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Weak Field Gravitation as a Composite Particle Effect

THOMAS BORNE[a] AND HARALD STUMPF[b]

[a]**Fondation Louis de Broglie, 23, Quai de Conti, F–75006 Paris**
[b]**Institut für Theoretische Physik der Universität Tübingen, Auf der Morgenstelle, D–72076 Tübingen**

ABSTRACT. We intend to explain geometrical gravitation theory as an effective theory for bound states of four subfermion fields. This is part of an atomistic program which is based on de Broglie's fusion idea and on Heisenberg's nonlinear spinorfield project. The fundamental spinorfield dynamics is nonperturbatively regularized, canonically quantized and represented in a functional formalism which allows a systematic derivation of effective dynamics for bound states by means of Weak Mapping. We apply this formalism to the graviton case. After the derivation and the discussion of bound 'graviton' states we obtain an effective linear gravitation theory in vacuum as well as the correct coupling of a Dirac field to the gravitational field. This may be regarded as a first promising step towards the derivation of a full effective gravitation theory.

RÉSUMÉ. Nous avons l'intention d'expliquer la théorie géométrique de la gravitation comme une théorie effective pour des états liés de quatre fermion champs élémentaires. C'est une partie d'un programme atomistique lequel s'appuie sur l'idée de fusion de de Broglie et sur la théorie des champs spinoriels nonlinéaire d' Heisenberg. La dynamique quantique des champs spinoriels fondamentaux est représentée dans un formalisme fonctionel lequel permet une déduction systématique de la dynamique effective des états liés par la méthode de Weak Mapping. Nous appliquons cette méthode au cas de graviton. Après la déduction et la discussion des états 'gravitons' liés nous obtenons une théorie linéaire effective de la gravitation en vide de même que le couplage correct du champ Dirac au champ de la gravitation. Nous regardons ce résultat comme un premier pas prometteur vers la déduction d'une théorie effective complète de la gravitation.

1 Introduction

Since the works of Einstein gravitation is regarded as the prototype of a geometrical theory. The successful geometrization of a physical interaction, i.e. the identification of field quantities with geometrical quantities has promoted many attempts to apply this principle of geometrization to other fields in physics. On the other hand the fundamental interactions which govern the physics of microscopic systems are rather successfully described by the standard model of elementary particles in terms of special relativistic fields. Thus there seems to be a conceptual difference between the large scale gravitation theory and microscopic physics which one is faced with for any attempt of a unification of gravitation and the other fundamental interactions.

In this paper we are concerned with an attempt to derive a gravitation theory as an effective theory for bound states of an underlying nonlinear spinorfield theory in flat space. Any theory of gravitation is strongly related to geometry, because the universal coupling of the gravitational interaction acts on all material standards of length and time. The consequence of our fieldtheoretical ansatz is, that the observable geometry turns out to be an effective geometry with respect to a fixed flat (and unobservable) background metric.

This fieldtheoretical ansatz for a gravitation theory is motivated by several well–known problems of conventional gravitation theory. In the Einstein theory of gravitation the energy–momentum of the gravitational field is described only by a pseudotensor which prevents the formulation of proper conservation laws. In contrast, such conservation laws are considered to be important properties of conventional field theories in flat Minkowski space.

Further fundamental difficulties arise if one tries to apply the microscopic concept of quantization to gravitation theories: The Einstein–Hilbert Lagrangian induces an unrenormalizable quantum theory and taking serious the geometrical interpretation of gravitation in the microscopic domain leads to contradictions: The quantization of a metric would lead to fluctuations of this metric and thus would destroy the possibility of conservation laws which in turn are conditions for the formulation of a quantum theory. In addition, the dependence of the metric on the amount of matter also destroys this possibility. On the other hand the quantum concept is well established and tested in microscopic physics; thus in the above mentioned sense we assume the primacy of quantum theory over geometrization; compare also [44], [35], [42].

There have been several attempts for the formulation of a gravitation theory in a fieldtheoretic flat space framework, see for instance [13], [36], [37], [40], [43], [19], [7], [6], [18]. We take over the fieldtheoretical ansatz of these works according to which gravitation is described by fields in an unobservable flat pseudo–euclidean space; if the equations of motion of some 'matter' coupled to the gravitational fields may be interpreted as equations in a curved (Riemann) geometry, this observable geometry is induced by the gravitational fields.

Our program is a part of an atomistic program which is based on de Broglie's fusion theory [4] and Heisenberg's unified spinorfield project [17]: We intend to derive the reactions between observable physical particles as effects of an underlying subfermion theory.

This nonlinear subfermion theory is nonperturbatively regularized and canonically quantized. The quantum system is represented in a functional space and the dynamical equations are given by a functional Schrödinger equation, which results from an algebraic formulation of Heisenberg dynamics. This functional formalism respects the quantum field theoretic feature of inequivalent representations and in principle admits an explicit state space construction (for details we refer to [35]).

The derivation of effective dynamics for bound states of subfermions is performed by means of Weak Mapping [35]. This method consists in a mapping of the basic subfermion functional equation onto functional equations for systems of bound states. In a low energy limit the effective equations are expected to be identical with phenomenological theories. This program was successfully applied for instance to an effective $SU(2)$ Yang–Mills theory [32], [24]. First applications to the problem of gravitation were given in [31], [34].

For the application to graviton states we will derive the effective functional equations for a coupled system of bound gravitons and elementary fermions. As we restrict ourselves to the discussion of a classical gravitation theory, we will extract the classical part of these equations. For the evaluation of the resulting equations we have to discuss carefully the bound 'graviton' states and the emergence of the effective gravitation quantities. The evaluation of the linear part of the bosonic equations will lead to the linear Ricci– and Bianchi identities of a curved Riemann (or Riemann–Cartan) space, while from the fermion equations we will obtain a general covariant Dirac equation, i.e. the minimal coupling of the gravitational field to a 'matter' field according to the principle of equivalence.

The effective equations turn out to be equations for geometrical quantities in an *anholonomic* coordinate system which allows the treatment of spinors in curved spaces; this is a result of the flat Minkowski structure of the underlying subfermion theory.

2 Nonlinear Spinorfield Theory

The basic fermions of our model are described by Dirac spinors with an additional isospin degree of freedom. They obey the nonlinear field equation

$$(i\gamma^\mu\partial_\mu - m)^{\mathrm{reg}}_{\alpha\beta}\delta_{AB}\Psi_{B\beta}(x) = gV_{\substack{ABCD\\ \alpha\beta\gamma\delta}}\Psi_{B\beta}(x)\bar{\Psi}_{C\gamma}(x)\Psi_{D\delta}(x) \qquad (1)$$

where $\alpha = 1, \ldots 4$ is the spinor index and $A = 1, 2$ denotes the isospin.

The vertex is given by

$$V_{\substack{ABCD\\ \alpha\beta\gamma\delta}} := \frac{1}{2}\sum_{h=1}^{2}\left(\delta_{AB}\delta_{CD}v^h_{\alpha\beta}v^h_{\gamma\delta} - \delta_{AD}\delta_{CB}v^h_{\alpha\delta}v^h_{\gamma\beta}\right)\ ,\ v^1_{\alpha\beta} := \delta_{\alpha\beta}\ ,\ v^2_{\alpha\beta} := i\gamma^5_{\alpha\beta}\ . \qquad (2)$$

Equation (1) is nonperturbatively regularized by the third order kinetic operator

$$(i\gamma^\mu\partial_\mu - m)^{\mathrm{reg}}_{\alpha\beta} := \Big[(i\gamma^\mu\partial_\mu - m_1)(i\gamma^\nu\partial_\nu - m_2)(i\gamma^\rho\partial_\rho - m_3)\Big]_{\alpha\beta} \qquad (3)$$

with

$$(m_i)_{\alpha\beta} := m_i\delta_{\alpha\beta}\ , \quad i = 1, 2, 3. \qquad (4)$$

For later applications we assume the masses m_i to be very large compared with the mass scale of observable particles and we assume small differences from a mean mass m.

According to a decomposition theorem [29], [12] the third order equation (1) is equivalent to the following set of first order equations:

$$(i\gamma^\mu\partial_\mu - m_i)_{\alpha\beta}\delta_{AB}\psi_{B\beta i}(x) \qquad (5)$$

$$= g\lambda_i V_{\substack{ABCD\\ \alpha\beta\gamma\delta}} \sum_{j,k,l=1}^{3} \psi_{B\beta j}(x)\bar{\psi}_{C\gamma k}(x)\psi_{D\delta l}(x),\ i=1,2,3,$$

with the auxiliary fields

$$\psi_{A\alpha i}(x) := \lambda_i \prod_{\substack{k=1\\ k\neq i}}^{3} (i\gamma^\mu\partial_\mu - m_k)_{\alpha\beta}\delta_{AB}\Psi_{B\beta}(x) \quad (6)$$

and with

$$\Psi_{A\alpha}(x) = \sum_{i=1}^{3} \psi_{A\alpha i}(x)\ . \quad (7)$$

The regularization parameters λ_i are given by

$$\lambda_i := \prod_{\substack{k=1\\ k\neq i}}^{3} (m_i - m_k)^{-1}\ ; \quad (8)$$

they fulfill the relations

$$\sum_{i=1}^{3} \lambda_i = 0\ ,\quad \sum_{i=1}^{3} \lambda_i m_i = 0\ ,\quad \sum_{i=1}^{3} \lambda_i m_i^2 = 1\ . \quad (9)$$

The regularization of an expression is performed by summarizing over the auxiliary field indices i by making use of (9).

