Extended Electrodynamics:
II. Properties and invariant characteristics of the non-linear vacuum solutions

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This paper is the second one of a series of three papers. It considers the general properties of the non-linear solutions to the vacuum equations of *Extended Electrodynamics* [1]. The $*$-invariance and the conformal invariance of the equations are identified. It is also proved that all non-linear solutions have zero invariants: $\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(*F)_{\mu\nu}F^{\mu\nu} = 0$. The three invariant characteristics of the non-linear solutions: *amplitude, phase* and *scale factor* are introduced and discussed.

1 General properties of the equations

Recall from [1] the basic relation of *Extended Electrodynamics* (EED)

$$\vee(\delta\Omega, *\Omega) = \vee(\Phi, *\pi_1\Omega) + \vee(\Psi, *\pi_2\Omega). \tag{1}$$

An external field is called vacuum with respect to the EM-field Ω if the right hand side of (1) is equal to zero. Then the left hand side of (1) will also be equal to zero, so we get the equations

$$\vee(\delta\Omega, *\Omega) = 0. \tag{2}$$

From this coordinate free compactly written expression we obtain the following equations for the components of Ω in the basis (e_1, e_2):

$$\begin{aligned}&(\delta F\wedge *F)\otimes e_1\vee e_1 + (\delta * F)\wedge * * F)\otimes e_2\vee e_2 + \\ &+(\delta F\wedge * * F + \delta * F\wedge *F)\otimes e_1\vee e_2 = 0.\end{aligned} \tag{3}$$

Hence, the field equations, expressed through the operator δ are as follows

$$\delta F\wedge *F = 0,\ \delta * F\wedge * * F = 0,\ \delta * F\wedge *F - \delta F\wedge F = 0. \tag{4}$$

These equations, expressed through the operator $\mathbf{d}$ have the form

$$*F\wedge *\mathbf{d}*F = 0,\ F\wedge *\mathbf{d}F = 0,\ F\wedge *\mathbf{d}*F + *F\wedge *\mathbf{d}F = 0. \tag{5}$$

Using the components $F_{\mu\nu}$, we obtain from equations (4)

$$\begin{aligned}&F_{\mu\nu}(\delta F)^\nu = 0,\quad (*F)_{\mu\nu}(\delta * F)^\nu = 0,\\ &F_{\mu\nu}(\delta * F)^\nu + (*F)_{\mu\nu}(\delta F)^\nu = 0.\end{aligned} \tag{6}$$

In the same way, from equations (5) we get

$$\begin{aligned}&(*F)^{\mu\nu}(\mathbf{d} * F)_{\mu\nu\sigma} = 0,\quad F^{\mu\nu}(\mathbf{d}F)_{\mu\nu\sigma} = 0,\\ &(*F)^{\mu\nu}(\mathbf{d}F)_{\mu\nu\sigma} + F^{\mu\nu}(\mathbf{d} * F)_{\mu\nu\sigma} = 0.\end{aligned} \tag{7}$$

Now we give the 3-dimensional form of the equations in the same order:

$$\begin{aligned}&B\times\left(\mathrm{rot}\,B - \tfrac{\partial E}{\partial\xi}\right) - E\,\mathrm{div}\,E = 0,\\ &E.\left(\mathrm{rot}\,B - \tfrac{\partial E}{\partial\xi}\right) = 0,\end{aligned} \tag{8}$$

$$\begin{aligned}&E\times\left(\mathrm{rot}\,E + \tfrac{\partial B}{\partial\xi}\right) - B\,\mathrm{div}\,B = 0,\\ &B.\left(\mathrm{rot}\,E + \tfrac{\partial B}{\partial\xi}\right) = 0,\end{aligned} \tag{9}$$

$$\begin{aligned}&\left(\mathrm{rot}\,E + \tfrac{\partial B}{\partial\xi}\right)\times B + \left(\mathrm{rot}\,B - \tfrac{\partial E}{\partial\xi}\right)\times E\\ &\qquad + B\,\mathrm{div}\,E + E\,\mathrm{div}\,B = 0,\\ &B.\left(\mathrm{rot}\,B - \frac{\partial E}{\partial\xi}\right) - E.\left(\mathrm{rot}\,E + \frac{\partial B}{\partial\xi}\right) = 0.\end{aligned} \tag{10}$$

