A nonlinear field equation for the simultaneous treatment of the dynamics of charge and matter

Robert H. Tyler

Applied Physics Laboratory, University of Washington, 1013 NE 40th St., Seattle WA, 98105-6698
and
Institut fuer Meereskunde; Abteilung Theoretische Ozeanographie, Duesternbrooker Weg 20, 24105 Kiel, Germany
Email:tyler@apl.washington.edu

ABSTRACT. We propose a theory whereby the dynamics of mass and charge are described simultaneously in a simple 4-tensor equation describing the conservation of potential flux in its own flow field. We show how this equation can be expanded into a set of four first-order non-linear equations, and how under appropriate limits these equations reproduce the conventional equations for electrodynamics, mass (inertial) dynamics, and the equation of motion for an idealized charged massive particle in a specified electromagnetic field.

RÉSUMÉ Nous proposons une théorie décrivant simultanément les dynamiques de la masse et de la charge, via une équation tensorielle (á 4 indices) simple exprimant la conservation du flux de potentiel dans son propre champ advectif. Nous démontrons ensuite comment cette équation peut être développée en un systéme de quatre équations non-linéaires du premier ordre, et comment, sous certaines hypothéses, ces équations reproduisent les équations conventionelles de l'électrodynamique, de la dynamique de la masse (inertielle), ainsi que l'équation du mouvement pour une particule chargée et massive dans un champ electromagnétique donné.

1 Introduction

In this paper we show that simple aspects of the dynamics of systems involving mass, electrical charge, momentum, and electrical currents can be derived from the following simple conservation equation:

$$\partial_\lambda (A_\mu A_\lambda) = 0. \tag{1}$$

The tensor $A_\mu A_\lambda$ is the product of components of a 4-vector A_μ having three space components and one time component as will be discussed in more detail in later sections. We shall refer to the 4-vector A_μ as the 'potential flux field' since in vacuum electrodynamic applications, A_μ will be shown to have space components identical to the conventional magnetic vector potential and a time component proportional to the electric potential scalar. Similarly, when rest mass is absent or negligible, the electrodynamic aspects of the tensor $A_\mu A_\lambda$ will be seen to have some obvious similarities with the Maxwell stress-energy tensor. In more general applications, however, other aspects of A_μ are unusual and suggest that a better name might be 'amplitude' (after similarities it bears to the amplitude function used in quantum mechanics) or 'photon flux density' (since it is the fundamental carrier of the electromagnetic field and energy). In fact (to give contrast to the purely electromagnetic case) for applications where mass advection dominates, the D'Alembertian of $A_\mu A_\lambda$ resembles the conventional stress-energy tensor for matter.

In the following sections, we will attempt to show that (1) is a fundamental equation describing the field dynamics of mass and charge. Charge density is related to curvature in the A_μ field, while energy and mass density are related to curvature in the square of the A_μ field. In the safest approach (which we will presume unless stated otherwise), we can follow a conventional approach and regard the material charge and mass as being imposed in the problem and that these 'cause' the fields. The tenor of the formulations we will show suggest, however, that it might be more natural to regard the mass and charge as aspects of the field. In fact, in the derivations we find relationships governing the form of field configurations that will produce mass and charge densities. In this paper we focus on simple fields which do not have such configurations since the main purpose of this paper is just to show that the formulation proposed reproduces conventional formulations in the appropriate limits.

In section 2 the general formulation is laid out together with a table listing the relationships of A_μ to the conventional quantities, and a subsection presents a discussion in terms of the Lagrangian density. In section 3 we verify that in the idealized limit where only electromagnetic fields are presumed to be important (no rest mass effects), a correct wave equation is obtained for the electric potential and in fact the present formulation can be reduced to a conventional description in terms of the vacuum Maxwell's equation. In section 4 we test the other limiting case where electrodynamic effects are neglected and only inertial effects due to rest mass are important. In this case both the correct unforced momentum equation for a particle, and the correct energy tensor description for a noninteracting 'dust' continuum are reproduced. In section 5 we treat a simple case which combines the inertial effects of a small particle and the electromagnetic effects of a background field which is presumed to be large enough that the effects of the particle on the total field can be neglected. In this case the correct momentum equation is obtained for a particle subject to electric and Lorentz forces (due to the magnetic field and the particle's velocity).