Introducing the charge conjugated spinor fields $\psi^c_{A\alpha i}(x)$ and the indices κ (isospin–/superspin index) and Z by

$$\begin{aligned} \psi_{\alpha\kappa i} &:= (\psi_{1\alpha i}, \psi_{2\alpha i}, \psi^c_{1\alpha i}, \psi^c_{2\alpha i}) \quad (10)\\ Z &:= (\alpha,\kappa,i) \end{aligned}$$

we combine equation (5) and its charge conjugated equation into one equation

$$\left(D^\mu_{Z_1Z_2}\partial_\mu - m_{Z_1Z_2}\right)\psi_{Z_2}(x) = U_{Z_1Z_2Z_3Z_4}\psi_{Z_2}(x)\psi_{Z_3}(x)\psi_{Z_4}(x) \quad (11)$$

with

$$D^\mu_{Z_1Z_2} := i\gamma^\mu_{\alpha_1\alpha_2}\delta_{i_1i_2}\delta_{\kappa_1\kappa_2}\ ,\ m_{Z_1Z_2} := m_{i_1}\delta_{\alpha_1\alpha_2}\delta_{i_1i_2}\delta_{\kappa_1\kappa_2}\ , \quad (12)$$

$$U_{Z_1Z_2Z_3Z_4} := \frac{g}{2}\lambda_{i_1}B_{i_2i_3i_4}\sum_{h=1}^{2}\left\{v^h_{\alpha_1\alpha_2}(v^hC)_{\alpha_3\alpha_4}\delta_{\kappa_1\kappa_2}\left[\gamma_5(1-\gamma^0)\right]_{\kappa_3\kappa_4}\right\}_{\mathrm{as}[2,3,4]}\ , \quad (13)$$

$$B_{i_2i_3i_4} = 1 \quad \text{for} \quad i_2,i_3,i_4 = 1,2,3. \quad (14)$$

Quantization of the system is obtained by deriving the canonical conjugated pairs of field variables from the corresponding Lagrange function and by assuming canonical anticommutation relations for them; these anticommutation relations read

$$[\psi_Z(\mathbf{r},t), \psi_{Z'}(\mathbf{r}',t)]_+ = A_{ZZ'}\delta(\mathbf{r}-\mathbf{r}') \quad (15)$$

with

$$A_{ZZ'} := \lambda_i\delta_{ii'}\gamma^5_{\kappa\kappa'}(C\gamma^0)_{\alpha\alpha'}\ . \quad (16)$$

We use a still more condensed notation by collecting the algebraic indices and the space coordinates in a superindex $I := (Z,\mathbf{r})$. With this abbreviation the field equations (11) may be written as

$$i\frac{\partial}{\partial t}\psi_{I_1}(t) = K_{I_1I_2}\psi_{I_2}(t) + W_{I_1I_2I_3I_4}\psi_{I_2}(t)\psi_{I_3}(t)\psi_{I_4}(t) \quad (17)$$

with

$$K_{I_1I_2} := iD^0_{Z_1Z_3}\left(D^k_{Z_3Z_2}\partial_k - m_{Z_3Z_2}\right)\delta(\mathbf{r}_1-\mathbf{r}_2)\ , \quad (18)$$

$$W_{I_1I_2I_3I_4} := -iD^0_{Z_1Z_5}U_{Z_5Z_2Z_3Z_4}\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_3)\delta(\mathbf{r}_1-\mathbf{r}_4) \quad (19)$$

and the quantization conditions (15) read now

$$[\psi_I(t), \psi_{I'}(t)]_+ = A_{II'} \quad (20)$$

with

$$A_{II'} = A_{ZZ'}\delta(\mathbf{r}-\mathbf{r}')\ . \quad (21)$$

The quantum dynamics of the fundamental spinor field system is formulated in terms of antisymmetrized matrix elements

$$\tau^{(n)}(I_1\dots I_n, t|a) = \langle 0|\mathcal{A}\left(\psi_{I_1}(t)\dots\psi_{I_n}(t)\right)|a\rangle \quad (22)$$

with the antisymmetrization operator $\mathcal{A}$. We remark, that we take the $\tau^{(n)}$–functions for equal times $t_1 = \dots = t_n = t$; this corresponds to the use of GNS–basis states in the algebraic description of quantum fields.

The usage of one–time functions leads to a Hamilton formalism instead of an explicit covariant formulation and can be shown to be crucial for the possibility of an explicit quantum field theoretic state space construction [35]. Nevertheless in order to obtain the Lorentz contractions of bound states for their calculation we shall use the covariant formalism as an intermediate step, which for these states results in generalized de Broglie–Bargmann–Wigner equations.

Any quantum state $|a\rangle$ of the spinor field system can equally well be described by a set of $\tau^{(n)}(a)$–functions or by a functional state

$$|\mathcal{A}(j;a)\rangle := \sum_{n=0}^{\infty} \frac{i^n}{n!} \sum_{I_1 \dots I_n} \tau^{(n)}(I_1 \dots I_n, t|a) j_{I_1} \dots j_{I_n} |0\rangle_F \ . \tag{23}$$

The functional creation operators j_I are, together with corresponding annihilation operators ∂_I, the generators of a CAR–algebra which is represented in a Fock space with the cyclic vector $|0\rangle_F$. We stress, that this functional Fock space structure is completely independent from the structure of the representation space of the spinorfields.

The system dynamics can be formulated by equations for the functional states (23). However, these functionals are not a suitable starting point for the derivation of a composite particle dynamics because they contain uncorrelated parts which prevent a proper composite particle interpretation. For the case under consideration it suffices to remove the fermion propagators $F_{I_1 I_2}$ by transition to normalordered functional states

$$|\mathcal{F}(j;a)\rangle := \exp\ (\frac{1}{2} j_{I_1} F_{I_1 I_2} j_{I_2}) |\mathcal{A}(j;a)\rangle \tag{24}$$

with the equal time propagator $F_{I_1 I_2}$.

The normalordered functional state $|\mathcal{F}(j;a)\rangle$ may be expanded as

$$|\mathcal{F}(j;a)\rangle = \sum_n \frac{i^n}{n!} \varphi^{(n)}(I_1 \dots I_n, t|a) j_{I_1} \dots j_{I_n} |0\rangle_F \ . \tag{25}$$

The system dynamics can be shown to be given by a Schrödinger equation in functional space [33]

$$\frac{\partial}{\partial t} |F(j;a)\rangle = \mathcal{H}_F(j,\partial) |F(j;a)\rangle \tag{26}$$

with the functional Hamiltonian

$$\begin{aligned} \mathcal{H}_F(j,\partial) &= K_{I_1 I_2} j_{I_1} \partial_{I_2} \\ &+ W_{I_1 I_2 I_3 I_4} \Big[j_{I_1} \partial_{I_4} \partial_{I_3} \partial_{I_2} - 3 F_{I_4 K} j_{I_1} j_K \partial_{I_3} \partial_{I_2} \\ &+ (3 F_{I_4 K_1} F_{I_3 K_2} + \frac{1}{4} A_{I_4 K_1} A_{I_3 K_2}) j_{I_1} j_{K_1} j_{K_2} \partial_{I_2} \\ &- (F_{I_4 K_1} F_{I_3 K_2} + \frac{1}{4} A_{I_4 K_1} A_{I_3 K_2}) F_{I_2 K_3} j_{I_1} j_{K_1} j_{K_2} j_{K_3} \Big] \ . \end{aligned} \tag{27}$$

Equation (27) may be regarded as an abbreviation of an infinite set of coupled equations for the $\varphi^{(n)}$–functions.

We consider equations (26) and (27) as a suitable starting point for the derivation of a composite particle dynamics; for details about the concept of functional space and the derivation of functional equations we refer to [35].

3 Effective Fermion–Graviton Equations

We consider the appearance of the gravitational force as a composite particle effect resulting from the subfermion dynamics (26). The gravitational force is assumed to be mediated by gravitons, which are generated by bound states of four elementary subspinor fields in accordance with the spin fusion theory of de Broglie [3], [4] and Tonnelat [38], [39] for gravitons. In order to perform our program we have to define four–particle bound states ('graviton' states) and to discuss the relation of these states to conventional graviton states with zero mass and spin 2, and we have to derive dynamical equations for these bound states from our basic spinorfield dynamics.

This subfermion dynamics is governed by equation (26), which describes all possible reactions and processes between the elementary fermions. Among these reactions there are bound state processes. The suitable means for the extraction of such processes, i.e. for the derivation of an effective dynamics of composite particles, is Weak Mapping.

Weak Mapping is defined as a reformulation of the subfermion dynamics with respect to certain bound states. This reformulation can be achieved by a mapping of the subfermion functional equation onto an effective functional equation for the corresponding bound states. As we intend to explain gravitation as a four–particle bound state effect, we choose these bound states to be just four–subfermion bound states which we couple to elementary subfermions[1].

In accordance with previous investigations [31] for a successfull derivation of an effective gravitation theory we have to deal with dressed bound states. The formal theory of Weak Mapping with dressed particle states was developed in [30] and [35]. As solutions of the full equation (26), dressed particle states contain an infinite number of polarization cloud parts, which are induced by a hard core part. For our application we consider the polarization cloud formalism only in the lowest order. This can be justified by an estimation of higher polarization cloud terms [35].

The effective coupled graviton–fermion system is formally described again in a functional space. The corresponding functional states are given by

$$|\mathcal{G}(b,f;a)\rangle := \sum_{m,n} \frac{i^n}{n!} \frac{1}{m!} \Theta^{(m,n)}(r_1 \dots r_m; l_1 \dots l_n, t|a) b_{r_1} \dots b_{r_m} f_{l_1} \dots f_{l_n} |v\rangle \tag{28}$$

[1] For simplicity we represent the matter coupling to gravitation by elementary subfermions; a more realistic model should deal with fermions as three–particle bound states [25].

with bosonic 'graviton' source operators b_r and fermion operators f_l. The functions $\Theta^{(m,n)}$ are the correlation functions of the coupled boson–fermion system; they are interpreted as matrix elements of corresponding phenomenological field operators describing these particles. The indices r and l are induced by the indices and the quantum numbers of corresponding bosonic and fermionic bound states.

Together with corresponding functional annihilation operators ∂_r^b and ∂_l^f, respectively, the bosonic functional operators b_r generate a CCR–algebra, whereas the effective fermion functional operators f_l generate a CAR–algebra. Both algebras are represented on the direct product of two Fock spaces with the cyclic vacuum state $|v\rangle$.

The transformation of the subfermion functional equation (26) into a functional equation for the effective states (28) is performed by means of the relations

$$b_r = C_{4,r}^{I_1 I_2 I_3 I_4} j_{I_1} j_{I_2} j_{I_3} j_{I_4} \tag{29}$$

and

$$f_l = C_{1,l}^{I_1} j_{I_1} \ . \tag{30}$$

Except for a center of mass part the functions $C_{4,r}$ and $C_{1,l}$ are four–particle and 'one–particle' hard core bound state functions, which are given as solutions of the diagonal part of (26) and which will be discussed in section 4; in particular the characterizing indices k and l have to be specified. Apart from bound states the solutions of the diagonal part of (26) contain also scattering states. However, due to the high subfermion masses and due to decoupling theorems these scattering parts will be suppressed in the final evaluations.

The polarization cloud terms appear in the inverse relations

$$j_{I_1} \dots j_{I_{2n-1}} = \sum_l R_{I_1 \dots I_{2n-1}}^{1,l} f_l \tag{31}$$

$$j_{I_1} \dots j_{I_{4n}} = \sum_k R_{I_1 \dots I_{4n}}^{4,r} b_r \tag{32}$$

for $n \in \mathbb{N}$.

The functions $R_{I_1 \dots I_{2n-1}}^{1,l}$ are duals corresponding to the fermion polarization cloud parts of the order $(2n-1)$, whereas the functions $R_{I_1 \dots I_{4n}}^{4,r}$ are duals for the bosonic polarization clouds. Together with the transformations (29) and (30) these relations define a consistent lowest order approximation of the full dressed particle formalism in the present case of a coupled system of four–particle bound states and elementary fermions.

For the hard core functions we assume the orthogonality relations

$$R_{I_1}^{1,l} C_{1,l'}^{I_1} = \delta_{ll'} \tag{33}$$

and

$$R_{I_1 I_2 I_3 I_4}^{4,r} C_{4,r'}^{I_1 I_2 I_3 I_4} = \delta_{rr'} \tag{34}$$

which have to be considered as approximations of corresponding dressed particle expressions in lowest order.