From the second equations of (8) and (9) the well known Poynting relation follows

$$\mathrm{div}\,(E\times B) + \frac{\partial}{\partial\xi}\frac{E^2 + B^2}{2} = 0,$$

and from the second equation of (10), if $E.B = g(x, y, z)$, we obtain the Maxwell theory relation

$$B.\,\mathrm{rot}\,B = E.\,\mathrm{rot}\,E.$$

The explicit form of equations (8) end (9) should not make us conclude, that the second (scalar) equations follow from the first (vector) equations. Here is an example: let $\mathrm{div}\,E = 0$, $\mathrm{div}\,B = 0$ and the time derivatives of E and B are zero. Then the system of equations reduces to

$$E \times \operatorname{rot} E = 0 \ , \quad B \times \operatorname{rot} B = 0,$$
$$B . \operatorname{rot} E = 0 \ , \quad E . \operatorname{rot} B = 0.$$

As it is seen, the vector equations do not require any connection between E and B in this case, therefore, the scalar equations, which require such connection, do not follow from the vector ones. The third equations of (5) and (6) determine (in equivalent way) the energy-momentum quantities, transferred from F to $*F$, and reversely, in a unit 4-volume, with the expressions, respectively

$$\begin{aligned} F_{\mu\nu}(\delta * F)^\nu &= -(*F)_{\mu\nu}(\delta F)^\nu, \\ F^{\mu\nu}(\mathbf{d} * F)_{\mu\nu\sigma} &= -(*F)^{\mu\nu}(\mathbf{d}F)_{\mu\nu\sigma}. \end{aligned}$$

From these relations it is seen, that the 1-forms δF and $\delta * F$ *play the role of external currents* respectively for $*F$ and F. In the same spirit we could say, that the energy-momentum quantities $F_{\mu\nu}(\delta F)^\nu$ and $(*F)_{\mu\nu}(\delta * F)^\nu$, which F and $*F$ exchange with themselves, are equal to zero. And this corresponds fully to our former statements, concerning the physical sense of the equations for the components of Ω.

From the first two equations of (6) and from the expression for the divergence $\nabla_\nu Q_\mu^\nu$ of Maxwell's energy-momentum tensor Q_μ^ν in CED

$$\nabla_\nu Q_\mu^\nu = \frac{1}{4\pi}\left[F_{\mu\nu}(\delta F)^\nu + (*F)_{\mu\nu}(\delta * F)^\nu\right] \qquad (11)$$

it is immediately seen from equations (6) that this divergence is also zero. In view of this *we assume the tensor* Q_μ^ν, *defined by*

$$\begin{aligned} Q_\mu^\nu &= \frac{1}{4\pi}\left[\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta_\mu^\nu - F_{\mu\sigma}F^{\nu\sigma}\right] \\ &= \frac{1}{8\pi}\left[-F_{\mu\sigma}F^{\nu\sigma} - (*F)_{\mu\sigma}(*F)^{\nu\sigma}\right], \end{aligned} \qquad (12)$$

to be the energy-momentum tensor in EED. We are interested in finding explicit time-stable solutions of finite type, i.e. $F_{\mu\nu}$ to be finite functions of the three spatial coordinates, therefore, if it turns out that such solutions really exist, then integral conserved quantities can be easily constructed and computed, making use of the 10 Killing vectors on the Minkowski space-time. We recall that in CED such finite and time-stable solutions in the whole space are not allowed by the Maxwell's equations.

We first note, that in correspondence with the requirement for *general covariance*, equations (2), given above and presented in different but equivalent forms, are written down in coordinate free manner. This requirement is universal, i.e. it concerns all basic equations of a theory and means simply, that the existence of real objects and the occurrence of real processes *can not* depend on the local coordinates used in the theory., i.e. on the way used to describe the local character of the evolution and structure of the natural objects and processes. Of course, in the various coordinate systems the equations and their solutions will appear quite different. Namely the covariant character of the equations allows us to choose the most appropriate coordinates, i.e those that reflect most fully the features of each particular case. A typical example for this is the usage of spherical coordinates in describing spherically symmetric fields. Let's not forget also, that the coordinate-free form of the equations permits an easy transfer of the same physical situation onto manifolds with more complicated structure and nonconstant metric tensor.

