Extended Electrodynamics: III. Free Photons and (3+1)-Soliton-like Vacuum Solutions

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This paper completes the series of three papers under the general title of *Extended Electrodynamics.* It aims to give explicitly all non- linear vacuum solutions to our non-linear field equations [1] in canonical coordinates, and to define in a coordinate free manner the important subclass of non-linear solutions, which we call *almost photon-like.* By means of a correct definition of the local and integral *intrinsic angular momentums* of these solutions, we separate the *photon-like* solutions through the requirement their integral intrinsic angular momentum to be equal to Planck's constant h. Finally, using standard spherical coordinates we consider such solutions moving radially to or from a given center.

1 Explicit non-linear vacuum solutions

As it was shown in [2] with every nonlinear solution F of our nonlinear equations (we use all notations from [1] and [2])

$$\delta F\wedge *F=0,\ \delta *F\wedge F=0,\ \delta *F\wedge *F-\delta F\wedge F=0 \quad (1)$$

a class of F-adapted coordinate systems is associated, such that F and $*F$ acquire the form respectively

$$F=\varepsilon u dx\wedge dz+u dx\wedge d\xi+\varepsilon p dy\wedge dz+p dy\wedge d\xi$$

$$*F=-p dx\wedge dz-\varepsilon p dx\wedge d\xi+u dy\wedge dz+\varepsilon u dy\wedge d\xi .$$

Since we look for non-linear solutions of (1), we substitute these F and $*F$ in (1) and after some elementary calculations we obtain

$$\begin{aligned}\delta F=&\ (u_\xi-\varepsilon u_z)dx+(p_\xi-\varepsilon p_z)dy\\ &+\varepsilon(u_x+p_y)dz+(u_x+p_y)d\xi,\end{aligned}$$

$$\begin{aligned}\delta *F=&-\varepsilon(p_\xi-\varepsilon p_z)dx+\varepsilon(u_\xi-\varepsilon p_z)dy\\ &-(p_x-u_y)dz-(p_x-u_y)d\xi,\end{aligned}$$

$$\begin{aligned}F_{\mu\nu}(\delta F)^\nu dx^\nu &=(*F)_{\mu\nu}(\delta *F)^\nu dx^\nu=\\ &=\varepsilon\Big[p(p_\xi-\varepsilon p_z)+u(u_\xi-\varepsilon u_z)\Big]dz\\ &+\left[p(p_\xi-\varepsilon p_z)+u(u_\xi-\varepsilon u_z)\right]d\xi,\end{aligned}$$

$$(\delta F)^2=(\delta *F)^2=-(u_\xi-\varepsilon u_z)^2-(p_\xi-\varepsilon p_z)^2$$

A simple direct calculation shows, that the equation

$$\delta *F\wedge *F-\delta F\wedge F=0$$

is identically fulfilled for any such F and $*F$ with arbitrary u and p. We infer that our equations reduce to only one equation, namely

$$\begin{aligned}p(p_\xi-&\ \varepsilon p_z)+u(u_\xi-\varepsilon u_z)\\ &=\tfrac{1}{2}\left[(u^2+p^2)_\xi-\varepsilon(u^2+p^2)_z\right]=0.\end{aligned} \quad (2)$$

The obvious solution to this equation is

$$u^2+p^2=\phi^2(x,y,\xi+\varepsilon z). \quad (3)$$

The solution obtained shows that the equations impose some limitations only on the amplitude function ϕ and that the phase function φ is arbitrary except that it is bounded: $|\varphi|\leq 1$. The amplitude ϕ is a running wave along the specially chosen coordinate z, which is common for all F-adapted coordinate systems.Considered as a function of the spatial coordinates, the amplitude ϕ is *arbitrary*, so it can be chosen *spatially finite.* The time-evolution does not affect the initial form of ϕ, so it will stay the same in time. Since $|\varphi|\leq 1$ and the two independent field components are given by $F_{14}=u=\phi.\varphi$, $F_{24}=p=\phi.\sqrt{1-\varphi^2}$ this shows, that *among the nonlinear solutions of our equations there are (3+1) soliton-like solutions.* The spatial structure of ϕ can be determined by initial condition, and the phase function φ can be used to describe additional *internal dynamics* of the solution.