The mapping of the subfermion functional equation (26) onto an effective functional Schrödinger equation

$$\frac{\partial}{\partial t} |\mathcal{G}(b,f;a)\rangle = \mathcal{H}_{\mathrm{BF}}\Big(b,f,\partial^b,\partial^f\Big) |\mathcal{G}(b,f;a)\rangle \tag{35}$$

is performed by means of the invariance relation

$$|\mathcal{F}(j;a)\rangle = |\mathcal{G}(b,f;a)\rangle \tag{36}$$

with respect to (29) and (30). In a second step the functional subfermion Hamiltonian $\mathcal{H}_{\mathrm{F}}$ is transformed into the effective Hamiltonian $\mathcal{H}_{\mathrm{BF}}$ by means of (31), (32) and the functional chain rule.

This chain rule is a short–cut method of Weak Mapping which already takes into account the neglection of exchange forces in a low energy limit [35]. In our case the functional chain rule reads

$$\begin{aligned} \partial_I |\mathcal{G}(b,f;a)\rangle &= \Big[(\partial_I b_r)\partial_r^b + (\partial_I f_l)\partial_l^f\Big] \, |\mathcal{G}(b,f;a)\rangle \\ &= \Big[4C_{4,r}^{IK_1K_2K_3} j_{K_1} j_{K_2} j_{K_3} \partial_r^b + C_{1,l}^I \partial_l^f\Big] \, |\mathcal{G}(b,f;a)\rangle \ . \end{aligned} \tag{37}$$

Repeated application of the chain rule yields $\partial_{I_1}\partial_{I_2}|F\rangle$ and $\partial_{I_1}\partial_{I_2}\partial_{I_3}|F\rangle$ in terms of effective boson and fermion operators acting on the functional state $|G\rangle$.

By means of these transformations we obtain the following effective functional equation:

$$\begin{aligned}\frac{\partial}{\partial t}|\mathcal{G}(b,f;a)\rangle \;=\; & \Big\{K_{I_1I_2}\Big[4C^{I_2I_3I_4I_5}_{4,r'}R^{4,r}_{I_1I_3I_4I_5}b_r\partial^b_{r'}+C^{I_2}_{1,l'}R^{1,l}_{I_1}f_l\partial^f_{l'}\Big]+W_{I_1I_2I_3I_4}\Big[24C^{I_2I_3I_4K}_{4,r}R^{1,l_1}_{I_1}R^{1,l_2}_{K}f_{l_1}f_{l_2}\partial^b_r \\ & +144C^{I_3I_4K_1K_2}_{4,r'_1}C^{I_2K_3K_4K_5}_{4,r'_2}R^{4,r}_{I_1K_1K_2K_3K_4K_5}b_r\partial^b_{r'_1}\partial^b_{r'_2}+36C^{I_3I_4K_1K_2}_{4,r}C^{I_2}_{1,l'}R^{1,l}_{I_1K_1K_2}f_l\partial^f_{l'}\partial^b_r \\ & -12C^{I_4K_1K_2K_3}_{4,r'}C^{I_3}_{1,l_1}C^{I_2}_{1,l_2}R^{4,r}_{I_1K_1K_2K_3}\partial^f_{l_2}\partial^f_{l_1}b_r\partial^b_{r'}+C^{I_4}_{1,l_1}C^{I_3}_{1,l_2}C^{I_2}_{1,l_3}R^{1,l'}_{I_1}f_{l'}\partial^f_{l_1}\partial^f_{l_2}\partial^f_{l_3} \\ & -3F_{I_4K_1}\Big(12C^{I_2I_3K_2K_3}_{4,r}R^{4,r'}_{I_1K_1K_2K_3}b_{r'}\partial^b_r \qquad +C^{I_2}_{1,l_1}C^{I_3}_{1,l_2}R^{1,l'_1}_{I_1}R^{1,l'_2}_{K_1}f_{l'_1}f_{l'_2}\partial^f_{l_2}\partial^f_{l_1}\Big) \\ & +(3F_{I_4K_1}F_{I_3K_2}+\frac{1}{4}A_{I_4K_1}A_{I_3K_2})\Big(C^{I_2}_{1,l'}R^{1,l}_{I_1K_1K_2}f_l\partial^f_{l'} \qquad +4C^{I_2K_3K_4K_5}_{4,r}R^{4,r'}_{I_1K_1K_2K_3K_4K_5}b_{r'}\partial^b_r\Big) \\ & -(F_{I_4K_1}F_{I_3K_2}+\frac{1}{4}A_{I_4K_1}A_{I_3K_2})F_{I_2K_3}R^{4,r}_{I_1K_1K_2K_3}b_r\Big]\Big\}|\mathcal{G}(b,f;a)\rangle\ ,\end{aligned}\tag{38}$$

where we have taken into account only first order polarization cloud terms.

We consider equation (38) to be a formulation of the complete dynamics of the system of four–particle bound states and elementary fermions in a low energy limit, including also quantization effects. For a first evaluation we intend to compare this dynamics with a phenomenological classical dynamics of gravitons and fermions; thus we have to extract a classical part from the functional equation (38).

This can be achieved by means of the ansatz [31]

$$|\mathcal{G}(b,f;a)\rangle = \exp\left(Z_0[b,f;a]\right)|v\rangle \tag{39}$$

with

$$Z_0[b,f;a] := if_l\Theta^f_l(t)+b_r\Theta^b_r(t)\ , \tag{40}$$

where Θ^f and Θ^b are classical field variables for the fermion and the boson dynamics. By comparing the ansatz (39) with the original functional states (28) one verifies that the effect of this ansatz is the transition from one–particle matrix elements $\langle 0|\chi_r|a\rangle$ to classical functions χ_r and a factorization of higher order correlation functions $\Theta^{(m,n)}$ with respect to these functions.

Substitution of (39), (40) into (38) and projection onto one–particle states yields the classical boson equation

$$\begin{aligned}i\frac{\partial}{\partial t}\Theta^b_r \;=\; & 4K_{I_1I_2}C^{I_2I_3I_4I_5}_{4,r'}R^{4,r}_{I_1I_3I_4I_5}\Theta^b_{r'}+W_{I_1I_2I_3I_4}\Big\{-12C^{I_4K_1K_2K_3}_{4,r'}C^{I_3}_{1,l_1}C^{I_2}_{1,l_2}R^{4,r}_{I_1K_1K_2K_3}\Theta^f_{l_2}\Theta^f_{l_1}\Theta^b_{r'} \\ & -36F_{I_4K_1}C^{I_2I_3K_2K_3}_{4,r'}R^{4,r}_{I_1K_1K_2K_3}\Theta^b_{r'}+144C^{I_3I_4K_1K_2}_{4,r'_1}C^{I_2K_3K_4K_5}_{4,r'_2}R^{4,r}_{I_1K_1K_2K_3K_4K_5}\Theta^b_{(r'_1}\Theta^b_{r'_2)} \\ & +4(3F_{I_4K_1}F_{I_3K_2}+\frac{1}{4}A_{I_4K_1}A_{I_3K_2})C^{I_2K_3K_4K_5}_{4,r'}R^{4,r}_{I_1K_1K_2K_3K_4K_5}\Theta^b_{r'}\Big\}\ ,\end{aligned}\tag{41}$$

and the classical fermion equation

$$\begin{aligned}i\frac{\partial}{\partial t}\Theta^f_l \;=\; & K_{I_1I_2}C^{I_2}_{1,l'}R^{1,l}_{I_1}\Theta^f_{l'}+W_{I_1I_2I_3I_4}\Big\{36C^{I_3I_4K_1K_2}_{4,r}C^{I_2}_{1,l'}R^{1,l}_{I_1K_1K_2}\Theta^f_{l'}\Theta^b_r \\ & -C^{I_4}_{1,l'_1}C^{I_3}_{1,l'_1}C^{I_2}_{1,l'_3}R^{1,l}_{I_1}\Theta^f_{l'_3}\Theta^f_{l'_2}\Theta^f_{l'_1} \qquad +(3F_{I_4K_1}F_{I_3K_2}+\frac{1}{4}A_{I_4K_1}A_{I_3K_2})R^{1,l}_{I_1K_1K_2}C^{I_2}_{1,l'}\Theta^f_{l'}\Big\}\ .\end{aligned}\tag{42}$$

One can easily understand the meaning of the various terms: The first term of (41) is the kinetic boson term, the third a linear and the fourth a quadratic boson term, while the second term describes the coupling of a fermion current to the boson equation and the last term is an additional linear term stemming from the subfermion quantization. Analogously the first term of (42) is the kinetic fermion term and the second describes the coupling of bosons to the fermion equations. The last terms are the residual fermion interaction and a linear correction term, which leads to a fermion mass correction and which we are not interested in for the moment.

4 Bound States and Dressed Particle States

For the evaluation of the various terms of the effective equations (41) and (42) we need the explicit form of the hard core states and the first polarization cloud parts. However, for a first examination we restrict ourselves to the linear part of the boson equation, without the fermion coupling term and without the linear quantization term. Due to this restriction we have to discuss only four–particle hard core states with their duals as well as fermion hard core states and the first fermion polarization cloud part with their respectively duals.

We start with the discussion of 'graviton' states, which according to the fusion idea of de Broglie [4] we assume to be bound states of four spin 1/2–subfermions. However, as the original de Broglie–Bargmann–Wigner equations [1] for spin–2 particles are too restrictive for the derivation of a gravitation theory[2], the original fusion concept of de Broglie has to be generalized. Our generalized fusion generates spin 2–states as well as spin 0– and spin 1–parts. As in principle we intend to derive a nonlinear gravitation theory we will not a priori restrict our graviton states to spin 2 but leave it to the full effective dynamics to select proper eigenstates.

'Graviton' bound states were calculated in [34] and [35]. We give an improved version of these treatments with respect to the regularization procedure, the numerical functions and the discussion of the quantum numbers.

In our functional framework hard core bound states are defined as solutions of the diagonal part of equation (26). However, in order to obtain the correct Lorentz transformation properties of the bound state 'wavefunctions' it is convenient to calculate bound states in a corresponding covariant many–time formalism and afterwards to perform a one–time limit. Thus four–particle hard core bound states are defined to be solutions of a set of generalized de Broglie–Bargmann–Wigner equations which directly follow from the covariant dynamics. For brevity we do not explicitly derive these equations but refer to [35].

In addition it can be shown [35] that indeed the one-time version of the diagonal part can be used to fix the states of the many–time covariant dynamics. So in principle the covariant dynamics is determined by the Hamilton formulation. The defining bound state equation in its one time version reads

$$\begin{aligned}\varphi^{(4)}(I_1,I_2,I_3,I_4,t|a) = 3G_{I_4K_1}W_{K_1K_2K_3K_4}\Big[F_{K_2I_3}\varphi^{(4)}(I_1,I_2,K_3,K_4,t|a) \\ -F_{K_2I_2}\varphi^{(4)}(I_1,I_3,K_3,K_4,t|a) + F_{K_2I_1}\varphi^{(4)}(I_2,I_3,K_3,K_4,t|a)\Big]_{\mathrm{as}[I_1\ldots I_4]}\end{aligned} \tag{43}$$

where $G_{I_1I_2}$ is the inverse of the kinetic operator $K_{I_1I_2}$.