Since the left hand sides of the equations are linear combinations of the first derivatives of the unknown functions with coefficients, depending linearly on these unknown functions, (6) presents a system of special type *quasilinear first order partial differential equations.* The number of the unknown functions $F_{\mu\nu} = -F_{\nu\mu}$ is 6, and in general, the number of the equations is $3.4 = 12$, but the number of the independent equations depends strongly on whether the two invariants $I_1 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ and $I_2 = \frac{1}{2}(*F)_{\mu\nu}F^{\mu\nu}$ are equal to zero or are not equal to zero. If $I_2 \neq 0$ then $det(F_{\mu\nu}) \neq 0$ and the first two equations of (6) are equivalent to $\delta F = 0$ and $\delta * F = 0$, which automatically eliminates the third equation of (6), i.e. in this case our equations reduce to Maxwell's equations.

It is clearly seen from the form of equations (7), that the metric tensor essentially participates (through the $*$-operator applied to 2-forms only) in the equations. If we use the δ-operator, then the metric participates also through the $*$-operator, applied to 3-forms, but this does not lead to a more complicated coordinate form of the equations. We note also that in nonlinear coordinates the metric tensor and its derivatives will participate, therefore, the solutions will depend strongly on the metric tensor chosen. This may cause existence or non-existence of solutions of a given class, e.g. soliton-like ones. In our framework such additional complications do not appear because of the opportunity to work in global coordinates with constant metric tensor.

We note two important invariance (symmetry) properties of our equations.

Property 1. *The transformation $F \to *F$ does not change the system.*

The proof is obvious, in fact, the first two equations interchange, and the third one is kept the same.

In terms of Ω this means that if Ω is a solution, then $*\Omega$ is also a solution, which means, in turn, that equations (2) are equivalent to the equations

$$\vee(\Omega, *\mathbf{d}\Omega) = 0. \tag{13}$$

Property 2. *Under conformal change of the metric the equations do not change.*

The proof of this property is also obvious and is reduced to the observation, that as can be seen from (7), the $*$-operator participates only through its reduction to 2-forms, and as is well known, $*_2$ is conformally invariant.

Summing up the first two equations of (8) and (9) we obtain the time derivative of the classical Poynting vector in our more general approach:

$$\frac{\partial}{\partial\xi}(E\times B) = E \operatorname{div} E + B \operatorname{div} B - E\times \operatorname{rot} E - B\times \operatorname{rot} B.$$

In CED the first and the second terms on the right are missing.

Here is an example of static solutions of (2), which are not solutions to Maxwell's equations.

$$E = (\arcsin\alpha z, \arccos\alpha z,\ 0),$$
$$B = (b\cos\alpha z,\ -b\sin\alpha z,\ 0),$$

where a, b and α are constants. We obtain

$$\operatorname{rot} E = (a\alpha\sin\alpha z,\ a\alpha\cos\alpha z,\ 0),$$
$$\operatorname{rot} B = (b\alpha cos\alpha z,\ -b\alpha\sin\alpha z,\ 0)$$

Obviously,

$$E\times\operatorname{rot} E = 0,\ B\times\operatorname{rot} B = 0,\ E.\operatorname{rot} B = 0,\ B.\operatorname{rot} E = 0.$$

For the Poynting vector we get $E \times B = (0, 0, -ab)$, and for the energy density $w = \frac{1}{2}(a^2 + b^2)$. Considered in a finite volume, these solutions could model some standing waves.

2 General properties of the solutions

It is quite clear that the solutions of our equations are naturally divided into two classes: *linear* and *nonlinear*. The first class consists of all solutions to Maxwell's vacuum equations, where the name *linear* comes from. These solutions are well known and won't be discussed here. The second class, called *nonlinear*, includes all other solutions. This second class is naturally divided into two subclasses. The first subclass consists of all nonlinear solutions, satisfying the conditions

$$\delta F \neq 0,\ \delta * F \neq 0, \tag{14}$$

and the second subclass consists of those nonlinear solutions, satisfying one of the two pairs of conditions:

$$\delta F = 0,\ \delta * F \neq 0;\ \delta F \neq 0,\ \delta * F = 0.$$

Further we assume that conditions (14) are fulfilled.

Our main purpose is to show explicitly, that among the nonlinear solutions there are soliton-like ones, i.e. the components $F_{\mu\nu}$ of which at any moment of time are *finite* functions of the three spatial variables with *connected compact* 3-dimensional support. We are going to study their properties and to introduce corresponding characteristics. First we shall establish some of their basic features, proving three propositions.