Making use of the above mentioned substitutions

$$u=\phi.\varphi,\ p=\phi\sqrt{1-\varphi^2},$$

and the equality $|A|=\phi$, we get

$$|\delta F|=|\delta *F|=\frac{|\phi||\varphi_\xi-\varepsilon\varphi_z|}{\sqrt{1-\varphi^2}},\ L=\frac{|A|}{|\delta F|}=\frac{\sqrt{1-\varphi^2}}{|\varphi_\xi-\varepsilon\varphi_z|}. \quad (4)$$

For the induced pseudoorthonormal bases (1-forms and vector fields) we find

$$\mathbf{A} = \varphi dx + \sqrt{1-\varphi^2}dy,\ \varepsilon\mathbf{A}^* = -\sqrt{1-\varphi^2}dx + \varphi dy,$$
$$\mathbf{R} = -dz,\ \mathbf{S} = d\xi,$$

$$\mathbf{A} = -\varphi\tfrac{\partial}{\partial x} - \sqrt{1-\varphi^2}\tfrac{\partial}{\partial y},\ \varepsilon\mathbf{A}^* = \sqrt{1-\varphi^2}\tfrac{\partial}{\partial x} - \varphi\tfrac{\partial}{\partial y},$$
$$\mathbf{R} = \tfrac{\partial}{\partial z},\ \mathbf{S} = \tfrac{\partial}{\partial \xi}.$$

Hence, the nonlinear solutions in canonical coordinates are parametrized by one function ϕ of 3 independent variables and one *bounded* function of 4 independent variables. Therefore, the separation of various subclasses of nonlinear solutions is made by imposing additional conditions on these two functions. In the next sections we are going to distinguish a subclass of solutions , the integral properties of which reflect the well known integral properties and characteristics of free photons. These solutions will be called *photon-like* and will be distinguished through imposing additional requirements on φ and L in a coordinate-free manner.

2 Almost photon-like solutions

We note first that we have three invariant quantities at hand: ϕ, φ and L. The amplitude function ϕ is to be determined by the initial conditions, which have to be *finite*. Hence, we may impose additional conditions on L and φ. These conditions *have to express some internal consistency among the various characteristics of the solution.* The kind of internal consistency to use comes from the observation that *the amplitude function ϕ is a first integral of the vector field* $\mathbf{V}$, i.e.

$$\mathbf{V}(\phi) = \left(-\varepsilon\tfrac{\partial}{\partial z} + \tfrac{\partial}{\partial \xi}\right)(\phi) =$$
$$-\varepsilon\tfrac{\partial}{\partial z}\phi(x,y,\xi+\varepsilon z) + \tfrac{\partial}{\partial \xi}\phi(x,y,\xi+\varepsilon z) = 0.$$

In order to extend this consistency between $\mathbf{V}$ and ϕ we require the two functions φ and L to be first integrals of some of the available F-generated vector fields. Explicitly, we require the following:

1^0. *The phase function φ is a first integral of the three vector fields* $\mathbf{A}, \mathbf{A}^*$ and $\mathbf{R}$: $\mathbf{A}(\varphi) = 0, \mathbf{A}^*(\varphi) = 0, \mathbf{R}(\varphi) = 0$.

2^0. *The scale factor L is a non-zero finite first integral of the vector field* $\mathbf{S}$: $\mathbf{S}(L) = 0$.

The requirement $\mathbf{R}(\varphi) = 0$ just means that in these coordinates φ does not depend on the coordinate z. The two other equations of 1^0 define the following system of differential equations for φ:

$$-\varphi\frac{\partial\varphi}{\partial x} - \sqrt{1-\varphi^2}\frac{\partial\varphi}{\partial y} = 0,\ \sqrt{1-\varphi^2}\frac{\partial\varphi}{\partial x} - \varphi\frac{\partial\varphi}{\partial y} = 0.$$

Noticing that the matrix

$$\left\| \begin{matrix} -\varphi & -\sqrt{1-\varphi^2} \\ \sqrt{1-\varphi^2} & -\varphi \end{matrix} \right\|$$

has non-zero determinant, we conclude that the only solution of the above system is the zero-solution:

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial y} = 0.$$

We conclude that in the coordinates used the phase function φ depends only on ξ. Therefore, in view of (4), for the scale factor L we get

$$L = \frac{\sqrt{1-\varphi^2}}{|\varphi_\xi|}.$$

Now, the requirement 2^0, which in these coordinates reads

$$\mathbf{S}(L) = \frac{\partial L}{\partial \xi} = \frac{\partial}{\partial \xi}\frac{\sqrt{1-\varphi^2}}{|\varphi_\xi|} = 0,$$

means that the scale factor L is a pure constant: $L = const$. In this way the defining relation for L turns into a differential equation for φ:

$$L = \frac{\sqrt{1-\varphi^2}}{|\varphi_\xi|} \ \rightarrow\ \frac{\partial\varphi}{\partial\xi} = \mp\frac{1}{L}\sqrt{1-\varphi^2}. \qquad (5)$$