For the solution of the integral equation (43) we assume that the four–particle bound states are built up by the fusion of two two–particle bound states with spin 1– and spin 0–parts. Such two–particle states as exact solutions of the defining diagonal equations were calculated in [22]. In a low energy limit and in a symmetric s–wave approximation these two–particle bound states read

$$\begin{aligned}\varphi^{(2)}_{\substack{i_1 i_2\\ \kappa_1\kappa_2\\ \alpha_1\alpha_2}}(\mathbf{r}_1,\mathbf{r}_2,t|k) = N^{(2)}T^a_{\kappa_1\kappa_2}e^{ik_0t}e^{-i\mathbf{k}\frac{\mathbf{r}_1-\mathbf{r}_2}{2}} \quad \times \\ \left[A_\mu U_{i_1i_2}\left(\frac{\mathbf{r}_1-\mathbf{r}_2}{2}\right)(\gamma^\mu C)_{\alpha_1\alpha_2} + F_{\mu\nu}V_{i_1i_2}\left(\frac{\mathbf{r}_1-\mathbf{r}_2}{2}\right)(\Sigma^{\mu\nu}C)_{\alpha_1\alpha_2}\right]\end{aligned} \tag{44}$$

with

$$U_{i_1i_2}(\mathbf{r}) = 4i\pi^2\lambda_{i_1}\lambda_{i_2}M_{i_1i_2}r^{-1}K_1(M_{i_1i_2}r) \tag{45}$$

$$V_{i_1i_2}(\mathbf{r}) = 2i\pi^2\lambda_{i_1}\lambda_{i_2}M_{i_1i_2}K_0(M_{i_1i_2}r) \tag{46}$$

where $M_{i_1i_2} := \frac{m_{i_1}+m_{i_2}}{2}$, $r := |\mathbf{r}|$ and $K_n(x)$ are the modified Bessel functions. The functions (45) and (46) are obtained by expansions with respect to small differences $\Delta_{i_1i_2} = m_{i_1} - m_{i_2}$ of the auxiliary masses m_i.

The bound state functions (44) are eigenstates to the momentum k, which is the center–of–mass momentum; the matrices T^a are antisymmetric matrices of the Dirac algebra which are related to the isospin and fermion quantum numbers.

For the four–particle bound states we assume that their dependence on the internal relative coordinates of the two constituting two–particle states is given by the functions (45) and (46), whereas the functional dependence on the relative coordinate of the centers–of–mass of these constituents is determined by the defining equation (43). Thus we make the 'graviton' state ansatz

[2]For the relations between de Broglie–Bargmann–Wigner equations and spin–2 graviton representations see [20], [26], [21], [27].

$$\varphi^{(4,g)}_{Z_1Z_2Z_3Z_4}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,t|k) = N^{(4)} t_{ab} T^a_{\kappa_1\kappa_2} T^b_{\kappa_3\kappa_4} e^{ik_0 t} e^{-i\frac{\mathbf{k}}{4}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4)} \times \tag{47}$$
$$\Big\{ U_{i_1i_2}(\mathbf{r}_1-\mathbf{r}_2)U_{i_3i_4}(\mathbf{r}_3-\mathbf{r}_4)(\gamma^\mu C)_{\alpha_1\alpha_2}(\gamma^\nu C)_{\alpha_3\alpha_4} X_{\mu\nu}\big[(\mathbf{r}_1+\mathbf{r}_2)-(\mathbf{r}_3+\mathbf{r}_4)|k\big]$$
$$+U_{i_1i_2}(\mathbf{r}_1-\mathbf{r}_2)V_{i_3i_4}(\mathbf{r}_3-\mathbf{r}_4)(\gamma^\mu C)_{\alpha_1\alpha_2}(\Sigma^{\rho\sigma} C)_{\alpha_3\alpha_4} Y_{\mu\rho\sigma}\big[(\mathbf{r}_1+\mathbf{r}_2)-(\mathbf{r}_3+\mathbf{r}_4)|k\big]$$
$$+V_{i_1i_2}(\mathbf{r}_1-\mathbf{r}_2)U_{i_3i_4}(\mathbf{r}_3-\mathbf{r}_4)(\Sigma^{\rho\sigma} C)_{\alpha_1\alpha_2}(\gamma^\mu C)_{\alpha_3\alpha_4} \bar{Y}_{\rho\sigma\mu}\big[(\mathbf{r}_1+\mathbf{r}_2)-(\mathbf{r}_3+\mathbf{r}_4)|k\big]$$
$$+V_{i_1i_2}(\mathbf{r}_1-\mathbf{r}_2)V_{i_3i_4}(\mathbf{r}_3-\mathbf{r}_4)(\Sigma^{\mu\nu} C)_{\alpha_1\alpha_2}(\Sigma^{\rho\sigma} C)_{\alpha_3\alpha_4} Z_{\mu\nu\rho\sigma}\big[(\mathbf{r}_1+\mathbf{r}_2)-(\mathbf{r}_3+\mathbf{r}_4)|k\big]\Big\}$$

with the normalization constant $N^{(4)}$, (complex) coefficients t_{ab} and the unknown functions $X_{\mu\nu}, \dots Z_{\mu\nu\rho\sigma}$, which have to be determined by (43). The right–hand side of (47) has to be simultaneously antisymmetrized in all indices $I_1 \dots I_4$.

With the ansatz

$$\begin{aligned} X_{\mu\nu}(\mathbf{r}|k) &= B^X(\mathbf{r})X_{\mu\nu}(k) \quad , \quad Y_{\mu\rho\sigma}(\mathbf{r}|k) = B^Y(\mathbf{r})Y_{\mu\rho\sigma}(k) \\ \bar{Y}_{\rho\sigma\mu}(\mathbf{r}|k) &= B^{\bar{Y}}(\mathbf{r})\bar{Y}_{\rho\sigma\mu}(k) \quad , \quad Z_{\mu\nu\rho\sigma}(\mathbf{r}|k) = B^Z(\mathbf{r})Z_{\mu\nu\rho\sigma}(k) \end{aligned} \tag{48}$$

for the tensor functions $X, Y, \bar{Y}, Z$ and by assuming the functions $B^X, B^Y, B^{\bar{Y}}, B^Z$ to be even functions[3] we can approximately calculate these functions (s. [34]); for the functions B^Y and B^Z these calculations yield

$$B^Y(\mathbf{r}) = -g\frac{1}{2^4\pi^2} r^2 K_1[2mr]^2 \ , \tag{49}$$
$$B^Z(\mathbf{r}) = -g\frac{3^2 5^2}{2^5\pi^2}(mr)^2 K_1[2mr]^2 \tag{50}$$

with the subfermion coupling constant g and the mean auxiliary field mass m.

The possible quantum numbers of the states (47) are the energy–momentum, spin, isospin and the subfermion number. The corresponding eigenvalue equations for transition matrix elements or functional states, respectively, are given in [35]. Due to our ansatz we have an eigenstate of the center–of–mass momentum k. In addition we assume $k^2 = m_G^2$, i.e. we assume our state to be on the mass shell m_G with a unique graviton mass m_G. Due to $k^2 = m_G^2$ we have $X_{\mu\nu}(k) = X_{\mu\nu}(\mathbf{k})\dots$. We do not fix this graviton mass m_G for the moment, but we remark that the effective dynamical equations induce a mass renormalization which enables us to derive equations for gravitons with an effective renormalized mass $m'_G = 0$.

An analysis of the spin content of the states (47) shows, that these states in general are not spin eigenstates but they contain spin 0–, spin 1– and spin 2–parts. We do not restrict this variety of spins in our bound states at this stage but leave it to the effective dynamics to select the corresponding spin properties.

The bound state equations (43) do not determine the isospin–superspin dependence of the bound states. Thus we have to make a suitable ansatz for this part of the graviton states. The application of the principle of universal coupling to the formation of the graviton bound states demands that these bound states are built up just from the two elementary subfermions with isospin $A = 1$ and $A = 2$ and their antiparticles. Thus the graviton states have to be eigenstates to isospin 0 and subfermion number 0; this determines the states (47) to be given by

$$t_{ab} T^a_{\kappa_1\kappa_2} T^b_{\kappa_3\kappa_4} = T^{(1}_{\kappa_1\kappa_2} T^{2)}_{\kappa_3\kappa_4} \tag{51}$$

in their isospin–superspin part with

$$\begin{aligned} T^1_{\kappa_1\kappa_2} &:= -\frac{1}{2}(\gamma^0\gamma^5 C + \gamma^5 C)_{\kappa_1\kappa_2} \\ T^2_{\kappa_1\kappa_2} &:= \frac{1}{2}(\gamma^0\gamma^5 C - \gamma^5 C)_{\kappa_1\kappa_2} \ . \end{aligned} \tag{52}$$

We collect these results and write for the 'graviton' states

$$\varphi^{(4,g)}_{Z_1Z_2Z_3Z_4}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,t|\mathbf{k}) = \tag{53}$$
$$= N^{(4)} T^{(1}_{\kappa_1\kappa_2} T^{2)}_{\kappa_3\kappa_4} e^{iE_k t} e^{-i\frac{\mathbf{k}}{4}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4)} \Big[U_{i_1i_2}(\mathbf{r}_1-\mathbf{r}_2)U_{i_3i_4}(\mathbf{r}_3-\mathbf{r}_4)B^X[(\mathbf{r}_1+\mathbf{r}_2)-(\mathbf{r}_3+\mathbf{r}_4)] \quad \times$$
$$(\gamma^\mu C)_{\alpha_1\alpha_2}(\gamma^\nu C)_{\alpha_3\alpha_4} X_{\mu\nu}(\mathbf{k}) + U_{i_1i_2}(\mathbf{r}_1-\mathbf{r}_2)V_{i_3i_4}(\mathbf{r}_3-\mathbf{r}_4)B^Y[(\mathbf{r}_1+\mathbf{r}_2)-(\mathbf{r}_3+\mathbf{r}_4)] \quad \times$$

[3] which corresponds to our restriction to states with vanishing orbital momentum

$$(\gamma^\mu C)_{\alpha_1\alpha_2}(\Sigma^{\rho\sigma} C)_{\alpha_3\alpha_4} Y_{\mu\rho\sigma}(\mathbf{k}) + V_{i_1i_2}(\mathbf{r}_1-\mathbf{r}_2)U_{i_3i_4}(\mathbf{r}_3-\mathbf{r}_4)B^{\bar{Y}}[(\mathbf{r}_1+\mathbf{r}_2)-(\mathbf{r}_3+\mathbf{r}_4)] \quad \times$$
$$(\Sigma^{\rho\sigma} C)_{\alpha_1\alpha_2}(\gamma^\mu C)_{\alpha_3\alpha_4} \bar{Y}_{\rho\sigma\mu}(\mathbf{k}) + V_{i_1i_2}(\mathbf{r}_1-\mathbf{r}_2)V_{i_3i_4}(\mathbf{r}_3-\mathbf{r}_4)B^{Z}[(\mathbf{r}_1+\mathbf{r}_2)-(\mathbf{r}_3+\mathbf{r}_4)] \quad \times$$
$$\left.(\Sigma^{\mu\nu} C)_{\alpha_1\alpha_2}(\Sigma^{\rho\sigma} C)_{\alpha_3\alpha_4} Z_{\mu\nu\rho\sigma}(\mathbf{k})\right]_{\mathrm{as}[I_1,I_2,I_3,I_4]} .$$

The bound states (53) are the starting point for the derivation of an effective gravitation theory, which may be formulated in terms of the (classical) field variables $g_{\mu\nu}$ (metrical tensor), $\Gamma_{\mu\rho\sigma}$ (affine connection) and $R_{\mu\nu\rho\sigma}$ (curvature tensor). Thus we have to discuss the relation between these gravitational variables and our bound states.