2 General formulation

Before applying (1) to specific cases, we discuss some of the general properties of A_μ together with the relationships with the conventional quantities. In the following, Einstein summation is used (summation over repeated indices is implicit), Latin subscripts refer to components in 3-space, and Greek indices refer to components in 4-space. Also, for simplicity, we will assume light speed $c = 1$. Throughout this paper, it will be assumed that $\mu = 1, 2, 3$ corresponds to orthogonal space components while $\mu = 4$ corresponds to the time component of the 4-vector.

First, the form of the 4-vector A_μ depends on the spacetime in which it is interpreted. The simplest general form of A_μ seems to be gained for a 4-space involving the proper time rather than the conventional time. However, for clarity here, we will interpret A_μ in a conventional 4-space described by the flat pseudo-Euclidean metric $g_{\mu\lambda}$ =diag$(-1, -1, -1, -1)$ wherein tensor indices can be easily raised or lowered to form inner products by simply multiplying by -1: (e.g., $A^\lambda = -A_\lambda$, $F^{\mu\lambda} = -F_{\mu\lambda}$; for a description of this and other flat spacetime geometries see the book by Arzelies[1]). The components of this 4-space are $x^\mu = \langle x^i, it \rangle$ and the fourth component A^4 of the 4-vector $A^\mu = \langle A^i, A^4 \rangle \equiv \langle A^i, i\phi \rangle$ is also imaginary. In this section the position of the indices is used in a conventional manner to indicate the covariant and contravariant components, and $\partial_\lambda = \frac{\partial}{\partial x^\lambda}$ refers to the contravariant derivative. In other sections, however it is not necessary to make this distinction.

Equation 1 can be expanded to give the following vector and scalar equations:

$$\partial_t(\phi A^i) + \partial_j(A^i A^j) = 0, \tag{2}$$

$$\partial_t(\phi^2) + \partial_j(\phi A^j) = 0. \tag{3}$$

As will be supported in later sections, the components of $A^\mu = \langle A^i, i\phi \rangle$ are related to conventional fields and sources. We list in Table

1 the general relationships which throughout the text will be implicitly referred to:

field potential	ϕ
field potential flux	A^i
field energy	ϕ^2
field momentum	ϕA^i
field rest energy (field mass)	$\phi_o^2 = \phi^2 - A_i A^i$
electric charge density	$\Box^2 \phi = -\rho_e/\epsilon_o$
electric current density	$\Box^2 A^i = -\mu_o J^i = -J^i/\epsilon_o$
energy density	$\Box^2 \phi^2 = -\rho_{\mathcal{E}}/\epsilon_o$
momentum density	$\Box^2 (\phi A^i) = -p^i/\epsilon_o$
rest energy density (mass density)	$\Box^2 \phi_o^2 = -\rho_m/\epsilon_o$
electric field	$\mathbf{E} = -\partial_t \mathbf{A} - \nabla\phi$
magnetic field	$\mathbf{B} = \nabla \times \mathbf{A}$
D'Alembertian	$\Box^2 = \partial_t \partial_t - \nabla^2$

Relationships between fields and densities

In the simplest sense, a field potential ϕ is associated with electric charges, and the flux of this potential is described by the vector A_i. A complicating factor in practical problems involving material media with charge and mass, however, is that usually the variables used to describe the dynamics are implicitly macroscopic averages. By analogy, the variables used in equations of fluid dynamics are often implicitly understood to be larger scale averages but pressure or viscous terms are included to reflect the parts of the unresolved small scale dynamics which have non-vanishing macroscopic averages.

We will also follow this approach in that the variables in (2) and (3) will often be implicitly treated as macroscopic averages but in some cases the unresolved scales must be considered.

For example, in many cases, ϕ can be viewed as being similar to the usual electric potential with a similar correspondence for the macroscopic variables. For an electrically neutral material, ϕ will be high near positive charges lower near negative charges and the macroscopic average of ϕ would be rather uniform (or zero, in the conventional description). But the macroscopic average of the field energy ϕ^2 is not simply the square of the macroscopic potential. The field energy, because it is the square of the potential, always includes energy from all scales: ϕ^2 *describes all* *of the energy present.* The curvature in ϕ^2 gives the total energy density as described in the table.