The obvious solution to this equation is

$$\varphi(\xi) = \cos\left(\kappa\frac{\xi}{L} + const\right), \qquad (6)$$

where $\kappa = \pm 1$. We note that the naturally arising *characteristic frequency* ν according to the equation

$$\nu = \frac{c}{L}, \qquad (7)$$

has nothing to do with the concept of frequency in CED. In fact, *the quantity L can not be defined in Maxwell's theory.*

Finally (recalling [2]) we note that the derived phase function $\varphi(\xi)$ leads to the following. Consider the 2-form $tr(\mathcal{R}^0)$, where $\mathcal{R}^0$ is the matrix of 2-forms, formed similarly to the matrix $\mathcal{R}$ in [2], but using the basis $(\mathbf{A}, \varepsilon\mathbf{A}^*, \mathbf{R}, \mathbf{S})$ instead of the basis $(A, \varepsilon A^*, \mathbf{R}, \mathbf{S})$. It turns out that $tr(\mathcal{R}^0)$ is a closed 2-form. In fact,

$$tr(\mathcal{R}^0) = \varphi dx\wedge d\xi + \varphi dy\wedge d\xi - dy\wedge dz + dz\wedge d\xi$$

and since $\varphi = \varphi(\xi)$, we get $\mathbf{d}tr(\mathcal{R}^0) = 0$. Note also that the above explicit form of $tr(\mathcal{R}^0)$ allows to define the phase function by

$$\varphi = \sqrt{\frac{|tr(\mathcal{R}^0)|^2}{2}}.$$

This class of solutions we call *almost photon-like.*

Remark. If one of the two functions u and p, for example p, is equal to zero: $p = 0$, then, formally, we again have a solution, which may be called *linearly polarized* by obvious reasons. Clearly, the phase function of such solutions will be *constant*: $\varphi = const$, so, the corresponding scale factor becomes infinitely large: $L \to \infty$, therefore, condition 2^0 is not satisfied. The reason for this is, that at $p = 0$ the function u becomes a *running wave* and we get $|\delta F| = |\delta * F| = 0$, so the scale factor can not be defined by the relation $L = |A|/|\delta F|$.

3 Intrinsic angular momentum (helicity) and photon-like solutions

The problem of describing the *intrinsic angular momentum* (IAM), or in short *helicity, spin* of the photon is of fundamental importance in modern physics. So, we are going to consider two approaches for its mathematical description. But first, some preliminary comments.

First of all, *there is no doubt that every free photon carries such an intrinsic angular momentum.* Since the angular momentum is a conserved quantity, the existence of the photon's intrinsic angular momentum can be easily established and, in fact, its presence has been experimentally proven by the immediate observation of its mechanical action and its value has been numerically measured. Assuming this is so, we have to understand its origin, nature and its significance for the existence of the photons.

So, we begin with the assumption:*every free photon carries an intrinsic angular momentum with integral value equal to Planck's constant h.* According to our understanding, the photon's IAM comes from an intrinsic *periodic process.* This point of view undoubtedly leads to the concept, that photons *are not* point-like structureless objects: they have a structure, i.e. they are *extended objects.* In fact, according to one of the basic principles of physics *all free objects move as a whole uniformly.* So, if the photon is a point-like object any characteristic of a periodic process, e.g. frequency, should come from an outside force field, i.e. it cannot be free: a free point-like (structureless) object cannot have a characteristic frequency.

This simple, but valid, conclusion presents the theoretical physics of the first quarter of this century with a serious dilemma: to keep the notion of structurelessness and to associate in a formal way a characteristic frequency to microobjects, or to leave off the notion of structurelessness, to assume the notion of extendedness and availability of intrinsically occurring periodic process and to build corresponding integral characteristics, determined by this periodic process. Reflection shows that the majority of physicists have adopted the first approach, which led to quantum mechanics as a computational method , and the wave-particle dualistic-probabilistic interpretation as a philosophical consequence. If we set aside the widespread and intrinsically controversial idea that all microobjects are at the same time (point-like) particles and (infinite) waves, and look impartially, in a fair-minded way, at the quantum mechanical wave function for a *free* particle, we see that *the only positive consequence* of its introduction is *the legalization of frequency*, as an inherent characteristic of the microobject. In fact, the probabilistic interpretation of the quantum mechanical wave function for a free object, obtained as a solution of the free Schroedinger equation, is impossible since its square is not an integrable quantity (the integral is infinite). The frequency is really needed not because of the dualistic-probabilistic nature of microobjects, it is needed because the Planck relation $W = h\nu$ which turns out to be universally true in microphysics, so there is no way to avoid the introduction of frequency. The question is, does the introduction of frequency necessarily require some (linear) wave equation and the simple complex exponentials of the kind $const.exp[i(\mathbf{k}.\mathbf{r} - \nu t)]$, i.e. infinite running waves, as "free solutions". Our answer to this question is no. The classical monochromatic wave does not seem to be the most adequate mathematical object needed. In fact, it does not represent the finite nature of photons, it even contradicts it. As for the frequency, there are non-linear finite waves, which also have this characteristic.

