{H}(r) = e^{-2}\mathcal{H}(R)$:

$$\begin{aligned} \mathcal{H}(r) = \left(\tfrac{d\theta}{dr}\right)^2 &+ \sin^2\theta\left[p + \tfrac{m^2(\alpha-1)^2}{r^2}\right] \\ &+ \tfrac{m^2}{2}\left(\tfrac{1}{r}\tfrac{d\alpha}{dr}\right)^2, \end{aligned} \quad (7)$$

$$H = \int_0^\infty \mathcal{H}(r) 2\pi r dr, \quad p = \frac{\beta}{e^2}. \quad (8)$$

Calculating $\delta H/\delta\theta$ and $\delta H/\delta\alpha$, we obtain coupled equations for $\theta(r)$ and $\alpha(r)$:

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - \sin\theta\cos\theta\left[\frac{m^2(\alpha-1)^2}{r^2} + p\right] = 0, \quad (9)$$

$$\frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} + 2\sin^2\theta(1-\alpha) = 0. \quad (10)$$

One can easily see that Eqs.(9),(10) are satisfied for $r \to \infty$, if $\frac{d\alpha}{dr}(\infty) = 0$ and $\theta(\infty) = 0$ and so it is natural to look for solutions of Eqs. (9),(10) under the following boundary conditions:

$$\theta(0) = \pi, \quad \theta(\infty) = 0, \quad (11)$$

$$\alpha(0) = 0, \quad \frac{d\alpha}{dr}(\infty) = 0. \quad (12)$$

Notice that Eqs.(4),(11) define the class of mappings $R^2_{comp} \to S^2$, such that $Q_t = m$, where Q_t is the topological index ("winding number") of localized distributions $s_a(\mathbf{x}), a = 1, 2, 3$, described by the mappings of this class. The results of numerical investigation of the boundary value problem (9)-(12) will be published elsewhere; here we note only that the 2D topological solitons of the model (2), which satisfy Eqs. (9)-(12), do exist if $0 < p < p_{crit} \approx 0.4$. We hope that topological solitons will be found within the 3D model (2) as well, which would be of more interest for the particle physics. We would like to note here that one of the 2D gauged sigma models discussed in [15] is similar to the model (2); the important distinction between these models is that the model (2) is characterized by the spontaneously broken $Z(2)$ symmetry of its Lagrangian for arbitrary D.

3. In the 2D case one can also consider minimal interaction of the A3-field with the Chern-Simons (CS) gauge field ("the A3CS model").The Lagrangian of the A3CS model is obtained when one replaces the Maxwell term $\mathcal{L}_M = -\frac{1}{4}F^2_{\mu\nu}$ in (2),(3) by the Chern-Simons term, $\mathcal{L}_{CS} = \epsilon_{\mu\nu\lambda}A^\mu \partial^\nu A^\lambda, \quad \mu,\nu,\lambda = 0,1,2.$ Further investigation of 2D topological solitons in the A3CS model is analogous to that for the A3M model and is also carried out using the ansatz given by Eqs.(4),(5) and variables α and r introduced by Eq.(6). As a result we obtain coupled equations for $\theta(r)$ and $\alpha(r)$:

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - \sin\theta\cos\theta\left[\frac{m^2(\alpha-1)^2}{r^2} + p\right] = 0, \quad (13)$$

$$\frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} + \sin^2\theta(1-\alpha) = 0. \quad (14)$$

Making scaling transformation $r' = r/\sqrt{2}$, we obtain coupled equations (9),(10) with p replaced by $2p$, thus the solitons of the A3CS model can be easily obtained from solitons of the A3M model by means of the above transformation.

Note that the same 2D stationary solitons can be found in the nonrelativistic analogs of the A3M and the A3CS models, in which the A3-field is replaced by the 3-component unit field of the easy-axis Heisenberg ferromagnet, described by the Landau-Lifshitz equation (see, e.g., [10]).

4. Further we shall consider another gauged sigma model, which describe minimal interaction of the easy-axis 4-component unit isovector field $q_\alpha(x^\mu)$ ("the A4-field") with the gauge $SU(2)$ Yang-Mills field $A^a_\mu(x^\nu)$, where $\alpha,\mu,\nu = 0,1,2,3, \quad a = 1,2,3.$ The Lagrangian density of this ("the A4YM") model is:

$$\mathcal{L} = \mathcal{D}_\mu q^a \mathcal{D}^\mu q^a + \partial_\mu q^0 \partial^\mu q^0 - V(q^0) - \frac{1}{4}(F^a_{\mu\nu})^2, \quad (15)$$

$$\mathcal{D}_\mu q^a = \partial_\mu q^a + g\varepsilon^{abc}A^b_\mu q^c, \quad a,b,c = 1,2,3,$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc}A^b_\mu A^c_\nu,$$

$$V(q^0) = \beta[1-(q^0)^2],$$

where β, g are coupling constants.

One can look for stationary topological solitons of the A4YM model using the following ansatzes for the A4- and the $SU(2)$ Yang-Mills fields:

$$q^0 = \cos\theta(R), \quad q^a = \sin\theta(R)\frac{x^a}{R}, \quad R^2 = x^2+y^2+z^2, \quad (16)$$

$$A^a_0 = 0, \quad A^a_i = c(R)\varepsilon^{iak}x^k. \quad (17)$$

Stationary solitons describe localized particle-like Hamiltonian density distributions $\mathcal{H}_{st}$ of time-independent fields $q^\alpha(\mathbf{x})$ and $A^a_i(\mathbf{x})$,

$$\mathcal{H}_{st} = (\mathcal{D}_k q^a)^2 + (\partial_k q^0)^2 + \frac{1}{4}(F^a_{ik})^2 + \beta[1-(q^0)^2]. \quad (18)$$