Proposition 1. *All nonlinear solutions have zero invariants*:

$$I_1 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \pm\sqrt{det[(F \pm *F)\mu\nu]} = 0,$$
$$I_2 = \frac{1}{2}(*F)_{\mu\nu}F^{\mu\nu} = \pm 2\sqrt{det(F_{\mu\nu})} = 0.$$

Proof. Recall the field equations in the form (2.8):

$$F_{\mu\nu}(\delta F)^\nu = 0,\quad (*F)_{\mu\nu}(\delta * F)^\nu = 0,$$
$$F_{\mu\nu}(\delta * F)^\nu + (*F)_{\mu\nu}(\delta F)^\nu = 0.$$

It is clearly seen that the first two groups of these equations may be considered as two linear homogeneous systems with respect to δF^μ and $\delta * F^\mu$ respectively. In view of the inequalities (14) these homogeneous systems have non-zero solutions, which is possible only if $det(F_{\mu\nu}) = det((*F)_{\mu\nu}) = 0$, i.e. if $I_2 = 2E.B = 0$. Further, summing up these three systems of equations, we obtain

$$(F + *F)_{\mu\nu}(\delta F + \delta * F)^\nu = 0.$$

If now $(\delta F + \delta * F)^\nu \neq 0$, then

$$0 = det(F + *F)_{\mu\nu} = \left[\tfrac{1}{4}(F + *F)_{\mu\nu}(*F - F)^{\mu\nu}\right]^2$$
$$= \left[-\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu}\right]^2 = (I_1)^2.$$

If $\delta F^\nu = -(\delta * F)^\nu \neq 0$, we sum up the first two systems and obtain $(*F - F)_{\mu\nu}(\delta * F)^\nu = 0$. Consequently,

$$0 = det(*F - F)_{\mu\nu} = \left[\tfrac{1}{4}(*F - F)_{\mu\nu}(-F - *F)^{\mu\nu}\right]^2$$
$$= \left[\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu}\right]^2 = (I_1)^2.$$

This completes the proof.

Recall [2] that in this case the energy-momentum tensor $Q_{\mu\nu}$ has just one isotropic eigen direction and all other eigen directions are space-like. Since all eigen directions of $F_{\mu\nu}$ and $*F_{\mu\nu}$ are eigen directions of $Q_{\mu\nu}$ too, it is clear that $F_{\mu\nu}$ and $(*F)_{\mu\nu}$ can not have time-like eigen directions. But the first two systems of (6) require δF and $\delta * F$ to be eigen vectors of F and $*F$ respectively, so we obtain

$$(\delta F).(\delta F) \leq 0,\ (\delta * F).(\delta * F) \leq 0. \qquad (15)$$

Proposition 2. *All nonlinear solutions satisfy the conditions*

$$(\delta F)_\mu (\delta * F)^\mu = 0,\ |\delta F| = |\delta * F| \qquad (16)$$

Proof. We form the inner product $i(\delta * F)(\delta F \wedge *F) = 0$ and get

$$(\delta * F)^\mu (\delta F)_\mu (*F) - \delta F \wedge (\delta * F)^\mu (*F)_{\mu\nu} dx^\nu = 0.$$

Because of the obvious nulification of the second term the first term will be equal to zero (at non-zero $*F$) only if $(\delta F)_\mu (\delta * F)^\mu = 0$.

Further we form the inner product $i(\delta * F)(\delta F \wedge F - \delta * F \wedge *F) = 0$ and obtain

$$(\delta * F)^\mu (\delta F)_\mu F - \delta F \wedge (\delta * F)^\mu F_{\mu\nu} dx^\nu -$$

$$-(\delta * F)^2 (*F) + \delta * F \wedge (\delta * F)^\mu (*F)_{\mu\nu} dx^\nu = 0.$$

Clearly, the first and the last terms are equal to zero. So, the inner product by δF gives

$$(\delta F)^2\ (\delta * F)^\mu F_{\mu\nu} dx^\nu - [(\delta F)^\mu (\delta * F)^\nu F_{\mu\nu}]\, \delta F$$

$$+(\delta * F)^2 (\delta F)^\mu (*F)_{\mu\nu} dx^\nu = 0.$$

The second term of this equality is zero. Besides, $(\delta * F)^\mu F_{\mu\nu} dx^\nu = -(\delta F)^\mu (*F)_{\mu\nu} dx^\nu$. So,

$$\left[(\delta F)^2 - (\delta * F)^2\right] (\delta F)^\mu (*F)_{\mu\nu} dx^\nu = 0.$$