These considerations made us turn to soliton-like objects: they present the two features of microobjects: (*localized spatial extendedness and time-periodicity*) simultaneously, and, therefore, seem to be more adequate theoretical models for those microobjects, obeying the Planck's relation $W = h\nu$. Of course, if we are interested only in the behaviour of the microobject as a whole, we can use the point-like notion, but any attempt to give a meaning to its integral characteristics without looking for their origin in the consistent internal dynamics and structure, in our opinion, is an incinsistent perspective. One of the basic "stumbling points" of such an approach is the existence of an intrinsic mechanical angular momentum, which can not be understood as an attribute of a free structureless object.

Having in mind the above considerations, we consider two ways to introduce and define the intrinsic angular momentum as a local quantity and to obtain by integration its integral value. Thus, these two approaches will be of use only for the spatially finite nonlinear solutions of our equations. Both approaches introduce (in different ways) 3-tensors (2-covariant and 1-contravariant). Although these two 3-tensors are built of quantities, connected in a definite way with the field F, their nature is quite different. The first approach is based on an appropriate tensor generalization of the classical Poynting vector. The second approach makes use of the concept of *torsion*, connected with the field F, considered as a 1-covariant and 1-contravariant tensor. The first approach is purely algebraic with respect to F, while the second one uses derivatives of $F_{\mu\nu}$. The spatially finite nature of the solutions F allows us to build corresponding integral conserved quantities, naturally interpreted as angular momentum. The scale factor $L = const$ appears as a multiple, so these quantities go to infinity for all linear solutions, i.e. the solutions of Maxwell equations.

3.1 The First Approach

In the first approach we consider just almost photon-like solutions and make use of the corresponding scale factors $L = const$, isotropic vector fields $\mathbf{V}$ and the two 1-forms A and A^*. By these four quantities we build the following 3-tensor field H:

$$H = \kappa\frac{L}{c}\mathbf{V}\otimes(A\wedge A^*). \tag{8}$$

The connection with the classical vector of Poynting comes through the exteriour product of A and A^*, the 3-dimensional meaning of which is just the Pointing's vector. In components we have

$$H^{\mu}_{\nu\sigma} = \kappa\frac{L}{c}\mathbf{V}^{\mu}(A_\nu A^*_\sigma - A_\sigma A^*_\nu).$$

In our system of coordinates we get

$$H = \kappa\frac{L}{c}\left(-\varepsilon\frac{\partial}{\partial z} + \frac{\partial}{\partial \xi}\right)\otimes(\varepsilon\phi^2 dx\wedge dy),$$

hence, the only non-zero components are

$$H^3_{12} = -H^3_{21} = -\kappa\frac{L}{c}\phi^2,\ H^4_{12} = -H^4_{21} = \kappa\varepsilon\frac{L}{c}\phi^2.$$

It is easily seen, that the divergence $\nabla_\mu H^\mu_{\nu\sigma} \rightarrow \nabla_\mu H^\mu_{12}$ is equal to 0. In fact,

$$\nabla_\mu H^\mu_{12} = \frac{\partial}{\partial z}H^3_{12} + \frac{\partial}{\partial \xi}H^4_{12} = \kappa\frac{L}{c}\left[-(\phi^2)_z + (\varepsilon\phi^2)_\xi\right] = 0$$

because ϕ^2 is a running wave along the coordinate z. Since for the Minkowski space-time the tangent bundle, the co-tangent bundle and their tensor and exterior products are trivial bundles, we may consider the tensor field $H_{\mu,\nu\sigma}$ as 1-form on the Minkowski manofold with values in the exterior product $K\wedge K$, where K is an algebraic Minkowski space: $H \in \Lambda^1(M, K\wedge K)$, $H = H_{\mu,\nu\sigma}dx^\mu\otimes e^\nu\wedge e^\sigma$, and $\{e^\nu\}$ is a basis of K. Now, making use of the Hodge $*$ on M we get the 3- form

$$*H = (*H_\mu dx^\mu)_{\nu\sigma}\otimes e^\nu\wedge e^\sigma,$$

where $*$ acts only on dx^μ. This 3-form has values in $K\wedge K$ and is **closed** because of the original zero divergence of H. Now, we can integrate $*H$ on the 3-space and we can form an antisymmetric 2-tensor $\mathbf{H}\in K\wedge K$:

$$\mathbf{H}_{\nu\sigma} = \int_{R^3} H_{4,\nu\sigma}dxdydz.$$

According to Stokes theorem, the components of this antisymmetric tensor shall not depend on the time coordinate, i.e. they are conserved quantities.