In the table, we have shown some definitions for rest energy quantities. From one standpoint, these definitions may be viewed as axioms which are necessary to satisfy the above requirement that ϕ^2 describe all of the energy. From another standpoint (which we prefer), ϕ_o^2 is viewed as a contribution to the correlation of A_iA^i due to unresolved subatomic scales. That is, our view is that if all scales were resolved, the energy ϕ^2 would always be equal to A_iA^i. Likewise, the magnitude of ϕ would be equal to the magnitude of A_i. To say then that 'rest energy is present' means then that we are implicitly working with macroscopic A_i which do not describe all of the potential flux, only parts which are organized on larger scales.

So what is A_i? In problems where there is either no rest mass (as in the case for radiation to be discussed) or when rest mass is present but the electrodynamic aspects dominate, A_i is much like the usual magnetic vector potential. In the case described above of a neutral material but now with some of the positive charges having a small relative velocity (constituting an electric current), A_i can be viewed as the macroscopic average of the flux of the Coulomb fields as they are advected with the moving charges.

We will consider a contrasting case now where there is no relative motion between the charges in the medium but the whole medium has a velocity. We must first consider though what a velocity means in our formulation because as can be seen, the quantities in the table can be used with the equations (2), (3) for calculations without the notion of a velocity entering. Also, since quantities such as momentum, energy, and **E**, **B** are usually what is measured rather than a velocity (fluxes are what is measured rather than velocities), calculations could be compared with experimental results without discussing velocities. Still, for the purposes of comparison and discussion we resolve to define a velocity in the following way:

Because A_i is a flux of ϕ, it seems reasonable to define a velocity as

$$u_i = A_i/\phi. \tag{4}$$

We can test the consistency of this assumption using items in the table. Using a rearranged version of the description of the field rest energy we have for the magnitude of the velocity

$$|u_i| = \frac{|A_i|}{|\phi|} = \frac{|A_i|}{(\phi_o^2 + A_i A^i)^{1/2}} \tag{5}$$

which has a maximum velocity of 1 (as mentioned, this is equal to light speed in our notation) and this only occurs when there is no field rest energy ϕ_o^2.

Now returning to our case of the neutral material moving with a uniform velocity which we will now call u_i, we can expect that the macroscopic average A_i is zero if ϕ is zero. The macroscopic momentum, $\phi A_i = \phi^2 u_i$, however, is not zero for the same reasons that ϕ^2 is not zero. In fact, because ϕ^2 is the total energy present, the momentum describes the advection of all energy—rest, kinetic, and electromagnetic.

Expanding further, the momentum density can also be written as

$$\phi A_i = \phi^2 u_i = (\phi_o^2 + A_i A^i) u_i = (\phi_o^2 + \phi^2 u^2) u_i \tag{6}$$

which in rearranging gives

$$\phi A_i = (1 - u^2)^{-1} \phi_o^2 u_i. \tag{7}$$

Taking the D'Alembertian $\square^2()$ and substituting with variables in the table, we have

$$p_i = \gamma^2 \rho_m u_i \tag{8}$$

where the Lorentz factor $\gamma = (1-u^2)^{-1/2}$. The right-hand side of (8) has a familiar form given by the relativistic mass density times velocity. (For relativistic mass *density*, a factor γ^2 appears rather than γ (which would be the coefficient relating relativistic mass to rest mass). Conventionally, the extra γ factor is regarded as due to the Lorentz contractions on the proper volume over which the density would be integrated [4] (see page 288).)

In a similar manner, a conventional form relating energy and mass can be obtained from

$$\phi^2 = \phi_o^2 + A_i A^i = \phi_o^2 + \phi^2 u^2 \tag{9}$$

which, after rearranging gives

$$\phi^2 = (1 - u^2)^{-1} \phi_o^2. \tag{10}$$

Taking $\Box^2()$ and replacing with variables in the table, we have

$$\rho_{\mathcal{E}} = \gamma^2 \rho_m \tag{11}$$

which relates energy to relativistic mass and in the usual notation is $E = mc^2$.