Now, if $(\delta F)^\mu (*F)_{\mu\nu} dx^\nu \neq 0$, then the relation $|\delta F| = |\delta * F|$ follows immediately. If $(\delta F)^\mu (*F)_{\mu\nu} dx^\nu = 0 = -(\delta * F)^\mu F_{\mu\nu} dx^\nu$ according to the third equation of (6), we shall show that $(\delta F)^2 = (\delta * F)^2 = 0$. In fact, forming the inner product $i(\delta F)(\delta F \wedge *F) = 0$, we get

$$(\delta F)^2 * F - \delta F \wedge (\delta F)^\mu (*F)_{\mu\nu} dx^\nu = (\delta F)^2 * F = 0.$$

In a similar way, forming the inner product $i(\delta * F)\delta * F \wedge F = 0$ we have

$$(\delta * F)^2 F - \delta(*F) \wedge (\delta * F)^\mu F_{\mu\nu} dx^\nu = (\delta * F)^2 F = 0.$$

This completes the proof. We note that in this last case the isotropic vectors δF and $\delta * F$ are also eigen vectors of $Q_{\mu\nu}$, and since $Q_{\mu\nu}$ has just one isotropic eigen direction, we conclude that δF and $\delta * F$ are colinear.

In order to formulate the third proposition, we recall [2] that at zero invariants $I_1 = I_2 = 0$ the following representation holds:

$$F = A \wedge \zeta,\ *F = A^* \wedge \zeta,$$

where A and A^* are 1-forms, ζ is the only (up to a scalar factor) isotropic eigen vector of Q_μ^ν (or the corresponding through the pseudometric η 1-form) and the relations $A.\zeta = 0,\ A^*.\zeta = 0$ hold. Having this in mind we prove the following

Proposition 3. *All nonlinear solutions satisfy the relations*

$$\zeta^\mu (\delta F)_\mu = 0,\ \zeta^\mu (\delta * F)_\mu = 0. \qquad (17)$$

Proof. We form the inner product $i(\zeta)(\delta F \wedge *F) = 0$:

$$[\zeta^\mu (\delta F)_\mu] * F - \delta F \wedge (\zeta)^\mu (*F)_{\mu\nu} dx^\nu =$$

$$= [\zeta^\mu (\delta F)_\mu]\, A^* \wedge \zeta - (\delta F \wedge \zeta) \zeta^\mu (A^*)_\mu + (\delta F \wedge A^*) \zeta^\mu \zeta_\mu = 0.$$

Since the second and the third terms are equal to zero and $*F \neq 0$, then $\zeta^\mu (\delta F)_\mu = 0$. Similarly, from the equation $(\delta * F) \wedge F = 0$ we get $\zeta^\mu (\delta * F)_\mu = 0$. The proposition is proved.

3 Algebraic properties of the nonlinear solutions

Since all nonlinear solutions have zero invariants $I_1 = I_2 = 0$ we can make a number of algebraic considerations, which clarify considerably the structure and make easier the study of the properties of these solutions. As we mentioned earlier, all eigen values of F, $*F$ and $Q_{\mu\nu}$ in this case are zero, and the eigen vectors can not be time-like. There is only one non-zero isotropic eigen direction of Q_μ^ν, defined by the isotropic vectors $\pm\zeta$ and the representations $F = A \wedge \zeta$, $*F = A^* \wedge \zeta$ hold, moreover, we have $A.A^* = 0,\ A^2 = (A^*)^2 \leq 0, A.\zeta = A^*.\zeta = 0$. Recall that the two 1-forms A and A^* are defined up to isotropic additive factors, colinear to ζ. The above representation of F and $*F$ through ζ shows that these factors do not contribute to F and $*F$, therefore, we assume further that, *these additive factors are equal to zero.*