$$\begin{aligned}\mathbf{H}_{12} = -\mathbf{H}_{21} &= \textstyle\int_{R^3} H_{4,12}dxdydz = \kappa\varepsilon\frac{L}{c}W\\ &= \kappa\varepsilon WT = \kappa\varepsilon\frac{W}{\nu}.\end{aligned}$$

The non-zero eigenvalues of $\mathbf{H}_{\nu\sigma}$ are purely imaginary and are equal to $\pm iWT$. This tensor has unique non-zero invariant $P(F)$,

$$P(F) = \sqrt{\frac{1}{2}\mathbf{H}_{\nu\sigma}\mathbf{H}^{\nu\sigma}} = WT. \tag{9}$$

The quantity $P(F)$ will be called *Planck's invariant* for the finite nonlinear solution F. All finite nonlinear solutions $F_1, F_2, , ...,$ satisfying the condition

$$P(F_1) = P(F_2) = ... = h,$$

where h is the Planck's constant, will be called further *photon-like*. The tensor field H will be called *the intrinsic angular momentum tensor* and the tensor $\mathbf{H}$ will be called *spin tensor* or *helicity tensor*. The Planck's invariant $P(F) = WT$, having the physical dimension of action, will be called *integral angular momentum*, or just *spin* or *helicity*.

The reason to use this terminology are quite clear: the time evolution of the two mutually orthogonal vector fields A and A^* is a rotationally-advancing motion around and along the z-coordinate (admissible are the right and the left rotations: $\kappa = \pm 1$), with the advancing velocity of c and the frequency of circulation $\nu = c/L$. We see the basic role of the two features of the solutions: their soliton-like character, giving finite value of all integral quantities, and their nonlinear character, allowing us to define the scale factor L correctly. From this point of view the spin of the photon is far from being an *incomprehensible* quantity, it appears as a normal integral characteristic of a solution, representing a model of the photon.

3.2 The Second Approach

We proceed to the second approach to introduce IAM. We recall the definition of the *torsion* of two (1,1) tensors. If G and K are 2 such tensors

$$G=G^{\nu}_{\mu}dx^{\mu}\otimes\frac{\partial}{\partial x^{\nu}},\quad K=K^{\nu}_{\mu}dx^{\mu}\otimes\frac{\partial}{\partial x^{\nu}},$$

their torsion is defined as a 3-tensor $S^{\sigma}_{\mu\nu}=-S^{\sigma}_{\nu\mu}$ by the relation

$$\begin{aligned}S(G,K)(X,Y)=&\ [GX,KY]+[KX,GY]+GK[X,Y]\\&+KG[X,Y]-G[X,KY]-G[KX,Y]\\&-K[X,GY]-K[GX,Y],\end{aligned}$$

where $[,]$ is the Lie-bracket of vector fields,

$$GX=G^{\nu}_{\mu}X^{\mu}\frac{\partial}{\partial x^{\nu}},\quad GK=G^{\nu}_{\mu}K^{\mu}_{\sigma}dx^{\sigma}\otimes\frac{\partial}{\partial x^{\nu}}$$

and X,Y are 2 arbitrary vector fields. If $G=K$, in general $S(G,G)\neq 0$ and

$$\begin{aligned}S(G,G)(X,Y)=&\ 2\big\{[GX,GY]+GG[X,Y]\\&-G[X,GY]-G[GX,Y]\big\}.\end{aligned}$$

This last expression defines at every point $x\in M$ the torsion $S(G,G)=S_G$ of G with respect to the 2-dimensional plane, defined by the two vectors $X(x)$ and $Y(x)$. Now we are going to compute the torsion S_F of the nonlinear solution F with respect to the intrinsically defined by the two unit vectors $\mathbf{A}$ and $\varepsilon\mathbf{A}^*$ 2-plane. In components we have

$$(S_F)^{\sigma}_{\mu\nu}=2\left[F^{\alpha}_{\mu}\frac{\partial F^{\sigma}_{\nu}}{\partial x^{\alpha}}-F^{\alpha}_{\nu}\frac{\partial F^{\sigma}_{\mu}}{\partial x^{\alpha}}-F^{\sigma}_{\alpha}\frac{\partial F^{\alpha}_{\nu}}{\partial x^{\mu}}+F^{\sigma}_{\alpha}\frac{\partial F^{\alpha}_{\mu}}{\partial x^{\nu}}\right].$$