So far we have only shown that the relationships described (or defined) in the table appear to be consistent. The task in later sections will be to show that these relationships used in the governing equations (2, 3) produce familiar dynamical equations. To finish this subsection, we discuss some of the general aspects of (2, 3) .

Equation 2 represents conservation of field momentum while (3) represents conservation of field energy. Also, from (3) we see that the time rate of change of the field energy is due to spatial convergence of the field momentum. As will be made more clear in later sections, similar conservation principles applied to discrete 'particles' or material densities are obtained by applying the operator $\Box^2 = \partial_t\partial_t - \nabla^2$ to (2, 3), replacing with the appropriate densities in the table, and integrating over the volume of the particle. Note, however, that such an operation neglects the contributions of the rest of the field.

Equation 1 can be written as

$$A^\lambda \partial_\lambda A^\mu = -A^\mu \partial_\lambda A^\lambda \equiv -A^\mu S \tag{12}$$

where S $(= \partial_\lambda A^\lambda)$ can be viewed as a source term. If A^μ were viewed as the conventional 4-potential, we could impose the Lorentz gauge which is equivalent to setting S to zero (i.e. in appropriate units, the Lorentz gauge is $\partial_\lambda A^\lambda = S = 0$). Our viewpoint, however, is that A^μ is physically real and thus S must be chosen in accordance. Note also that the governing equations (2, 3) require conservation of energy and momentum but do not require conservation of the potential amplitude ϕ, hence any constraint imposed on S is additional. In most of this paper, we will make the simplest assumption $S = 0$ since a more sophisticated form for S does not appear to be required to gain agreement with simple conventional results. This assumption seems consistent with an assumption of conservative wave field dynamics with material charge and matter sources prescribed. We do not propose however, that this assumption is valid for more complicated cases, particularly those where the dynamics is dominated by interactions between the rest mass and electromagnetic fields.

2.1 Lagrangian

In this subsection we make comparisons based on Lagrangian density formulations. The material presented is not needed, however, for later sections and some readers may wish to skip this part.

A departure point for theories that treat both electromagnetic fields and matter is often to start with a Lagrangian density which is created by simply adding together energy terms describing Lagrangian densities from the various dynamical processes. In particular, the Maxwell's equations with sources are derivable from the Lagrangian

$$\mathcal{L}_1 = \mathcal{L}_p + \mathcal{L}_I + \mathcal{L}_m \tag{13}$$

where $\mathcal{L}_p = -\frac{1}{4}F_{\mu\lambda}F^{\mu\lambda}$ is an energy relating the electromagnetic field, and $\mathcal{L}_I = A_i J^i - \rho_e \phi$ is called an 'electromagnetic interaction energy' between the particles and field. The last term $\mathcal{L}_m = -\rho_m$ representing rest energy has sometimes been included since together with $\mathcal{L}_I$, the equations of motion for charged incoherent matter can be produced when the time track is varied[3]. (Note, however, that in the usual functional variation approach, this term is assumed constant and is often omitted since it would have no effect.)

It is useful to show here that the terms in the Lagrangian above can be produced from (1) using a physical argument. Recall that (1) describes the advection of potential flux A^μ (by the potential flux). For a bound universe, we suppose that the flux of A^j through the boundaries D of the universe is zero. Hence,

$$\oint_D \partial_\lambda (A^\mu A^\lambda) da^\mu = 0 \tag{14}$$

(the integrand is of course exactly zero by (1); the integration constant is also made zero by the boundary assumptions above.) A 4-D extension of Gauss' Theorem[3] can be used to write (14) as an integral over the 4-space approaching an infinite volume V and arbitrary time interval T :

$$\int_T \int_V \partial_\mu (\partial_\lambda (A^\mu A^\lambda)) dV dt = 0. \tag{15}$$

Assume first the case where there is only radiation (no mass), and we expect $S = 0$. We can differentiate the governing equation 1 with respect to μ (calling the result $-\mathcal{L}_2$) to get

$$\mathcal{L}_2 = -\partial_\mu \partial_\lambda (A^\mu A^\lambda) = -\partial_\mu A^\lambda \partial_\lambda A^\mu = 0. \tag{16}$$

It can be shown that a variation of the functional dependence of the integral (equation 15, using the integral $\mathcal{L}_2$) on A_μ, $\partial_\lambda A_\mu$, is also zero. Hence, the integrand $\mathcal{L}_2$ satisfies the variational condition for a Lagrangian density but is more restrictive since the integrand is also exactly zero.

