Partial decoupling of electrodynamics

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ABSTRACT. We discuss a reformulation of the electrodynamics of a Dirac spinor field, ψ, coupled to the electromagnetic field, A_μ, in a formalism in which covariant Green functions are used to express A_μ in terms of ψ in the Lorentz gauge. This leads to a non-local fermion self-coupling interaction. The quantized version of this reduced theory is discussed.

RÉSUMÉ. On étudie une reformulation de l'électrodynamique d'un champ spinoriel de Dirac, ψ, couplé à un champ électromagnétique, A_μ, dans un formalisme dans lequel les fonctions de Green covariantes sont utilisées pour exprimer A_μ en fonction de ψ dans la jauge de Lorentz. Cela conduit à une interaction d'auto-couplage d'un fermion non local. On discute la version quantifiée de cette théorie réduite.

1. Introduction.

It is now almost a cliché to say that quantum electrodynamics (QED) is the most accurate theory known, since it predicts such things as the magnetic moment of the electron to an accuracy of better than one part in a million. QED is the quantized (or 'second quantized') theory of the Dirac spinor field, representing spin one-half fermions, interacting with a massless real vector field, the electromagnetic field (EM). It is based on the Lagrangian of classical electrodynamics (CED), namely

$$\mathcal{L}_{ED} = \overline{\psi}(x)\,(i\,\gamma^\mu\, D_\mu - m)\,\psi(x) - \frac{1}{4}F^{\alpha\beta}(x)F_{\alpha\beta}(x), \tag{1}$$

where, as usual, ψ is a Dirac spinor and $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$, with A^α being the EM field. We use the standard notational conventions in eq. (1), with $D_\mu = \partial_\mu + iqA_\mu$, and $\hbar = c = 1$. CED and so QED are among the simplest examples of gauge theories that can be written down.

The classical, or Euler-Lagrange, equations of motion for this theory, obtained from the stationary action principle

$$\delta\, S(A^\mu, \psi) = \delta \int \mathcal{L}_{ED}\; d^4x = 0, \tag{2}$$

are the well-known coupled Dirac-Maxwell equations

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = q\;\gamma^\mu A_\mu(x)\psi(x), \tag{3}$$

and

$$\partial_\mu F^{\mu\,\nu}(x) = q\;\overline{\psi}(x)\gamma^\nu\psi(x). \tag{4}$$

These "classical" equations of motion for ED are frequently written down (e.g. ref. [1]) but their solution is not often discussed, since most works go on to discuss the quantized version of this theory. Yet, as Jackiw has pointed out [2], classical field theories are both relevant to quantum field theory and interesting in themselves. In this paper we discuss a partially decoupled formulation of this theory in the Lorentz gauge, and consider the consequences of this decoupling for the (second) quantized version.

2. Partial reduction of electrodynamics.

We begin by recalling some well-known, but pertinent for our discussion, characteristics of ED. The gauge invariance of ED allows one to choose a gauge in which the equations of motion can be written in a more convenient form. Thus, in the Lorentz gauge, $\partial_\mu A^\mu = 0$, equation (4) is modified to the form

$$\partial^\mu\partial_\mu A^\nu = q\;\overline{\psi}(x)\gamma^\nu\psi(x). \tag{5}$$

These Lorentz gauge equations (3) and (5) for ED are derivable from the action principle (2) but with a modified Lagrangian density, namely

$$\mathcal{L}^{LG}_{ED} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu. \tag{6}$$

Henceforth we shall employ the Lorentz gauge only and shall, therefore, suppress the superscript LG of eq. (6) in all that follows.

As is well known from electromagnetic theory (e.g. ref. [3]) the Maxwell equation (5) can be solved formally to yield the result

$$A^\nu(x) = A_0^\nu(x) + q \int d^4x' D(x-x')\, \overline{\psi}(x')\gamma^\nu\psi(x'), \tag{7}$$

where $A_0^\nu(x)$ is a solution of the homogeneous (or "free field") equation (5) with $q = 0$, i.e. such that $\partial_\mu\partial^\mu A_0^\nu(x) = 0$, while $D(x-x')$ is an appropriate Green function, defined as a solution of

$$\partial_\mu\partial^\mu D(x-x') = \delta^4(x-x'). \tag{8}$$

Note that this formal solution holds for classical fields, and also for quantized fields, where the field amplitudes are operators. We recall that equation (8) does not define the Green function uniquely, since any solution of the homogeneous equation can be added to it without invalidating eq. (8). Boundary conditions and physical considerations are used to pin down $D(x-x')$. For example, in conventional QFT the Feynman prescription is used (e.g. refs. [1] and [3]), and $D(x-x')$ is then called the Feynman propagator. Recall that Green functions corresponding to advanced and retarded fields are