In our coordinate system

$$\mathbf{A}=-\varphi\frac{\partial}{\partial x}-\sqrt{1-\varphi^2}\frac{\partial}{\partial y},\quad \varepsilon\mathbf{A}^*=\sqrt{1-\varphi^2}\frac{\partial}{\partial x}-\varphi\frac{\partial}{\partial y},$$

so,

$$(S_F)^{\sigma}_{\mu\nu}\mathbf{A}^{\mu}\varepsilon\mathbf{A}^{*\nu}=(S_F)^{\sigma}_{12}(\mathbf{A}^1\varepsilon\mathbf{A}^{*2}-\mathbf{A}^2\varepsilon\mathbf{A}^{*1}).$$

For $(S_F)^{\sigma}_{12}$ we get

$$(S_F)^1_{12}=(S_F)^2_{12}=0,$$

$$(S_F)^3_{12}=-\varepsilon(S_F)^4_{12}=2\varepsilon\{p(u_\xi-\varepsilon u_z)-u(p_\xi-\varepsilon p_z)\}.$$

Remark. In our case $(S_F)^{\sigma}_{12}=(S_{*F})^{\sigma}_{12}$, so further we shall work with S_F only.

It is easily seen that the following relation holds: $\mathbf{A}^1\varepsilon\mathbf{A}^{*2}-\mathbf{A}^2\varepsilon\mathbf{A}^{*1}=1$. Now, for the almost photon-like solutions

$$u=\phi(x,y,\xi+\varepsilon z)\cos\left(\kappa\frac{\xi}{L}+const\right),$$

$$p=\phi(x,y,\xi+\varepsilon z)\sin\left(\kappa\frac{\xi}{L}+const\right)$$

we obtain

$$(S_F)^3_{12}=-\varepsilon(S_F)^4_{12}=-2\varepsilon\frac{\kappa}{L}\phi^2,$$

$$(S_F)^{\sigma}_{\mu\nu}\mathbf{A}^{\mu}\varepsilon\mathbf{A}^{*\nu}=\left[0,0,-2\varepsilon\frac{\kappa}{L}\phi^2,2\frac{\kappa}{L}\phi^2\right].$$

Since ϕ^2 is a running wave along the z-coordinate, the vector field $S_F(\mathbf{A},\varepsilon\mathbf{A}^*)$ has zero divergence: $\nabla_\nu\left[S_F(\mathbf{A},\varepsilon\mathbf{A}^*)\right]^\nu=0$. Now we define the *helicity vector* of the solution F by

$$\Sigma_F=\frac{L^2}{2c}S_F(\mathbf{A},\varepsilon\mathbf{A}^*).$$

Since $L=const$, then Σ_F has also zero divergence, so the integral quantity

$$\int(\Sigma_F)_4dxdydz$$

does not depend on time and is equal to κWT. The photon-like solutions are distinguished by the condition $WT=h$.

Here are three more integral expressions for the quantity WT. We form the 4- form

$$-\frac{1}{L}\mathbf{S}\wedge*\Sigma_F=\frac{\kappa}{c}\phi^2\omega_\circ$$

and integrate it over the 4-volume $\mathcal{R}^3\times[0,L]$, the result is κWT. Besides, we easily verify the relations

$$\begin{aligned}\tfrac{1}{c}\textstyle\int_{R^3\times[0,L]}|A\wedge A^*|\omega_\circ&=\tfrac{L^2}{c}\textstyle\int_{R^3\times[0,L]}|\delta F\wedge\delta*F|\omega_\circ\\&=WT.\end{aligned}$$

Since we distinguish the photon-like solutions by the relation $WT=h$, the last expression suggests the following interpretation of Planck's constant h. Since $|A\wedge A^*|$ is proportional to the area of the square, defined by the two mutually orthogonal vectors A and εA^*, the above integral sums up all these areas over the whole 4- volume, occupied by the solution F during the intrinsically determined time period T, in which the couple $(A,\varepsilon A^*)$ completes a full rotation. The same can be said for the couple $(\delta F,\delta*F)$ with some different factor in front of the integral. This shows quite clearly the helical origin of the full energy $W=h\nu$ of the single photon.