Of course The Lagrangian is not unique and the major requirement is that it must also produce the correct dynamical equations when used in the Lagrangian equations.

With $F^{\mu\lambda} = \partial_\mu A^\lambda - \partial_\lambda A^\mu$, (16) can be written as

$$\begin{aligned} \mathcal{L}_2 &= -\partial_\mu \partial_\lambda (A^\mu A^\lambda) = 0 \\ &= \frac{1}{2} F^{\mu\lambda} F^{\mu\lambda} - A^\lambda \partial_\mu \partial_\mu A^\lambda + \frac{1}{2} \partial_\mu \partial_\mu (A^\lambda A^\lambda) \end{aligned} \tag{17}$$

which, noting the rules above for lowering and raising indices, multiplying through by ϵ_o and replacing with appropriate densities can be written as

$$\mathcal{L}_2 = -\frac{1}{2} F_{\mu\lambda} F^{\mu\lambda} - A_i J^i + \rho_e \phi + \frac{1}{2} \rho_m \tag{18}$$

Equation 18 shows a balance of energy between the electromagnetic field, the rest mass, and the electromagnetic interaction of mass with the field. Note that the terms in $\mathcal{L}_2$ are similar to those in $\mathcal{L}_1$ but

$$\mathcal{L}_2 = 2\mathcal{L}_p - \mathcal{L}_I - \frac{1}{2} \mathcal{L}_m \tag{19}$$

which can be compared with (13). Under the assumption that no mass is present, $\mathcal{L}_I = \mathcal{L}_m = 0$. Therefore, $\mathcal{L}_2$ is proportional to $\mathcal{L}_1$, and like $\mathcal{L}_1$ is also the correct Lagrangian density giving the source-free Maxwell's equations. The additional constraint that $\mathcal{L}_2$ is also exactly zero gives the appropriate equipartition of energy since , $\mathcal{L}_2 = -\frac{1}{2} F_{\mu\lambda} F^{\mu\lambda} = \epsilon_o (E^2 - B^2)$ where E and B are the magnitudes of the electric and magnetic fields, respectively.

Although with the assumption $S = 0$ above, we discounted formally including massive particles, we find nonetheless the correct form $\mathcal{L}_I$ for the density describing the electromagnetic interaction of the particles, as well as $\mathcal{L}_m$ describing the rest energy, suggesting that the assumption of a conservative ϕ field might also extend to cases involving rest mass. Note, however, that these terms do not appear in the conventional ratios. For any one term considered independently, the multiplier is unimportant since the Lagrangian is not unique. But the 'Lagrangian' of (19) (integral of the sum of the Lagrangian densities) can be converted to the conventional one of (13) by adding or multiplying by constants only if $\mathcal{L}_p$, $\mathcal{L}_I$, $\mathcal{L}_m$ have universal integrals which are independent of each other. This additional constraint that would be required to reproduce the conventional Lagrangian might be partially expected since the conventional form (13) does not generally accommodate the conversion of particle mass to electromagnetic energy or vice versa.

Another difference between the formulation here and the conventional one is that in treating the dynamics of fields and matter simultaneously, the mechanical energy of the particles have been automatically included. Conventionally, another Lagrangian density must be added in or the variational scheme must be elaborated to account for this. When we assume that A_μ is dominated by particle mass dynamics and electromagnetic fields are neglected, it is easy to show that the correct Lagrangian describing the mechanical process can be obtained from (1). A more simple and general demonstration of the adequacy of (1) in describing basic mechanics is given below in the discussion relating to the energy tensor of dust.