$$\begin{aligned} D_{ret}(x-x') &= \frac{1}{2\pi}\theta(t-t')\delta((x-x')^2) \\ &= \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta(t-t'-|\mathbf{x}-\mathbf{x}'|), \end{aligned} \tag{9}$$

and

$$\begin{aligned} D_{adv}(x-x') &= \frac{1}{2\pi}\theta(t'-t)\delta((x-x')^2) \\ &= \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta(t-t'+|\mathbf{x}-\mathbf{x}'|), \end{aligned} \tag{10}$$

where $\theta(u) = 1$ if $u > 0$ and $\theta(u) = 0$ if $u < 0$, as usual. A symmetric Green function, usually called the principal-value Green function, is

$$\begin{aligned} D(x-x') = D(x'-x) &= \frac{1}{2}(D_{ret} + D_{adv}) \\ &= \frac{1}{4\pi}\delta((x-x')^2). \end{aligned} \tag{11}$$

The Feynman propagator is also symmetric. Its explicit representation in coordinate space is

$$D_F(x-x') = \frac{1}{4\pi}\delta((x-x')^2) - \frac{i}{4\pi^2(x-x')^2}, \tag{12}$$

which is just the principal value form with an extra term that is a solution of the homogeneous equation ((5) with $q = 0$). (Differences among the Green functions are solutions of the homogeneous equation.)

Substitution of the formal solution (7) into (3) yields the equation

$$\begin{aligned} (i\gamma^\mu\partial_\mu - m)\psi(x) &= q\gamma_\mu A_0^\mu(x)\psi(x) \\ &+ q^2 \int d^4x' D(x-x')\overline{\psi}(x')\gamma^\mu\psi(x')\gamma_\mu\psi(x), \end{aligned} \tag{13}$$

where $A_0^\mu(x)$ is a (known) solution of the free ($q = 0$) equation (5). Equation (13) is a covariant Dirac equation with a nonlinear and nonlocal 'self-coupling' interaction. As such, it is not easy to solve and, to our knowledge, no exact (analytic or numeric) solutions of equation (13) have been reported in the literature, though approximate, particularly iterative, solutions have been discussed by various authors, particularly Barut and co-workers (see ref. [4] and citations therein).

It is straightforward to write down a Lagrangian for the decoupled equation (13), such that this equation is obtained from the action principle (2). Thus, for symmetric Green functions, $G(x-x') = G(x'-x)$, the Lagrangian corresponding to eq. (13) is

$$\begin{aligned} \mathcal{L} &= \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m - q\gamma_\mu A_0^\mu(x))\psi(x) \\ &- \frac{1}{2}q^2 \int d^4x' D(x-x')\overline{\psi}(x')\gamma^\mu\psi(x')\overline{\psi}(x)\gamma_\mu\psi(x). \end{aligned} \tag{14}$$

This has the structure of a Lagrangian density for a Dirac field in the presence of a given free EM field $A_0^\mu(x)$, and with an interaction term that is reminiscent of the static Coulomb interaction in the Coulomb gauge (e.g. ref. [1], p. 88), except that it is not static. Indeed, it is covariant and includes relativistic ("retardation") effects.

Stationary solutions of equation (13), with

$$\psi(x) = \phi(\mathbf{r})e^{-iEt},$$

require that $\phi(\mathbf{r})$ satisfy the time-independent equation

$$\begin{aligned} &\Bigg(-i\vec{\alpha}\cdot\nabla + m\gamma^0 + q\gamma^0\gamma_\mu A_0^\mu(\mathbf{r}) \\ &\qquad + \frac{q^2}{4\pi}\int d^3r' \frac{\overline{\phi}(\mathbf{r}')\gamma_\mu\phi(\mathbf{r}')\gamma^0\gamma^\mu}{|\mathbf{r}'-\mathbf{r}|}\Bigg)\phi(\mathbf{r}) \\ &\qquad = E\phi(\mathbf{r}), \end{aligned} \tag{15}$$

which exhibits a nonlocal static self-coupling term. Note that this static self-coupling interaction is not purely Coulombic, since

$$\frac{\overline{\phi}(\mathbf{r}')\gamma_\mu\phi(\mathbf{r}')\gamma^0\gamma^\mu}{|\mathbf{r}'-\mathbf{r}|} = \frac{\phi^\dagger(\mathbf{r}')\phi(\mathbf{r}') - \phi^\dagger(\mathbf{r}')\,\vec{\alpha}\,\phi(\mathbf{r}')\cdot\vec{\alpha}}{|\mathbf{r}'-\mathbf{r}|}. \tag{16}$$

The second term on the right of eq. (16) is a relativistic "correction" to the Coulomb term.