In [35] it was demonstrated, that the center–of–mass amplitudes $X_{\mu\nu}, \ldots Z_{\mu\nu\rho\sigma}$ are in close relation to the phenomenological field variables or their expectation values, respectively, provided that the expansion functions in (29) and (30) differ from the bound state functions (53) just by these center–of–mass amplitudes. It is the separation of the center–of–mass amplitudes from the bound state functions (53) which provides for the generation of the correct effective gravitation variables. Hence if Weak Mapping has to allow for a nontrivial field dynamics, these quantities must not be fixed by (43). Rather we have to give up the couplings between these quantities following from (43) and regard them as independend quantities for the application of the Weak Mapping procedure.

In the graviton case these amplitudes are given by the set

$$\overset{\circ}{X}_r(\mathbf{k}) \in \Big\{X_{\mu\nu}(\mathbf{k}),\ Y_{\mu\rho\sigma}(\mathbf{k}),\ \bar{Y}_{\rho\sigma\mu}(\mathbf{k}),\ Z_{\mu\nu\rho\sigma}(\mathbf{k})\Big\} \tag{54}$$

with

$$r = (r_1, r_2, r_3, r_4) = \Big((\mu\nu), (\mu\rho\sigma), (\rho\sigma\mu), (\mu\nu\rho\sigma)\Big) . \tag{55}$$

The separation of these quantities from the bound state functions (53) yields the four sets of expansion functions

$$C^{I_1I_2I_3I_4}_{4,r,k} := N^{(4)} T^{(1}_{\kappa_1\kappa_2} T^{2)}_{\kappa_3\kappa_4} e^{-i\frac{\mathbf{k}}{4}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4)} \quad \times \tag{56}$$
$$\left\{\begin{array}{l} U_{i_1i_2}(\mathbf{r}_1-\mathbf{r}_2)U_{i_3i_4}(\mathbf{r}_3-\mathbf{r}_4)B^X\Big(\frac{\mathbf{r}_1+\mathbf{r}_2}{4}-\frac{\mathbf{r}_3+\mathbf{r}_4}{4}\Big)(\gamma^\mu C)_{\alpha_1\alpha_2}(\gamma^\nu C)_{\alpha_3\alpha_4} \\ U_{i_1i_2}(\mathbf{r}_1-\mathbf{r}_2)V_{i_3i_4}(\mathbf{r}_3-\mathbf{r}_4)B^Y\Big(\frac{\mathbf{r}_1+\mathbf{r}_2}{4}-\frac{\mathbf{r}_3+\mathbf{r}_4}{4}\Big)(\gamma^\mu C)_{\alpha_1\alpha_2}(\Sigma^{\rho\sigma} C)_{\alpha_3\alpha_4} \\ V_{i_1i_2}(\mathbf{r}_1-\mathbf{r}_2)U_{i_3i_4}(\mathbf{r}_3-\mathbf{r}_4)B^{\bar{Y}}\Big(\frac{\mathbf{r}_1+\mathbf{r}_2}{4}-\frac{\mathbf{r}_3+\mathbf{r}_4}{4}\Big)(\Sigma^{\rho\sigma} C)_{\alpha_1\alpha_2}(\gamma^\mu C)_{\alpha_3\alpha_4} \\ V_{i_1i_2}(\mathbf{r}_1-\mathbf{r}_2)V_{i_3i_4}(\mathbf{r}_3-\mathbf{r}_4)B^Z\Big(\frac{\mathbf{r}_1+\mathbf{r}_2}{4}-\frac{\mathbf{r}_3+\mathbf{r}_4}{4}\Big)(\Sigma^{\mu\nu} C)_{\alpha_1\alpha_2}(\Sigma^{\rho\sigma} C)_{\alpha_3\alpha_4} \end{array}\right\}$$

We have to give some comments with respect to the quantities $\overset{\circ}{X}_r$. According to (53) they have the symmetry properties

$$\begin{aligned} Y_{\mu\rho\sigma} &= Y_{\mu[\rho\sigma]} , \\ \bar{Y}_{\rho\sigma\mu} &= \bar{Y}_{[\rho\sigma]\mu} , \\ Z_{\mu\nu\rho\sigma} &= Z_{[\mu\nu][\rho\sigma]} . \end{aligned} \tag{57}$$

The complete antisymmetrization of the bound state functions (53) would induce the further conditions

$$\begin{aligned} X_{\mu\nu} &= X_{\nu\mu} , \\ Y_{\mu\rho\sigma} &= \bar{Y}_{\rho\sigma\mu} , \\ Z_{\mu\nu\rho\sigma} &= Z_{\rho\sigma\mu\nu} . \end{aligned} \tag{58}$$

By Weak Mapping these symmetry properties induce the symmetry properties of the effective gravitation quantities. These conditions are just the symmetry properties for the metric tensor, the affine connection and the curvature tensor of a Riemann space V_4 in anholonomic coordinates [28]. However, for the derivation of an effective gravitation theory coupled to elementary fermions the concept of the Riemann space has to be enlarged; we assume a Riemann–Cartan geometry to be a suitable framework for the formulation of a generalized gravitation theory[4]. The symmetry conditions (58) induced by the antisymmetrization of the bound state functions are just those which do not hold in a Riemann–Cartan space. Thus in order to induce a generalized geometry we have to break this antisymmetrization. This can be achieved by a suitable breaking of the isospin symmetry of the fundamental subspinors [11], which lifts the degeneracy of particles in multipletts and makes them distinguishable, i.e. lifts antisymmetrization.

We do not perform this symmetry breaking conse-

[4]For an application of Riemann–Cartan geometry to Poincaré gauge theory see [16].

quently but give up only the second of the conditions (58), i.e. we consider the quantities Y and $\bar{Y}$ as independent. Concerning X and Z this means that for a first evaluation we restrict ourselves to a Riemann space V_4. This is no contradiction to our treatment of a combined graviton–fermion system as in this paper we will derive only the linear gravitation equations which are formally identical for Riemann and for Riemann–Cartan space. We stress, however, that breaking the isospin symmetry of the subfermions induces a natural mechanism for the derivation of an effective Riemann–Cartan geometry.

For the evaluation of the linear part of the effective graviton equations (41) we need the dual functions $R^{4,r,k}_{I_1I_2I_3I_4}$, which are defined by the orthogonality relations

$$R^{4,r,k}_{I_1I_2I_3I_4} C^{I_1I_2I_3I_4}_{4,r',k'} = \delta_{kk'}\delta_{rr'} \ . \tag{59}$$

It can be shown that the following functions fulfill the conditions (59):

$$R^{4,r,k}_{I_1I_2I_3I_4} := \frac{1}{4(2\pi)^3}\frac{1}{N^{(4)}} T^{(1}_{\kappa_1\kappa_2} T^{2)}_{\kappa_3\kappa_4} e^{i\frac{\mathbf{k}}{4}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4)} \quad \times \tag{60}$$

$$\begin{Bmatrix} N_1 \hat{U}_{i_1i_2}(\mathbf{r}_1-\mathbf{r}_2)\hat{U}_{i_3i_4}(\mathbf{r}_3-\mathbf{r}_4)\hat{B}^X\left(\frac{\mathbf{r}_1+\mathbf{r}_2-\mathbf{r}_3-\mathbf{r}_4}{4}\right)(\gamma^\mu C)^+_{\alpha_1\alpha_2}(\gamma^\nu C)^+_{\alpha_3\alpha_4} \\ N_2 \hat{U}_{i_1i_2}(\mathbf{r}_1-\mathbf{r}_2)\hat{V}_{i_3i_4}(\mathbf{r}_3-\mathbf{r}_4)\hat{B}^Y\left(\frac{\mathbf{r}_1+\mathbf{r}_2-\mathbf{r}_3-\mathbf{r}_4}{4}\right)(\gamma^\mu C)^+_{\alpha_1\alpha_2}(\Sigma^{\rho\sigma} C)^+_{\alpha_3\alpha_4} \\ N_3 \hat{V}_{i_1i_2}(\mathbf{r}_1-\mathbf{r}_2)\hat{U}_{i_3i_4}(\mathbf{r}_3-\mathbf{r}_4)\hat{B}^{\bar{Y}}\left(\frac{\mathbf{r}_1+\mathbf{r}_2-\mathbf{r}_3-\mathbf{r}_4}{4}\right)(\Sigma^{\rho\sigma} C)^+_{\alpha_1\alpha_2}(\gamma^\mu C)^+_{\alpha_3\alpha_4} \\ N_4 \hat{V}_{i_1i_2}(\mathbf{r}_1-\mathbf{r}_2)\hat{V}_{i_3i_4}(\mathbf{r}_3-\mathbf{r}_4)\hat{B}^Z\left(\frac{\mathbf{r}_1+\mathbf{r}_2-\mathbf{r}_3-\mathbf{r}_4}{4}\right)(\Sigma^{\mu\nu} C)^+_{\alpha_1\alpha_2}(\Sigma^{\rho\sigma} C)^+_{\alpha_3\alpha_4} \end{Bmatrix}$$

with

$$\hat{B}^X(\mathbf{r}) \ = \ \hat{B}^Y(\mathbf{r}) = \hat{B}^Z(\mathbf{r}) = 1 \ , \tag{61}$$

$$\hat{U}_{i_1i_2}(\mathbf{r}) \ := \ (\lambda_{i_1}\lambda_{i_2})^{-1} M_{i_1i_2} \ , \tag{62}$$

$$\hat{V}_{i_1i_2}(\mathbf{r}) \ := \ (\lambda_{i_1}\lambda_{i_2})^{-1} M^2_{i_1i_2} \tag{63}$$

where $M_{i_1i_2} = \frac{m_{i_1}+m_{i_2}}{2}$.

The normalization constants $N_k,\ k=1,2,3$, are uniquely fixed by (59) and can be approximately calculated.