So far we have only shown that the governing equation (1) can be used to produce an energy balance with terms similar to those appearing in a conventional composite Lagrangian density describing electrodynamics with charged matter. In following sections we will show that (1) also gives the correct dynamical equations for some particular applications.

3 Radiation

Consider first the idealized case where no field rest mass is present. Then, $\phi_o^2 = 0$, the field energy $\phi^2 = A_i A_i$ and consequently $|A_i| = |\phi|$. As mentioned, in this and later sections we no longer need the raised index notation.

For this fundamental example, we will assume that the potential ϕ is an intrinsically positive quantity. In the conventional sense, this might be regarded as simply a choice for the reference value. In the tenor of this paper which regards ϕ as a physical quantity, however, this assumption is more a statement for the non-existence of 'anti-potential'. With this, we define a velocity $c_i = A_i/\phi$ which we see has unit (light velocity) magnitude and simply indicates the direction of positive potential flux.

Using now $A_i = \phi c_i$ in equations 2, 3 the first thing that is apparent in this case is that the two equations become redundant because (3) can be obtained by taking the product of (2) with c_i. This should not be surprising because in this case where the field momentum advects all of the energy present in the problem, the scalar momentum conservation equation (obtained by taking the product with c_i) is also an equation for the total energy conservation.

Now let us rewrite (2) as

$$\phi\partial_t A_i + A_j\partial_j A_i{=}0, \tag{20}$$

where we have assumed $S = \partial_t\phi + \partial_j A_j = 0$. Dividing (20) by ϕ we have the description that the material derivative of the potential flux moving with c_i is zero.

We can also rewrite (20) while using vector notation and utilizing some vector identities as

$$\partial_t\mathbf{A} - \frac{\mathbf{A}\times\nabla\times\mathbf{A}}{\phi} + \frac{1}{2\phi}\nabla A^2 = 0. \tag{21}$$

Because $|\mathbf{A}| = \phi$ in this example, and referring to the table, (21) also gives

$$\mathbf{E} = -\,\partial_t\mathbf{A} - \nabla\phi = -\frac{\mathbf{A}\times\mathbf{B}}{\phi} = -\mathbf{c}\times\mathbf{B}, \tag{22}$$

showing correctly both the relationship between the electric field and the potentials, and the relationship between $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{c}$ which is expected for a plane wave at least.

Now we take the divergence of (22) using again $S = \partial_t\phi + \nabla\cdot\mathbf{A} = 0$ and items from the table to give:

$$-\nabla\cdot\mathbf{E} = \Box^2\phi = -\frac{\rho_e}{\epsilon_o} = \nabla\cdot\left(\frac{\mathbf{A}\times\mathbf{B}}{\phi}\right) \tag{23}$$

which reproduces both the conventional relationship between the electric charge density and the electric field $\nabla\cdot\mathbf{E} = \frac{\rho_e}{\epsilon_o}$, and (when there is no charge density as expected for radiation) the correct wave equation $\Box^2\phi = 0$.

The additional equality relating the cross-product term in (23) can be viewed in two ways. If we view the charge density as imposed, then we have a condition for the way the fields must behave. With radiation, for example, the charge density is zero and therefore we must have

$$\nabla\cdot\left(\frac{\mathbf{A}\times\mathbf{B}}{\phi}\right) = \nabla\cdot(\mathbf{c}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{c} - \mathbf{c}\cdot\nabla\times\mathbf{B} = 0 \tag{24}$$

which would appear to be satisfied for typical radiation fields.

In the second approach, (23) suggests that a more complex field configuration can itself create an electric charge density

$$-\frac{\rho_e}{\epsilon_o} = \nabla \cdot (\frac{\mathbf{A} \times \mathbf{B}}{\phi}) \tag{25}$$

in a manner similar to the way electric (Schiff's) charge densities are created by non-zero vorticity of conducting fluid in a background magnetic field [6]. It is important to remember that these calculations assume $S = 0$ on which we have made reservations.

In general terms, when rest energy is discounted we should find consistency with the usual equations for electrodynamics. Of the four Maxwell equations, two are implicitly contained in the descriptions in the table, and a third is reproduced in (23). The fourth (Ampere's law) can also be reproduced by taking the curl of (22).