It is straightforward to show that the Hamiltonian density distributions of localized field bunches given by Eqs. (16),(17) are spherically symmetric:

$$\begin{aligned}\mathcal{H}_{st}(R) = \; & \left(\tfrac{d\theta}{dR}\right)^2 + \tfrac{2\sin^2\theta}{R^2} + 4gc\sin^2\theta + 2g^2c^2R^2\sin^2\theta \\ & +6c^2 + \left(\tfrac{dc}{dR}\right)^2 R^2 + \tfrac{1}{2}g^2c^4R^4 \\ & +4Rc\tfrac{dc}{dR} + 2gR^2c^3 + \beta\sin^2\theta.\end{aligned} \quad (19)$$

Equating variational derivatives $\delta H/\delta\theta$ and $\delta H/\delta c$ to zero ($H = 4\pi\int_0^\infty \mathcal{H}_{st}R^2dR$) one finds coupled equations for functions $c(R)$ and $\theta(R)$:

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{d\theta}{dR}\right) - \sin\theta\cos\theta\left(\frac{2}{R^2} + 4gc + 2g^2R^2c^2 + \beta\right) = 0, \quad (20)$$

$$\frac{1}{R^2}\frac{d}{dR}\left(R^4\frac{dc}{dR}\right) - 2g\sin^2\theta - 2g^2R^2c\sin^2\theta - g^2c^3R^4 - 3gR^2c^2 = 0. \quad (21)$$

We shall study localized field distributions described by the functions $\theta(R)$ and $c(R)$ which are solutions to Eqs. (20),(21) and satisfy the following boundary conditions:

$$\theta(0) = \pi, \quad \theta(\infty) = 0, \quad (22)$$

$$c(0) = 0, \quad c(\infty) = 0. \quad (23)$$

It is useful to introduce dimensionless variables,

$$r = gR, \quad b(r) = g^{-1}cr^2. \quad (24)$$

Then the Hamiltonian density takes the form

$$\begin{aligned}\mathcal{H}(r) = g^{-2}\mathcal{H}(R) = \; & \left(\tfrac{d\theta}{dr}\right)^2 + 2\sin^2\theta\tfrac{1}{r^2} + 4\sin^2\theta\tfrac{b}{r^2} + 2\sin^2\theta\tfrac{b^2}{r^2} + \tfrac{6b^2}{r^4} + \left(\tfrac{d(br^{-2})}{dr}\right)^2 r^2 \\ & + \tfrac{b^4}{2r^4} + 4\tfrac{b}{r}\tfrac{d(br^{-2})}{dr} + \tfrac{2b^3}{r^4} + P\sin^2\theta \\ = \; & \left(\tfrac{d\theta}{dr}\right)^2 + 2\sin^2\theta\tfrac{1}{r^2}(1+b)^2 + \tfrac{b^2}{2r^4}(b+2)^2 + \left[r\tfrac{d(br^{-2})}{dr} + \tfrac{2b}{r^2}\right]^2 + P\sin^2\theta,\end{aligned} \quad (25)$$

$$P = \frac{\beta}{g^2}. \qquad (26)$$

Note that $H = \int_0^\infty \mathcal{H}(R) 4\pi R^2 dR = g^{-1} H_d$, where $H_d = \int_0^\infty \mathcal{H}(r) 4\pi r^2 dr$.

Calculating $\delta H_d/\delta\theta$ and $\delta H_d/\delta b$, we arrive at coupled equations for $\theta(r)$ and $b(r)$:

$$\frac{d^2\theta}{dr^2} + \frac{2}{r}\frac{d\theta}{dr} - \sin\theta\cos\theta\left[\frac{2(b+1)^2}{r^2} + P\right] = 0, \quad (27)$$

$$\frac{d^2 b}{dr^2} - \frac{2b}{r^2} - 2\sin^2\theta(1+b) - \frac{b^2}{r^2}(b+3) = 0. \quad (28)$$

We shall look for localized solutions to Eqs. (27),(28), setting the following boundary conditions:

$$\theta(0) = \pi, \quad \theta(\infty) = 0, \qquad (29)$$

$$b(0) = 0, \quad b(\infty) = B, \quad B = 0, -1, -2. \qquad (30)$$

Eqs (16), (29) define localized distributions $q^\alpha(x^k), \alpha = 0,1,2,3, k = 1,2,3,$ of the A4-field possessing unit topological charge, $Q_t = 1$. Here Q_t is the "winding number", or the "mapping degree", of continuous maps $R^3_{comp} \to S^3$, corresponding to localized field distributions $q^\alpha(x^k)$; note that R^3 is compactified since the unique value of q^α at $|\mathbf{x}| = \infty$ is set by the boundary condition $\theta(\infty) = 0$. Investigation of the boundary value problems (27)-(30) is in progress.

5. In this paper we propose gauged sigma models with broken $Z(2)$ symmetry in $(D+1)$-dimensional space-time, which admit existence of cylindrically or spherically symmetric particle-like solutions. These localized solutions are topological solitons, since they describe bunches of anisotropic N-component ($N = 3, 4$) unit isovector fields possessing integer topological charges. They can be considered as soliton analogs of the Abrikosov-Nielsen-Olesen vortices (strings)[16] (for $D = 2, N = 3$) and of the 't Hooft-Polyakov monopoles [17] (for $D = 3, N = 4$) . Three-dimensional topological soliton solutions within the A3M and the A4YM models proposed above can be considered as classical prototypes of massive quantum particles (quarks, leptons, baryons, W- and Z-bosons); one can consider integrals of energy, isotopic charge and internal angular momentum of 3D solitons in these models as classical analogs of masses, charges and spins of corresponding quantum particles.

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(Manuscrit reçu le 26 mars 1996)