For the evaluation of the effective fermion equations (42) we also have to calculate dressed fermion states. These calculations were done in [34] by means of an iteration procedure. In a strong coupling limit one obtains for the first polarization cloud term

$$C^{I_1I_2I_3}_{1,l} = N^{(3)} A_{[I_1I_2I_3]I_4} C^{I_4}_{1,l} \tag{64}$$

with

$$A_{I_1I_2I_3I_4} = g\left[C_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} - (\gamma^5 C)_{\alpha_1\alpha_2}\gamma^5_{\alpha_3\alpha_4}\right]\gamma^5_{\kappa_1\kappa_2}\delta_{\kappa_3\kappa_4} \quad \times \tag{65}$$

$$\lambda_{i_1}\lambda_{i_2}\lambda_{i_3}\frac{m^2_{i_1}m^2_{i_2}}{m_{i_3}}\delta_{i_3i_4}\frac{K_1(m_{i_1}|\mathbf{r}_3-\mathbf{r}_1|)}{|\mathbf{r}_3-\mathbf{r}_1|}\frac{K_1(m_{i_2}|\mathbf{r}_3-\mathbf{r}_2|)}{|\mathbf{r}_3-\mathbf{r}_2|}\delta(\mathbf{r}_3-\mathbf{r}_4)$$

and the free Dirac fields (the fermion hard core states) $C^I_{1,l}$, which read in our notation

$$\begin{aligned} C^I_{1,l} &\equiv C(\alpha,\kappa,i,\mathbf{r}|s,\rho,j,\mathbf{k}) \\ &= \lambda_j\delta_{ij}e^{-i\mathbf{r}\mathbf{k}}\delta_{\kappa\rho}\chi^s_\alpha(\mathbf{k}) \end{aligned} \tag{66}$$

with the ordinary Dirac spinors $\chi^s_\alpha(\mathbf{k})$ for spin $s = \pm 1/2$. Compared with [34] the functions (64), (65) contain a modified coordinate dependence because of an improved realization of the regularization.

For the calculation of the corresponding duals $R^{1,l}_{I_1I_2I_3}$ we do not apply the full formalism of [34]. We make the ansatz

$$R^{1,l}_{I_1I_2I_3} = N^{(3)} g^{-2}\frac{A_{[I_1I_2I_3]I_4}}{\lambda_{i_1}\lambda_{i_2}\lambda_{i_3}}\frac{R^{1,l}_{I_4}}{\lambda_j} \tag{67}$$

with the fermion hard core dual

$$R^{1,l}_I := (\lambda_j)^{-2} C^I_{1,l} \ , \tag{68}$$

where we remember $l = (s,\rho,j,\mathbf{k})$.

This ansatz differs from the dual calculated in [34] with respect to the auxiliary field dependence; it can, however, be shown, that (67) fulfills exactly the orthogonality conditions

$$R^{1,l}_{I_1I_2I_3} C^{I_1I_2I_3}_{1,l'} = N\delta_{ll'} \ . \tag{69}$$

For the evaluation of the effective dynamical equations we also need the subfermion propagator $F_{I_1I_2}$. In principle this propagator has to be a part of the solution of the full subfermion functional equation.

However, in [35] it was argued that the free propagator can be used as a suitable approximation. This free propagator reads [23], [35]

$$F_{Z_1Z_2}(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{(2\pi)^2}\lambda_{i_1}\delta_{i_1i_2}\gamma^5_{\kappa_1\kappa_2} \quad \times \tag{70}$$
$$\left\{-m_{i_1}^2\frac{K_1(m_{i_1}r)}{r}C_{\alpha_1\alpha_2} + im_{i_1}(\gamma^k C)_{\alpha_1\alpha_2}r^k\left[\frac{2K_1(m_{i_1}r)}{r^3} + m_{i_1}\frac{K_0(m_{i_1}r)}{r^2}\right]\right\}$$

with $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2,\ r := |\mathbf{r}|$.

With the help of these functions we are able to evaluate the linear part of the effective graviton–fermion equations (41) and (42).

5 Gravitational Equations

The classical part of the effective dynamical boson equation is given by (41). As we indicated above, for a first examination we omit the nonlinear parts and the quantization term of this equation and restrict ourselves to the evaluation only of the linear part (without coupling to the fermion matter), which we expect to give a linearized vacuum gravitation theory. This linear vacuum part is given by

$$\begin{aligned} i\frac{\partial}{\partial t}\Theta^b_r &= 4K_{I_1I_2}C^{I_2I_3I_4I_5}_{4,r'}R^{4,r}_{I_1I_3I_4I_5}\Theta^b_{r'} \\ &\quad -36W_{I_1I_2I_3I_4}F_{I_4K_1}C^{I_2I_3K_2K_3}_{4,r'}R^{4,r}_{I_1K_1K_2K_3}\Theta^b_{r'}\ . \end{aligned} \tag{71}$$

The effective boson quantities of equation (41) were denoted by Θ^b_r with some indices r. These indices are specified by the definition of the Weak Mapping expansion functions (56), which are characterized by the indices

$$r = (r_1,r_2,r_3,r_4) = \Big((\mu\nu),(\mu\rho\sigma),(\rho\sigma\mu),(\mu\nu\rho\sigma)\Big) \tag{72}$$

and by the three–momentum $\mathbf{k}$. By the defining Weak Mapping relations (29) these indices are transferred to the boson operators b_r and by the classical ansatz (39), (40) the effective boson quantities are specified to read

$$\Theta^b_r(\mathbf{k},t) \in \left\{\Theta^X_{\mu\nu}(\mathbf{k},t),\Theta^Y_{\mu\rho\sigma}(\mathbf{k},t),\Theta^{\bar{Y}}_{\rho\sigma\mu}(\mathbf{k},t),\Theta^Z_{\mu\nu\rho\sigma}(\mathbf{k},t)\right\} \tag{73}$$

with the symmetries (see section 4)

$$\begin{aligned} \Theta^X_{\mu\nu} &= \Theta^X_{\nu\mu}\ , \\ \Theta^Y_{\mu\rho\sigma} &= \Theta^Y_{\mu[\rho\sigma]}\ , \\ \Theta^{\bar{Y}}_{\rho\sigma\mu} &= \Theta^{\bar{Y}}_{[\rho\sigma]\mu}\ , \\ \Theta^Z_{\mu\nu\rho\sigma} &= \Theta^Z_{[\mu\nu][\rho\sigma]}\ , \\ \Theta^Z_{\mu\nu\rho\sigma} &= \Theta^Z_{\rho\sigma\mu\nu}\ . \end{aligned} \tag{74}$$

Thus we are prepared to evaluate the various terms of equation (41) by inserting the kinetic operator (18), the vertex (19), the propagator (70), the expansion functions (56) and their duals (60). For this evaluation one has to take into account the antisymmetrization of the expansion functions according to section 4. The integrations in coordinate space are performed in a strong coupling limit for the Bessel functions; the nonperturbative subfermion regularization, i.e. the summation over the auxiliary field indices with (9) and the subsequent transition to the mean subfermion mass m, guarantees the occurence of finite constants only. The algebraic calculations, which determine the structure of the resulting effective equations, are performed exactly.

We obtain for the fouriertransformed quantities $\Theta^b_r(\mathbf{r},t)$ of $\Theta^b_r(\mathbf{k},t)$ the equations

$$\partial_0\Theta^X_{\mu\nu}(\mathbf{r},t) = \partial_k\left[\delta_{0\mu}\Theta^X_{k\nu}(\mathbf{r},t) + \delta_{k\mu}\Theta^X_{0\nu}(\mathbf{r},t)\right] + (ma_1+gc_1)\,\Theta^{\bar{Y}}_{0\mu\nu}(\mathbf{r},t)\ , \tag{75}$$

$$\begin{aligned} \partial_0\Theta^Y_{\mu\rho\sigma}(\mathbf{r},t) &= \partial_k\left[\delta_{0\mu}\Theta^Y_{k\rho\sigma}(\mathbf{r},t) + \delta_{k\mu}\Theta^Y_{0\rho\sigma}(\mathbf{r},t)\right] + (ma_2+gc_2)\,\Theta^Z_{0\mu\rho\sigma}(\mathbf{r},t) \\ &\quad +\frac{1}{4}gc_2\,\varepsilon^{0\mu\mu'\nu'}\varepsilon_\rho{}^{\sigma\rho'\sigma'}\Theta^Z_{\mu'\nu'\rho'\sigma'}(\mathbf{r},t)\ , \end{aligned} \tag{76}$$

$$\partial_0\Theta^{\bar{Y}}_{\rho\sigma\mu}(\mathbf{r},t) = 2\partial_k\left[\delta_{0\rho}\Theta^{\bar{Y}}_{k\sigma\mu}(\mathbf{r},t) + \delta_{k\rho}\Theta^{\bar{Y}}_{0\sigma\mu}(\mathbf{r},t)\right] + (ma_3+gb_1+gc_3)\,\delta_{0\rho}\Theta^X_{\mu\sigma}(\mathbf{r},t)\ , \tag{77}$$

$$\begin{aligned} \partial_0\Theta^Z_{\mu\nu\rho\sigma}(\mathbf{r},t) &= 2\partial_k\left[\delta_{0\mu}\Theta^Z_{k\nu\rho\sigma}(\mathbf{r},t) + \delta_{k\mu}\Theta^Z_{0\nu\rho\sigma}(\mathbf{r},t)\right] + (ma_4+gb_2+gc_4)\,\delta_{0\mu}\Theta^Y_{\nu\rho\sigma}(\mathbf{r},t) \\ &\quad +\frac{1}{4}gc_4\,\varepsilon^{\mu\nu0\mu'}\varepsilon_\rho{}^{\sigma\rho'\sigma'}\Theta^Y_{\mu'\rho'\sigma'}(\mathbf{r},t) \end{aligned} \tag{78}$$

where the right–hand sides have to be symmetrized or antisymmetrized according to (74).

The constants $a_1 \dots c_4$ may be approximately calculated; for those which we will use in the following we obtain

$$\begin{aligned} a_2 &= m\,3^2 \cdot 5^2 \qquad (79)\\ a_4 &= -m^{-1}\frac{2^2}{3^2 \cdot 5^2}\\ b_2 &= \frac{1}{m^2}\frac{1}{\pi^2 2 \cdot 3^2 \cdot 5^2}\\ c_2 &= \frac{2^4 \cdot 5}{\pi^3}\\ c_4 &= -\frac{1}{m^2}\frac{2^7}{\pi^3 3^4 \cdot 5^3}\ . \end{aligned}$$

We stress again, that the applied approximations effect only the values of these constants but not the structure of equations (75)—(78).

As a consequence of breaking the symmetry $Y = \bar{Y}$ (section 4) we obtain two sets of coupled linear equations for Θ^X and $\Theta^{\bar{Y}}$ and for Θ^Y and Θ^Z. We start with the discussion of the coupled equations (76) and (78).

One observes that the terms containing b_i and some of the terms containing c_i are corrections to the mass terms with a_i. However, there are two terms, which seem not to fit into this scheme; they have to be discussed separately.