4 Mass dynamics

Now let us demonstrate the case where A_i is presumed to be dominated by the advection of the rest mass field. For this purpose we will make use of the concept of a velocity u_i as it was defined and discussed in the General formulation section. The case we treat could represent, for example, the electrically neutral moving medium also described in that section. Here, however, we no longer require that the velocities are uniform. Then we are considering a fluid material; rigid body or particle dynamics can then be obtained as a subset.

In this case the field momentum is $\phi A_i = \phi^2 u_i$ and (2,3) can be written as

$$\partial_t(\phi^2 u_i) + \partial_j(\phi^2 u_i u_j) = 0, \tag{26}$$

$$\partial_t(\phi^2) + \partial_j(\phi^2 u_j) = 0. \tag{27}$$

Applying the $\Box^2()$ operator to (26) and (27) while neglecting the curvature in the velocity fields, and substituting for densities described in the table, we obtain

$$\partial_t(\rho_{\mathcal{E}} u_i) + \partial_j(\rho_{\mathcal{E}} u_i u_j) = 0, \tag{28}$$

$$\partial_t(\rho_{\mathcal{E}}) + \partial_j(\rho_{\mathcal{E}} u_j) = 0. \tag{29}$$

Because the dominant form of energy is assumed to be due to the mass (and for simplicity assuming velocities much less than that of light) the total energy density $\rho_{\mathcal{E}}$ is equivalent to the rest mass density ρ_m and (28, 29) are simply the usual unforced momentum and mass conservation equations of fluid dynamics. A more typical form is obtained by using (29) in (28) to give

$$\rho_m D_t u_i = 0, \tag{30}$$

where $D_t = \partial_t + u_j \partial_j$ is the material derivative (the time rate of change following the material).

Typically, (28, 29) will include other terms such as pressure and viscous terms when the velocities appearing in the equations are implicitly averaged over the molecular (or greater) scale. A pressure gradient term $-\partial_i P$, for example, may be added to the right side of (28). This term arises because when averaging (28, 29) to obtain equations in terms of the larger-scale variables, the larger-scale average of $u_i u_j$ is typically $[u_i][u_j] + \delta_{ij} u_i^* u_j^*$ where [] denotes the large-scale variables, δ_{ij} is the delta function (with unit value when $i = j$ and zero otherwise), and the starred quantities are the unresolved small-scale velocities. Then (28, 29) may be used with implicit large-scale variables while keeping the same notation but a pressure term $\partial_j(\delta_{ij} \rho_m u_i^* u_j^*) = \partial_i P$ must be added. If the medium is isotropic, then the pressure term can be described as essentially the gradient of thermal energy: $\partial_i(\frac{1}{6}\rho_m (u^*)^2)$. The macroscopic momentum equation then gains a force tending to cause flow down the gradient of thermal energy. In a related way, viscous terms can be obtained. Hence, in comparing (28, 29) with other formulations for fluid dynamics it should be understood which terms result from simply averaging and are implicitly included in (28, 29).

The terms which don't appear in the momentum equation (28) and maybe should are those describing external forces, and gravity. We might get around the lack of external forces, certainly electromagnetic ones, by arguing that our initial assumption that A_i is described basically by the mass advection precluded having external forces. (In fact, in the next section we give an example which includes electromagnetic forces.) Such an argument for the lack of gravity is much weaker. We leave a proper formulation for gravity beyond the scope of this paper. We will discuss, however, how we expect these additional forces may appear:

We discussed in the General formulation section that we felt that the field rest energy ϕ_o^2 resulted from an unresolved correlation $A_i A_i$.

That is, when we describe a problem using rest energy then the potential flux A_i as well as other variables are implicitly averaged over the atomic scales. We can keep the equations with similar notation for the macro-atomic averages but then need, in analogy with the pressure effect described above, to include one or more terms on the right side of the momentum equation which describe the components of the atomic-scale dynamics which have non-vanishing larger-scale averages. If, as speculated, the field rest energy density ϕ_o^2 is in essence an unresolved correlation of $A_i A_i$, then in the simplest guess we might expect a term proportional to $-\nabla\phi_o^2$ to appear on the right side of the field momentum equation. The dynamical effect of this additional term would then be to create motion which reduces the field rest energy.