3. Quantization.

The Lagrangian of eq. (14) can serve as the basis for a quantized field theory, using any formulation (canonical, light-front, path integral, etc.). Such (second) quantization is needed for many-body generalizations and for the incorporation of particles-antiparticle phenomena. We note that the reformulated QED based on the Lagrangian (14) (with its nonlocal self-coupling) is renormalizable, which is not generally the case with locally self-coupled fermion fields in $3+1$ dimensions, such as that considered by Heisenberg [5].

The Hamiltonian density corresponding to the Lagrangian density (14) takes the form

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_\psi^0 + \mathcal{H}_I^{(1)} + \mathcal{H}_I^{(2)} \\ &= \psi^\dagger(x)(-i\,\vec{\alpha}\cdot\nabla + m\beta)\psi(x) + q\overline{\psi}(x)\gamma_\mu A_0^\mu(x)\psi(x) \\ &+ \frac{1}{2}q^2\int d^4x' D(x-x')\overline{\psi}(x')\gamma^\mu\psi(x')\overline{\psi}(x)\gamma_\mu\psi(x),\end{aligned} \tag{17}$$

where $D(x-x') = D(x'-x)$. (We suppress the Hamiltonian of the free EM field.)

In conventional canonical S-matrix perturbation theory, the time-evolution operator,

$$U = \sum_n \frac{(-1)^n}{n!}\int d^4x_1...d^4x_n\, T(\mathcal{H}_I(x_1)...\mathcal{H}_I(x_n)), \tag{18}$$

involves two interaction terms,

$$\mathcal{H}_I^{(1)} = q\overline{\psi}(x)\gamma_\mu A_0^\mu(x)\psi(x), \tag{19}$$

and

$$\mathcal{H}_I^{(2)} = \frac{1}{2}q^2\int d^4x' D(x-x')\overline{\psi}(x')\gamma^\mu\psi(x')\overline{\psi}(x)\gamma_\mu\psi(x). \tag{20}$$

The second term contains the covariant photon propagator and contributes to processes represented by Feynman diagrams without any external photon lines. The first term (eq. (19)), which is linear in the photon field $A_0^\mu(x)$, contributes to processes not represented by (20), specifically diagrams with external photon lines.

Since the Lorentz gauge is assumed in the present formulation, care must be taken to ensure that the gauge condition, $\partial_\mu A^\mu = 0$, is accounted for. This can be done through the Gupta-Bleuler prescription in which the condition $\langle\chi|\partial_\mu A^\mu|\chi\rangle = 0$ is imposed on the states $|\chi\rangle$ under consideration (see, for example, ref. [6], sect. 3.2).

As is well known, QED has infrared divergences which appear in intermediate steps of calculations of observables, due to the fact that the photons are massless. One standard way to regulate such divergences is to add to the Lagrangian (1) or (6) a mass term of the form

$$\frac{1}{2}\mu^2 A^\nu(x)A_\nu(x), \tag{21}$$

where the mass parameter μ is ultimately set to zero. The inclusion of (21) modifies equation (5) and (8) in that the operator $\partial^\nu\partial_\nu$ is replaced by $\partial^\nu\partial_\nu + \mu^2$. This leads to a corresponding modification of the Green functions from "massless" to "massive" form. Explicitly, in the integral representation, this is

$$D^{(\mu)}(x-x') = \int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-x')}\frac{1}{\mu^2 - k\cdot k}, \tag{22}$$

where the "massless" case corresponds to $\mu = 0$. Evidently, this regularization scheme, if used, leads to no difficulties in the reduced theory, since one needs only to replace D by $D^{(\mu)}$ in expressions like (14) and (17).

Apart from perturbation theory, the Hamiltonian (17) is also amenable to variational solution for two-body (or more) bound states, as has been demonstrated on the case of the static Coulomb interaction [7-11]. However, the reduction of the eigenvalue equations in this case leads to modified potentials (kernels, in momentum space), because of the relativistic nature of the interaction.

4. Concluding remarks.

We have examined partially reduced electrodynamic field theory in the Lorentz gauge, in which the electromagnetic field amplitudes, A^μ, are expressed in terms of the fermion amplitudes, ψ, using covariant Green functions. The resulting equation for the fermion field, ψ, is a Dirac equation with nonlinear and nonlocal self-coupling. The formalism is covariant and the interaction includes relativistic ("retardation") effects. The Lagrangian corresponding to this nonlinear fermion equation leads to a reformulated QED with interactions that are analogous in structure to conventional QED in the Coulomb gauge, except that in this case they involve manifestly covariant expressions.

Acknowledgement.

The financial support of the Natural Sciences and Engineering Research Council of Canada for this work is gratefully acknowledged.

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(Manuscrit reçu le 2 janvier 1997, révisé le 7 juillet 1997)