We express now $Q_{\mu\nu}$ through A, A^* and ζ. First we normalize the vector ζ. This is possible, because it is an isotropic vector, so its time-like component ζ_4 is always different from zero. We divide ζ_μ by ζ_4 and get the vector $\mathbf{V} = (\mathbf{V}^1, \mathbf{V}^2, \mathbf{V}^3, 1)$, defining, of course, the same isotropic direction. Now we make use of the identity,

$$\frac{1}{2} F_{\alpha\beta} G^{\alpha\beta} \delta_\mu^\nu = F_{\mu\sigma} G^{\nu\sigma} - (*G)_{\mu\sigma} (*F)^{\nu\sigma}, \qquad (18)$$

where we put $F_{\mu\nu}$ instead of $G_{\mu\nu}$. Having in mind that $I_1 = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} = 0$, we obtain $F_{\mu\sigma} F^{\nu\sigma} =$

$(*F)_{\mu\sigma}(*F)^{\nu\sigma}$. Hence, the energy-momentum tensor is as follows

$$Q_\mu^\nu = -\frac{1}{4\pi}F_{\mu\sigma}F^{\nu\sigma} = -\frac{1}{4\pi}(*F)_{\mu\sigma}(*F)^{\nu\sigma} =$$

$$= -\frac{1}{4\pi}(A)^2\mathbf{V}\mu\mathbf{V}^\nu = -\frac{1}{4\pi}(A^*)^2\mathbf{V}_\mu\mathbf{V}^\nu. \qquad (19)$$

This choice of $\zeta = \mathbf{V}$ determines the following energy density $4\pi Q_4^4 = |A|^2 = |A^*|^2$.

We consider now the influence of the local conservation law $\nabla_\nu Q_\mu^\nu = 0$ on the vector field $\mathbf{V}$.

$$\nabla_\nu Q_\mu^\nu = -A^2\mathbf{V}^\nu\nabla_\nu\mathbf{V}_\mu - \mathbf{V}_\mu\nabla_\nu\left(A^2\mathbf{V}^\nu\right) = 0.$$

This relation holds for every $\mu = 1,2,3,4$. We consider it for $\mu = 4$ and get $\mathbf{V}^\nu\nabla_\nu(1) = \mathbf{V}^\nu\partial_\nu(1) = 0$. Therefore, $\mathbf{V}_4\nabla_\nu\left(A^2\mathbf{V}^\nu\right) = \nabla_\nu\left(A^2\mathbf{V}^\nu\right) = 0$. Since $A^2 \neq 0$, we obtain that $\mathbf{V}$ satisfies the equation

$$\mathbf{V}^\nu\nabla_\nu\mathbf{V}^\mu = 0,$$

which means, that $\mathbf{V}$ is a *geodesic* vector field, i.e. the integral trajectories of $\mathbf{V}$ are isotropic geodesics, or isotropic straight lines. Hence, *every nonlinear solution F defines unique isotropic geodesic direction in the Minkowski space-time.* This important consequence allows a special class of coordinate systems, called further F-adapted, to be introduced. These coordinate systems are defined by the requirement, that the trajectories of $\mathbf{V}$ are parallel to the (z,ξ)-coordinate plain. In such a coordinate system we have $\mathbf{V}_\mu = (0,0,\varepsilon,1)$, $\varepsilon = \pm 1$. Further on, we shall work in such arbitrary chosen but fixed F-adapted coordinate system, defined by the corresponding F under consideration.

We write down now the relations $F = A \wedge \mathbf{V}$, $*F = A^* \wedge \mathbf{V}$ component-wise, take into account the values of $\mathbf{V}_\mu$ in the F-adapted coordinate system and obtain the following explicit relations:

$$F_{12} = F_{34} = 0, \ F_{13} = \varepsilon F_{14}, \ F_{23} = \varepsilon F_{24},$$

$$(*F)_{12} = (*F)_{34} = 0, \quad (*F)_{13} = \varepsilon(*F)_{14} = -F_{24},$$
$$(*F)_{23} = \varepsilon(*F)_{24} = F_{14},$$

$$\begin{aligned} A &= (F_{14}, F_{24}, 0, 0), \ A^* = (-F_{23}, F_{13}, 0, 0) \\ &= (-\varepsilon A_2, \varepsilon A_1, 0, 0). \end{aligned} \qquad (20)$$