4 Solutions in spherical cordinates

The soliton-like solutions obtained so far describe objects, coming from infinity and going to infinity. Of interest are also soliton like solutions "radiated" from, or absorbed by some central source, and propagating radially from or to the center of this source. We are going to show, that our equations (1) admit such solutions. We assume this central source to be a small ball R^0 with radius $r_\circ$, and put the origin of the coordinate system at the center of the source-ball. The standard spherical coordinates (r,θ,φ,ξ) will be used and all considerations will be carried out in the region out of the ball R^0. In these coordinates we have

$$ds^2=-dr^2-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+d\xi^2,\ \sqrt{|\eta|}=r^2\sin\theta.$$

The $*$-operator acts in these coordinates as follows:

$$\begin{array}{ll} *dr=r^2\sin\theta d\theta\wedge d\varphi\wedge d\xi & *(dr\wedge d\theta\wedge d\varphi)=(r^2\sin\theta)^{-1}d\xi \\ *d\theta=-\sin\theta dr\wedge d\varphi\wedge d\xi & *(dr\wedge d\theta\wedge d\xi)=\sin\theta d\varphi \\ *d\varphi=(\sin\theta)^{-1}dr\wedge d\theta d\xi & *(dr\wedge d\varphi\wedge d\xi)=-(\sin\theta)^{-1}d\theta \\ *d\xi=r^2\sin\theta dr\wedge d\theta d\varphi & *(d\theta\wedge d\varphi\wedge d\xi)=(r^2\sin\theta)^{-1}dr \end{array}$$

$$\begin{array}{ll} *(dr\wedge d\theta)=-\sin\theta d\varphi\wedge d\xi & *(d\theta\wedge d\varphi)=-(r^2\sin\theta)^{-1}dr\wedge d\xi \\ *(dr\wedge d\varphi)=(\sin\theta)^{-1}d\theta\wedge d\xi & *(d\theta\wedge d\xi)=-\sin\theta dr\wedge d\varphi \\ *(dr\wedge d\xi)=r^2\sin\theta d\theta\wedge d\varphi & *(d\varphi\wedge d\xi=(\sin\theta)^{-1}dr\wedge d\theta. \end{array}$$

We look for solutions of the following kind:

$$F=\varepsilon u dr\wedge d\theta+ud\theta\wedge d\xi+\varepsilon p dr\wedge d\varphi+pd\varphi\wedge d\xi, \tag{10}$$

where u and p are spatially finite functions. We get

$$*F=\frac{p}{\sin\theta}dr\wedge d\theta+\varepsilon\frac{p}{\sin\theta}d\theta\wedge d\xi-usin\theta dr\wedge d\varphi-\varepsilon\sin\theta d\varphi\wedge d\xi.$$

The following relations hold:

$$F\wedge F=2\varepsilon(up-up)dr\wedge d\theta\wedge d\varphi\wedge d\xi=0,$$

$$F\wedge *F=\left(-u^2\sin\theta+u^2\sin\theta-\frac{p^2}{\sin\theta}+\frac{p^2}{\sin\theta}\right)dr\wedge d\theta\wedge d\varphi\wedge d\xi=0,$$

i.e. the two invariants are equal to zero: $(*F)_{\mu\nu}F^{\mu\nu}=0,\ F_{\mu\nu}F^{\mu\nu}=0.$

After some elementary computation we obtain

$$\delta F\wedge F=\delta *F\wedge *F=\varepsilon\left[u\left(\varepsilon p_r+p_\xi\right)-p\left(\varepsilon u_r+u_\xi\right)\right]dr\wedge d\theta\wedge d\varphi++\left[u\left(\varepsilon u_r+u_\xi\right)-u\left(\varepsilon p_r+p_\xi\right)\right]d\theta\wedge d\varphi\wedge d\xi,$$

$$F\wedge *\mathbf{d}F=\varepsilon\left[u\left(\varepsilon u_r+u_\xi\right)\sin\theta+\frac{p\left(\varepsilon p_r+p_\xi\right)}{\sin\theta}\right]dr\wedge d\theta\wedge d\varphi--\varepsilon\left[u\left(\varepsilon u_r+u_\xi\right)\sin\theta+\frac{p\left(\varepsilon p_r+p_\xi\right)}{\sin\theta}\right]d\theta\wedge d\phi\wedge d\xi,$$

$$(*F)\wedge *\mathbf{d}*F=\left[u\left(\varepsilon u_r+u_\xi\right)\sin\theta+\frac{p\left(\varepsilon p_r+p_\xi\right)}{\sin\theta}\right]dr\wedge d\theta\wedge d\varphi--\left[u\left(\varepsilon u_r+u_\xi\right)\sin\theta+\frac{p\left(\varepsilon p_r+p_\xi\right)}{\sin\theta}\right]d\theta\wedge d\phi\wedge d\xi.$$