Concerning the last term of (76) an explicit evaluation shows, that this term is equal to $-gc_2\Theta^Z_{0\mu\rho\sigma}$, provided the quantities Θ^Z fulfill the symmetry conditions (74) and the additional constraint

$$\Theta^{Z}{}_{\mu\nu}{}^{\mu}{}_{\sigma} = 0\ . \qquad (80)$$

These are just the symmetry conditions of the Weyl tensor in a Riemann space V_4 [41]. With (80) and the transformation $ma_2\Theta^Z_{\mu\nu\rho\sigma} \to \Theta^Z_{\mu\nu\rho\sigma}$ we obtain for (76)

$$\begin{aligned} \Theta^Z_{0\mu\rho\sigma}(\mathbf{r},t) &= \partial_0\Theta^Y_{\mu\rho\sigma}(\mathbf{r},t) \qquad (81)\\ &\quad -\partial_k\left[\delta_{0\mu}\Theta^Y_{k\rho\sigma}(\mathbf{r},t) + \delta_{k\mu}\Theta^Y_{0\rho\sigma}(\mathbf{r},t)\right]\ . \end{aligned}$$

It is easy to show that (81) is equivalent to the equations

$$\Theta^Z_{\mu\nu\rho\sigma} = 2\partial_{[\mu}\Theta^Y_{\nu]\rho\sigma} \qquad (82)$$

and

$$\partial^\mu\Theta^Y_{\mu\rho\sigma}(\mathbf{r},t) = 0\ , \qquad (83)$$

where again the Weyl tensor condition (80) has to be taken into account. The first of these equations is equivalent to the linearized Ricci identities (107) for the Weyl tensor in anholonomic coordinates in a Riemann space V_4, provided Θ^Y is interpreted as the corresponding anholonomic affine connection.

Equation (83) may be interpreted as the linear part of a condition which expresses the redundancy of the C–/Γ–formulation of the Einstein theory [10].

Concerning the last term of (78) an explicit evaluation shows that all mass terms of (78) can be removed by the condition

$$ma_4 + g\left(b_2 + \frac{3}{2}c_4\right) = 0 \qquad (84)$$

provided again that Θ^Z has the symmetry properties (74) and (80) of a Weyl tensor in a V_4.

By means of this mass–0 condition we obtain for (78)

$$\partial_0\Theta^Z_{\mu\nu\rho\sigma}(\mathbf{r},t) = 2\partial_k\left[\delta_{0\mu}\Theta^Z_{k\nu\rho\sigma}(\mathbf{r},t) + \delta_{k\mu}\Theta^Z_{0\nu\rho\sigma}(\mathbf{r},t)\right] \qquad (85)$$

where the symmetries of Θ^Z have to be observed. By direct calculation it can be shown that (85) is equivalent to

$$\partial^\mu\Theta^Z_{\mu\nu\rho\sigma}(\mathbf{r},t) = 0 \qquad (86)$$

if the Weyl tensor symmetries (74) and (80) are satisfied. Equation (86) is just the linearized Bianchi identity for the curvature tensor (100), (108) (holonom or anholonom) in a Riemann space.

There is, however, still another symmetry for the determination of the 10 independent components of a Weyl tensor in a V_4, which reads

$$\varepsilon^{\mu\nu\rho\sigma}\Theta^Z_{\mu\nu\rho\sigma}(\mathbf{r},t) = 0\ . \qquad (87)$$

In contrast to the Weyl tensor condition (74) this condition holds only in a V_4 and is not induced by our calculations nor necessary for the results which we obtained above. However, it can be shown to be consistent with (86) and (80). We take this as a further hint that our formalism induces a generalized geometry (a Riemann–Cartan geometry) in a natural way.

Summarizing our results the effective equations for the bound state quantities Θ^Z and Θ^Y may be interpreted as linear Bianchi– and Ricci–identities for the anholonomic Weyl tensor and the anholonomic affine connection in a Riemann space V_4.

Comparing these results with the formulation of the Einstein gravitation theory in terms of the Weyl tensor and the affine connection, equations (109) and (110), we can claim to have derived the linearized Einstein theory in the vacuum, provided we postulate the equations (104) for the tetrads, which in principle may be resolved with respect to the tetrads and which in turn induce the metrical tensor by means of (103).

An analogous evaluation of the effective equations (75) and (77) for the quantities Θ^X and $\Theta^{\bar{Y}}$ yields equations which have a structure similar to (86) and (80). One is tempted to interpret these equations as equations for the Ricci–tensor or the metric tensor and the affine connection. In spite of these hints we do not succeed in giving those equations a consistent meaning in terms of gravitation quantities. As with Θ^Z and Θ^Y we have already obtained a set of quantities which suffices for the derivation of an effective gravitation theory, we may set $\Theta^X = \Theta^{\bar{Y}} = 0$ without contradictions.

As we already remarked, the approximations we used for the evaluation of the effective equations (75)–(78) affect only some constants in the resulting equations, which are collected in the mass condition (84). In particular the covariant form of the resulting equations is an exact result of Weak Mapping in the one–time formalism.

The mass condition (84) fixes the coupling constant g, one of the parameters of our subfermion theory; with (79) one obtains

$$g = -8m^2\pi^2\left(1 - \frac{2^7}{\pi 3 \cdot 5}\right) \approx -46.0m^2 \ . \tag{88}$$

The condition (84) was necessary for the interpretation of (78) as a linearized Bianchi identity. By an analogous condition resulting from the independent derivation of an effective $SU(2)$–dynamics for composite vector bosons in [23] and [32] the value $g = -10\pi^2 m^2$ was obtained. With regard to the rough approximations of our coordinate functions this seems to be a good agreement with our value of g. This (previous) result may be regarded as an additional hint for the success of our atomistic program.

6 Coupling of Gravitation to Fermions

According to the principle of equivalence field equations of matter in a gravitational field are obtained by the transition from Lorentz covariant equations to equations, which are covariant with respect to general coordinate transformations. The general covariance is achieved by minimal coupling of the matter fields to the affine connection of the Riemann–(Cartan–)space. In the case of spinorial matter the general covariant derivative is given by

$$D_\mu \Psi(x) = \partial_\mu \Psi(x) + \frac{i}{4}\Gamma_{\mu\nu\sigma}(x)\Sigma^{\nu\sigma}\Psi(x) \tag{89}$$

with the anholonomic affine connection $\Gamma_{\mu\nu\sigma}$ and the generators of the Lorentz group $\Sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]_-$ [2]. Thus the free dirac equation in a Riemann space V_4 (and in a Riemann Cartan space) reads [14], [5]

$$\left[i\gamma^\mu\partial_\mu - m - \frac{1}{4}\Gamma_{\mu\rho\sigma}(x)\gamma^\mu\Sigma^{\rho\sigma}\right]\Psi(x) = 0 \ . \tag{90}$$

For a confirmation of the interpretation of our bound state quantity Θ^Y as the anholonomic affine connection of a Riemann space V_4 we have to show that it couples to matter fields according to the principle of equivalence. As in the preceeding sections we derived an effective theory for a coupled graviton–fermion system, the evaluation of the linear part of our effective classical fermion equation (42) should yield equation (90).

We are only interested in the covariant derivative terms of (42). Thus we omit the residual fermion self-interaction and quantization parts and are left with

$$\begin{aligned} i\frac{\partial}{\partial t}\Theta^f_l(t) = {} & K_{I_1I_2}C^{I_2}_{1,l'}R^{1,l}_{I_1}\Theta^f_{l'}(t) \\ & +36W_{I_1I_2I_3I_4}C^{I_3I_4K_1K_2}_{4,r}C^{I_2}_{1,l'}R^{1,l}_{I_1K_1K_2}\Theta^f_{l'}(t)\Theta^b_r(t) \ . \end{aligned} \tag{91}$$

The evaluation of this equation is performed by inserting the kinetic operator (18), the vertex (19), the graviton expansion function (56), the dual of the fermion polarization cloud term (67), (65) and the fermion hard core state (66). We remember the abbreviations $I = (\alpha, \kappa, i, \mathbf{r})$, $l = (s, \rho, j, \mathbf{k})$ as well as the bosonic indices (72).

Multiplying (91) with the Dirac spinors $C^I_{1,l}$ from (66), summarizing over l and taking into account the completeness relations for elementary Dirac spinors we obtain equations for the fields $\Theta^f_I := C^I_{1,l}\Theta^f_l$:

$$\begin{aligned} i\frac{\partial}{\partial t}\Theta^f_{I_1} = {} & K_{I_1I_2}\Theta^f_{I_2} \\ & +36W_{I_2I_3I_4I_5}C^{I_4I_5K_1K_2}_{4,r}C^{I_1}_{1,l}R^{1,l}_{I_2K_1K_2}\Theta^f_{I_3}\Theta^b_r \ . \end{aligned} \tag{92}$$

For the calculations we have to take into account the antisymmetrization of the vertex and the expansion functions according to section 4. Due to the partial subsymmetries in effect we have to calculate $18 \cdot 3 \cdot 3 = 162$ terms for the interaction part. Analogously to the bosonic equation in section 5 the algebraic parts are exactly calculated. Due to the structure of the involved functions the auxiliary field indices, i.e. the subfermion regularization, are connected with the space coordinates. Regularization is again performed by a systematic expansion of the various terms with respect to the deviations of the auxiliary masses from a mean value m, by application of the regularization relations (9) and a subsequent approximate evaluation of the remaining integrals. In this manner we obtain for (92):

$$i\frac{\partial}{\partial t}\Theta^f_{\alpha\kappa i}(\mathbf{r},t) = \left[-i(\gamma^0\gamma^k)_{\alpha\beta}\partial_k(\mathbf{r}) + \gamma^0_{\alpha\beta}m_i\right]\Theta^f_{\beta\kappa i}(\mathbf{r},t) + \delta_{ii_1}\Big[d_1\eta^{\mu\nu}\Theta^X_{\mu\nu}(\mathbf{r},t)\gamma^0_{\alpha\beta}\sum_{i_2}\Theta^f_{\beta\kappa i_2}(\mathbf{r},t) + d_2\Theta^Y_{\mu\rho\sigma}(\mathbf{r},t)\left(\gamma^0\Sigma^{\rho\sigma}\gamma^\mu\right)_{\alpha\beta}\sum_{i_2}\Theta^f_{\beta\kappa i_2}(\mathbf{r},t) \quad + d_2\Theta^{\bar{Y}}_{\rho\sigma\mu}(\mathbf{r},t)\left(\gamma^0\Sigma^{\rho\sigma}\gamma^\mu\right)_{\alpha\beta}\sum_{i_2}\Theta^f_{\beta\kappa i_2}(\mathbf{r},t)\Big] \tag{93}$$

with some constants d_1, d_2.