Clearly, the 1-forms A and $-A^*$ can be interpreted as *electric* and *magnetic* fields respectively. Only 4 of the components Q_μ^ν are different from zero, namely: $Q_4^4 = -Q_3^3 = \varepsilon Q_3^4 = -\varepsilon Q_4^3 = |A^2|$. Introducing the notations $F_{14} \equiv u$, $F_{24} \equiv p$, we can write

$$F = \varepsilon u dx \wedge dz + u dx \wedge d\xi + \varepsilon p dy \wedge dz + p dy \wedge d\xi$$

$$*F = -p dx \wedge dz + \varepsilon p dx \wedge d\xi + u dy \wedge dz + \varepsilon u dy \wedge d\xi.$$

In the important for us *spatially finite* case, i.e. when the functions u and p are finite with respect to the spatial variables (x,y,z), for the integral energy W and momentum $\mathbf{p}$ we obtain

$$W = \int Q_4^4 dxdydz = \int (u^2 + p^2) dxdydz < \infty,$$

$$\mathbf{p} = \left(0, 0, \varepsilon\frac{W}{c}\right), \ \to c^2|\mathbf{p}|^2 - W^2 = 0. \qquad (21)$$

Now we show how the nonlinear solution F defines at every point a pseudoorthonormal basis in the corresponding tangent and cotangent spaces. The nonzero 1-forms A and A^* are normed to $\mathbf{A} = A/|A|$ and $\mathbf{A}^* = A^*/|A^*|$. Two new unit 1-forms $\mathbf{R}$ and $\mathbf{S}$ are introduced through the equations:

$$\mathbf{R}^2 = -1, \quad \mathbf{A}^\nu\mathbf{R}_\nu = 0, \quad (\mathbf{A}^*)^\nu\mathbf{R}_\nu = 0,$$
$$\mathbf{V}^\nu\mathbf{R}_\nu = \varepsilon, \quad \mathbf{S} = \mathbf{V} + \varepsilon\mathbf{R}.$$

The only solution of the first 4 equations in the F-adapted coordinate system is $\mathbf{R}_\mu = (0,0,-1,0)$. Then for $\mathbf{S}$ we obtain $\mathbf{S}_\mu = (0,0,0,1)$, $\mathbf{S}^2 = 1$. This pseudoorthonormal co-tangent basis $(\mathbf{A}, \mathbf{A}^*, \mathbf{R}, \mathbf{S})$ is carried over to a tangent pseudoorthonormal basis by means of the pseodometric η.

We proceed further to introduce the concepts of *amplitude* and *phase* in a coordinate-free manner. First, of course, we look at the invariants, we have: $I_1 = I_2 = 0$. But in our case we have got another invariant, namely, the modulus of the 1-forms A and A^*: $|A| = |A^*|$. Let's begin with the *amplitude*, which shall be denoted by ϕ. As is seen from the obtained above expressions, the magnitude $|A|$ of A coincides with the square root of the energy density in any F-adapted coordinate system. And this is the meaning of the amplitude quantity. Hence, we define it by $|A| = |A^*|$. We give now two more coordinate-free ways to define the amplitude.

Recall first, that at every point, where the field is different from zero, we have three bases: the pseudoopthonormal coordinate basis $(dx, dy, dz, d\xi)$, the pseudoorthonormal basis $\chi^0 = (\mathbf{A}, \varepsilon\mathbf{A}^*, \mathbf{R}, \mathbf{S})$ and the pseudoorthogonal basis $\chi = (A, \varepsilon A^*, \mathbf{R}, \mathbf{S})$. The matrix $\chi_{\mu\nu}$ of χ with respect to the coordinate basis is

$$\chi_{\mu\nu} = \left\| \begin{matrix} u & -p & 0 & 0 \\ p & u & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\|.$$

We define now the amplitude ϕ of the field by

$$\phi = \sqrt{|det(\chi_{\mu\nu})|}. \qquad (22)$$

We consider now the matrix $\mathcal{R}$ of 2-forms

$$\mathcal{R} = \left\| \begin{array}{cccc} udx\wedge d\xi & -pdx\wedge d\xi & 0 & 0 \\ pdy\wedge d\xi & udy\wedge d\xi & 0 & 0 \\ 0 & 0 & -dy\wedge dz & 0 \\ 0 & 0 & 0 & dz\wedge d\xi \end{array} \right\|,$$

or, equivalently:

$$\mathcal{R} = udx\wedge d\xi\otimes(dx\otimes dx) - pdx\wedge d\xi\otimes(dx\otimes dy)$$
$$+pdx\wedge d\xi\otimes(dy\otimes dx) + +udy\wedge d\xi\otimes(dy\otimes dy)$$
$$-dy\wedge dz\otimes(dz\otimes dz) + dz\wedge d\xi\otimes(d\xi\otimes d\xi).$$