So, the two functions u and p have to satisfy the equation

$$u\left(\varepsilon u_r+u_\xi\right)\sin\theta+\frac{p\left(\varepsilon p_r+p_\xi\right)}{\sin\theta}=0, \tag{11}$$

which is equivalent to the equation

$$\left(u^2\sin\theta+\frac{p^2}{\sin\theta}\right)_\xi+\varepsilon\left(u^2\sin\theta+\frac{p^2}{\sin\theta}\right)_r=0. \tag{12}$$

The general solution of this equation is

$$u^2\sin\theta+\frac{p^2}{\sin\theta}=\phi^2(\xi-\varepsilon r,\theta,\phi). \tag{13}$$

For the non-zero components of the energy-momentum tensor we obtain

$$-Q_1^1=-Q_1^4=Q_4^1=Q_4^4=\frac{1}{4\pi r^2\sin\theta}\left(u^2\sin\theta+\frac{p^2}{\sin\theta}\right). \tag{14}$$

It is seen that the energy density is not exactly a running wave but when we integrate to get the integral energy, the integrand is exactly a running wave:

$$W = \frac{1}{4\pi}\int_{R^3-R^0} *\left(Q^4_\mu d\xi\right)$$
$$= \frac{1}{4\pi}\int_{R^3-R^0}\left(u^2\sin\theta + \frac{p^2}{\sin\theta}\right)dr\wedge d\theta\wedge d\phi.$$

Since the functions u and p are spatially finite, the integral energy W is finite, and from the explicit form of the energy-momentum tensor it follows the well known relation between the integral energy and momentum: $W^2 - c^2\mathbf{p}^2 = 0$.

5 Conclusion

Let's try to summurize the rezults obtained so far in EED.

Starting with some analysis of CED, we came to the conclusion for the 2-vector component character of the electromagnetic field, so the modeling mathematical object was chosen to be a 2-form on Minkowski space valued in $\mathcal{R}^2$. Some further analysis, mainly of the local conservation laws, clarified the ways of energy-momentum exchange between the field an some other (continuous) physical object. Then the idea for physical interpretation of the Frobenius integrability equations as lack of dissipation was realized. All this culminated in writing down the local energy-momentum balance equation (12) in [1] and the integrability equations for the all six 2-dimensional Pfaff systems generated by the four 1-forms α^i. These in general 24 equations for the 22 unknown functions $(F_{\mu\nu},\ \alpha^i_\mu)$ were assumed for dynamical equations in EED.

The nonlinear solutions of the vacuum equations were extensively studed in [2] and in this paper. After noting the conformal invariance of the equations in this case and pointed out a 3-parameter family of static non-linear solutions, we proved the important Proposition 1 in [2], according to which all non-linear solutions have zero invariants

$$I_1 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = 0,\ I_2 = \frac{1}{2}F_{\mu\nu}(*F)^{\mu\nu} = 0.$$

So, from the eigen properties of the the field and from the local conservation laws it followed that every nonlinear solution defines intrinsically unique light-like (or isotropic) direction in the 4-dimensional Minkowski space-time. This direction determines its straight-line propagation as a whole with the velocity of light. So, it becomes possible to introduce F-adapted coordinate systems, in which all relations simplify significantly. Then the electromagnetic frames were built and the amplitude function ϕ and the phase function φ of the solution were introduced in a coordinate free manner. The scale factor L as a special characteristic of the nonlinear solutions, and having no sense for the Maxwell solutions, was also defined in a coordinate free way. So we obtained 3 invariant local scalar characteristics of the nonlinear solutions.

Introducing all these features in an appropriate way into the dynamical equations we found them reduced to only one equation for the amplitude ϕ. Hence, the general solution in an F-adapted coordinate system was expressed through ϕ and φ: $\phi = \phi(x, y, \xi \pm z)$ should be a running wave along the direction of propagation, and φ should be bounded: $|\varphi|^2 \le 1$. Then the almost photon-like solutions were distinguished by means of imposing some internal consistency conditins on φ and L in a coordinate free way. The important quantity of spin was introduced in two independent ways and photon-like solutions were named those having Planck's invariant $P(F)$ equal to the Planck constant h. We showed four ways to define and compute this very important invariant characteristic of the photon. Finally, localized solutions in spherical coordinataes were explicitly found.

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