We stress that the quantities Θ^Z, which we interpreted as Weyl tensor components in a Riemann space, do not couple to the fermion fields; this is an exact result of the involved algebra. According to section 5 we set $\Theta^X = \Theta^{\bar{Y}} = 0$ and obtain for the regularized spinor fields $\Theta^f_{\alpha\kappa}(\mathbf{r},t) = \sum_i \Theta^f_{\alpha\kappa i}(\mathbf{r},t)$:

$$i\frac{\partial}{\partial t}\Theta^f_{\alpha\kappa}(\mathbf{r},t) = \left[-i(\gamma^0\gamma^k)_{\alpha\beta}\partial_k(\mathbf{r}) + \gamma^0_{\alpha\beta}m\right]\Theta^f_{\beta\kappa}(\mathbf{r},t) + d_2\Theta^Y_{\mu\rho\sigma}(\mathbf{r},t)\left(\gamma^0\Sigma^{\rho\sigma}\gamma^\mu\right)_{\alpha\beta}\Theta^f_{\beta\kappa}(\mathbf{r},t)\ . \tag{94}$$

After transforming $d_2\Theta^Y \to \frac{1}{4}\Theta^Y$, which is consistent with the results from section 5, this equation is equivalent to the parity transformed equation (90) [34], [35]. *Thus the effective fermion–graviton equation (42) may be interpreted as the general covariant Dirac equation for elementary fermions in a Riemann space V_4.*

We remark that we did not use the restricted V_4–symmetries (57) for the derivation of this result. Thus it holds also for the extended Riemann–Cartan space.

7 Conclusions

We have derived effective equations for the dynamics of a coupled system of composite four–fermion bound states and elementary fermions from an underlying nonlinear subfermion equation. We showed that the linear and classical parts of the resulting boson equation can be interpreted as the linear Bianchi– and Ricci identities of a Riemann geometry and as well as the linearized (classical) Einstein gravitation theory in vacuum. This interpretation was confirmed by the derivation of the correct coupling of the gravitation quantities to elementary fermions.

We started with a Lorentz covariant subfermion field theory in Minkowski space, thus our effective graviton–fermion theory is also referred to a Minkowski space. In the framework of this effective field theory we derived effective gravitation quantities which may be interpreted as geometrical quantities: as the anholonomic Weyl tensor and the anholonomic affine connection of a Riemann space. In this sense we have derived an effective curved geometry as the framework for a gravitation theory, resulting from a field theory in flat Minkowski space.

In the course of our investigations the Weyl tensor condition (74) turned out to be necessary in order to give an appropriate interpretation to our effective graviton equations. Thus we obtained a formulation of a gravitation theory in terms of the Weyl tensor and the affine connection [10]. According to [20] the Weyl tensor supports a unitary representation of the Poincaré group for spin 2 and mass 0, where the linearized Bianchi identities are just the unitarity conditions. Thus although we started with massive generalized 'graviton' states (53) in the linear approximation we arrived at conventional massless gravitons with spin $s = 2$.

Our formalism of Weak Mapping yields a full nonlinear theory for gravitons together with their coupling to other composite particles, thus in principle we should be able to derive a full nonlinear gravitation theory. For a first investigation we took into account only the linear and the vacuum part of the effective graviton equations. There are, however, arguments that the full nonlinear Einstein theory can be induced already from the linearized Einstein vacuum equations by means of a consistent coupling to their sources [8], [9].

We restricted ourselves to the case of a Riemann space V_4 by a certain choice of the symmetry conditions of our effective gravitation quantities. This restriction turned out to be an artificial one; our formalism seemed to induce a Riemann–Cartan space as an effective geometry in a natural way. This is in accordance with the generalization of Einstein gravitation to microscopic domains according to Poincaré gauge theories [15], [16]. The consequences of our effective theory with respect to a Riemann–Cartan geometry as well as the coupling of the fermion 'matter' to the gravitation equations will have to be investigated in forthcoming papers.

A Anholonomic gravitation theory

We give a short introduction into the 'phenomenological' gravitation theory in anholonomic coordinates (see for instance [28]), thereby restricting ourselves to the case of the Einstein theory. This theory is for-

[5] In this section small latin indices $i, j \ldots = 0, 1, 2, 3$ denote holonomic quantities.

mulated in a Riemann space V_4 with metrical tensor[5] $g_{ij}(x)$ and with a metrical affine connection $\Gamma^k_{ij}(x)$, which fulfills the metricity condition

$$\overset{\Gamma}{\nabla}_k \, g_{ij}(x) = 0 \tag{95}$$

with the general covariant derivative $\overset{\Gamma}{\nabla}$. For a Riemann space one has

$$\Gamma^k_{ij} = \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} \tag{96}$$

with the Christoffel symbol of the metric tensor

$$\left\{ \begin{matrix} k \\ ij \end{matrix} \right\} := \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \tag{97}$$

and one has a symmetric connection $\Gamma^k_{[ij]} = 0$. The Riemann–Christoffel– or curvature tensor may be defined by the so–called Ricci identities

$$R_{ijk}{}^l(x) := 2\partial_{[i}\Gamma^l_{j]k}(x) + 2\Gamma^l_{[i|m|}(x)\Gamma^m_{j]k}(x) \ . \tag{98}$$

In a Riemann space V_4 one has

$$\begin{aligned} R_{(ij)kl} &= 0 \ , \\ R_{ij(kl)} &= 0 \ , \\ R_{[ijk]l} &= 0 \ , \\ R_{ijkl} &= R_{klij} \ . \end{aligned} \tag{99}$$

The integrability condtitions for the curvature tensor are the Bianchi identities

$$\overset{\Gamma}{\nabla}_{[i} \, R_{jk]l}{}^m = 0 \ . \tag{100}$$

The Einstein field equations are given by

$$R_{ij} - \frac{1}{2} g_{ij} R = \kappa T_{ij} \tag{101}$$

with the matter energy–momentum tensor T_{ij}, the Ricci tensor $R_{ij} := R_{ikj}{}^k$ and the curvature scalar $R := R_i{}^i$.

The Weyl tensor C_{ijkl} is given by the tracefree part of the curvature tensor:

$$C_{ijkl} := R_{ijkl} - g_{i[k}R_{l]j} - g_{j[l}R_{k]i} + \frac{1}{3} R \, g_{i[k}g_{l]j} \tag{102}$$

with $C_{ij}{}^i{}_l = 0$.

We introduce an anholonomic coordinate system on the manifold by means of the tetrads $e_\mu{}^i$ with the holonomic indices $\mu = 0,1,2,3$. Together with their duals $e^\mu{}_i$ they fulfill the following set of equations:

$$\begin{aligned} e_\mu{}^i(x) e^\mu{}_j(x) &= \delta^i_j \ , \\ e_\mu{}^i(x) e^\nu{}_i(x) &= \delta^\mu_\nu \ , \\ e_\mu{}^i(x) e_\nu{}^j(x) g_{ij}(x) &= \eta_{\mu\nu} \ , \\ g_{ij}(x) &= e^\mu{}_i(x) e^\nu{}_j(x) \eta_{\mu\nu} \end{aligned} \tag{103}$$

with $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$.

With respect to their anholonomic indices tetrads transform as vectors under Lorentz transformations and thus allow the representation of spinors.

The anholonomic affine connection is given by resolving the anholonomic metricity condition as

$$\Gamma_{\mu\nu\rho} = -\Omega_{\mu\nu\rho} + \Omega_{\nu\rho\mu} - \Omega_{\rho\mu\nu} \tag{104}$$

with the object of anholonomity

$$\Omega^\rho_{\mu\nu} := e_\mu{}^i e_\nu{}^j \partial_{[i} e^\rho{}_{j]} \tag{105}$$

and $\Omega_{\mu\nu\rho} := g_{\rho\sigma}\Omega^\sigma_{\mu\nu}$. Some important properties of the anholonomic connection are

$$\begin{aligned} \Gamma_{\mu(\nu\rho)} &= 0 \ , \\ \Gamma_{[\mu\nu]\rho} &= -\Omega_{\mu\nu\rho} \ , \\ \Gamma_{[\mu\nu\rho]} &= -\Omega_{[\mu\nu\rho]} \ , \\ \eta^{\mu\nu}\Gamma_{\mu\nu\rho} &= -2\Omega^\mu_{\rho\mu} \ . \end{aligned} \tag{106}$$

The curvature tensor $R_{\mu\nu\rho\sigma}$ in anholonomic coordinates is given by the anholonomic Ricci identities

$$R_{\mu\nu\rho\sigma} = 2\partial_{[\mu}\Gamma_{\nu]\rho\sigma} + 2\Gamma_{[\mu|\lambda\rho|}\Gamma^\lambda_{\nu]\sigma} - 2\Gamma^\lambda_{[\mu\nu]}\Gamma_{\lambda\rho\sigma} \tag{107}$$

and the anholonomic Bianchi identities read

$$\begin{aligned} \overset{\Gamma}{\nabla}_{[\alpha} \, R_{\mu\nu]\rho\sigma} &\equiv \partial_{[\alpha}R_{\mu\nu]\rho\sigma} + \Gamma_{[\alpha|\lambda\rho}R^\lambda{}_{\sigma|\mu\nu]} - \Gamma_{[\alpha|\lambda\sigma}R^\lambda{}_{\rho|\mu\nu]} + 2R_{\rho\sigma}{}^\lambda{}_{[\nu}\Gamma_{\alpha|\lambda|\mu]} \\ &= 0 \ . \end{aligned} \tag{108}$$

With respect to our Weak Mapping procedure we prefer a formulation of Einstein's gravitation theory in terms of the Weyl tensor and the affine connection in anholonomic coordinates. This can be achieved by inserting the anholonomic versions of (102) and of the field equations (101) into (107) and (108); one obtains the equations

$$C_{\mu\nu\rho\sigma} - 2\partial_{[\mu}\Gamma_{\nu]\rho\sigma} - 2\Gamma_{[\mu|\lambda\rho}\Gamma^\lambda_{\nu]\sigma} + 2\Gamma^\lambda_{[\mu\nu]}\Gamma_{\lambda\rho\sigma} = \kappa\big(- \eta_{\mu[\rho}T_{\sigma]\nu} - \eta_{\nu[\sigma}T_{\rho]\mu} + \frac{2}{3} T \eta_{\mu[\rho}\eta_{\sigma]\nu} \big) \tag{109}$$

and

$$\partial_\sigma C_{\mu\nu\rho}{}^\sigma + 2\Gamma_{\sigma\lambda[\mu}C^\lambda{}_{\nu]\rho}{}^\sigma + \Gamma_{\sigma\lambda\rho}C_{\mu\nu}{}^{\lambda\sigma} + \Gamma^\sigma_{\lambda\sigma}C_{\mu\nu\rho}{}^\lambda = \kappa\big(2\partial_{[\mu}T_{\nu]\rho} - 2\Gamma^\lambda_{[\mu|\rho}T_{\nu]\lambda} - 2\Gamma^\lambda_{[\mu\nu]}T_{\lambda\rho} + \frac{1}{3}\partial_{[\mu}T\eta_{\nu]\rho}\big) \tag{110}$$

According to [10] together with (104) these equations are a redundant but complete set of equations for the formulation of the Einstein theory of gravitation. It is this formulation which is suitable for the comparison with our effective gravitation equations.

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