Now we can write

$$\phi = \sqrt{\frac{1}{2}\,|R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}|}.$$

We proceed further to define the *phase* of the nonlinear solution F. We shall need the matrix $\chi^0_{\mu\nu}$ of the basis χ^0 with respect to the coordinate basis. We obtain

$$\chi^0_{\mu\nu} = \left\| \begin{array}{cccc} \frac{u}{\sqrt{u^2+p^2}} & \frac{-p}{\sqrt{u^2+p^2}} & 0 & 0 \\ \frac{p}{\sqrt{u^2+p^2}} & \frac{u}{\sqrt{u^2+p^2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\|.$$

The *trace* of this matrix is

$$tr(\chi^0_{\mu\nu}) = \frac{2u}{\sqrt{u^2+p^2}}.$$

Obviously, the inequality $|\frac{1}{2}tr(\chi^0_{\mu\nu})| \leq 1$ is fulfilled. Now, by definition, the quantity $\varphi = \frac{1}{2}tr(\chi^0_{\mu\nu})$ will be called *the phase function* of the solution, and the quantity

$$\theta = \arccos(\varphi) = \arccos\left(\frac{1}{2}tr(\chi^0_{\mu\nu})\right) \qquad (23)$$

will be called *phase* of the solution.

Making use of the amplitude ϕ and the phase function φ we can write

$$u = \phi.\varphi,\ p = \phi.\sqrt{1-\varphi^2}. \qquad (24)$$

We note that the pair of 1-forms $A = udx + pdy$, $A^* = -pdx + udy$ defines a completely integrable Pfaff system, i.e. the following equations hold:

$$\mathbf{d}A\wedge A\wedge A^* = 0,\ \mathbf{d}A^*\wedge A\wedge A^* = 0.$$

In fact, $A\wedge A^* = (u^2+p^2)dx\wedge dy$, and in every term of $\mathbf{d}A$ and $\mathbf{d}A^*$ at least one of the basis vectors dx and dy will participate, so the above exterior products will vanish.

Remark. These considerations stay in force also for those linear solutions, which have zero invariants $I_1 = I_2 = 0$. But Maxwell's equations require u and p to be *running waves*, so the corresponding phase functions will be also running waves. As we'll see further, the phase functions for nonlinear solutions are arbitrary bounded functions.

We proceed further to define the new and important concept of *scale factor* L for a given nonlinear solution. It is defined by

$$L = \frac{|A|}{|\delta F|} = \frac{|A^*|}{|\delta * F|} = \frac{\phi}{\sqrt{|(\delta F\wedge\delta * F)|}}. \qquad (25)$$

Clearly, L can not be defined for the linear solutions, and in this sense it is *new* and we shall see that it is really *important*.

From the corresponding expressions $F = A\wedge\mathbf{V}$ and $*F = A^*\wedge\mathbf{V}$ it follows that the physical dimension of A and A^* is the same as that of F. We conclude that the physical dimension of L coincides with the dimension of the coordinates, i.e. $[L] = length$. From the definition it is seen that L is an *invariant* quantity, and depends on the point, in general. The invariance of L allows us to define a time-like 1-form (or vector field) $f(L)\mathbf{S}$, where f is some real function of L. Hence, every nonlinear solution determines a time-like vector field on M.

If the scale factor L, defined by the nonlinear solution F, is a *finite* and *constant* quantity, we can introduce a *characteristic* finite time-interval $T(F)$ by the relation

$$cT(F) = L(F),$$

and also a corresponding *characteristic frequency* by

$$\nu(F) = 1/T(F).$$

In these wave terms the scale factor L acquires the meaning of wave length, but this interpretation is arbitrary and we shall not make use of it.

It is clear, that the subclass of nonlinear solutions, which define constant scale factors, factor over the admissible values of the invariant $f(L)$. This makes possible comparing with experiment. For example, at constant scale factor L if we choose $f(L) = L/c$, then the scalar product of $(L/c)\mathbf{S}$ with the integral energy-momentum vector, which in the F-adapted coordinate system is $(0, 0, \varepsilon W, W)$, gives the invariant quantity $W.T$, having the physical dimension of action, and its numerical value could be easily measured.

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