Whether such additional forces would act in the sense of gravity is not clear and certainly requires further study. We feel that including these speculations here is important, however, because if gravity cannot also be extracted from the theory proposed then the theory is largely inconsistent. This is because in general relativity it is the sum of the energy tensors of matter and electromagnetic fields that acts directly as the source of the gravitational field.

To conclude this section describing mass dynamics, we show that a conventional formulation for a mechanical energy tensor can be reproduced directly from the governing equation (1).

An elegant description of the dynamics of continua which includes internal stresses is given by the following energy tensor equation for a 'non-interacting dust':

$$\partial_\lambda T_{\mu\lambda} = \partial_\lambda(\rho_o U_\mu U_\lambda) = 0, \tag{31}$$

where ρ_o is the usual mass density and U_μ is the proper four-velocity [5]. This equation is easily reproduced directly from the governing equation (1): For this case involving a non-interacting dust, we expect $A_\mu = \langle \phi u_i, i\phi \rangle$ which can also be written (using the velocity u_i and Lorentz factor γ as described in the General formulation section) as

$$A_\mu = \langle \phi u_i, i\phi \rangle = \phi_o \langle \gamma u_i, i\gamma \rangle = \phi_o U_\mu. \tag{32}$$

We take $\Box^2(1)$ (neglecting curvature in the velocity field) while using (32) to replace for A_μ. Replacing for the mass density as described in the table, (31) is obtained.

5 Particle in an electromagnetic field

In this section we will attempt to consider the simplest case where both electromagnetic and mass effects are important. We test the theory for a simple case where the electromagnetic aspects of A_μ are prescribed over a large scale and are assumed to be unaffected by the presense of a small material particle, although the mass dynamics of the particle depend on the dominant electromagnetic fields.

First, the field momentum equation (2) can be written using vector notation together with some vector identities as

$$\partial_t(\phi\mathbf{A}) - \mathbf{A}\times\nabla\times\mathbf{A}+\frac{1}{2}\nabla A^2 + \mathbf{A}(\nabla\cdot\mathbf{A}) = 0. \tag{33}$$

Now consider a small particle having momentum $m\mathbf{u}$, non-relativistic velocity $\mathbf{u}$ and charge q in a steady and uniform imposed background electromagnetic field $\mathbf{E}_b = -\nabla\phi_b \approx -\nabla\phi$, $\mathbf{B}_b = \nabla\times\mathbf{A}_b \approx \nabla\times\mathbf{A}$.

In consistency with the cases described in the Radiation section, we take $S = \partial_t\phi - \nabla\cdot\mathbf{A} = 0$ and the potential ϕ is viewed as positive. In this case, we will assume $\nabla\cdot\mathbf{A} = \nabla\cdot(\mathbf{A}_b+\mathbf{A}_p) = 0$, where $\mathbf{A}_p = \phi_p\mathbf{u}$ is the potential flux due to the advection of the potential of the particle. Also, consistent with work in the Radiation section, we will assume $A^2 \approx \phi^2$.

Using these approximations, (33) can be written as

$$\partial_t(\phi\mathbf{A}) - \mathbf{A}_p\times\mathbf{B}_b - \mathbf{A}_b\times\mathbf{B}_b - \phi\mathbf{E}_b = 0. \tag{34}$$

We should note that a bit of a puzzle occurs. If we consider the steady-state of (34) without the particle ($\mathbf{A}_p = 0$) we see that we cannot choose the background fields $\mathbf{E}$ and $\mathbf{B}$ arbitrarily but rather that they must be related in the same way as appeared in the radiation section. Conventionally, however, we are used to being able to consider background fields where only $\mathbf{E}$ or $\mathbf{B}$ is present, for example. What is the source of this discrepancy?

To limit the scope of this study, we have not explored this completely but note that we view this problem as similar to the question, "why aren't the relationships between $\mathbf{E}$ and $\mathbf{B}$ the same for static fields as they are for radiation?". One possibility is that they are at an appropriately small scale. This is a very important point and should be studied in conjunction with the structure of A_μ at the atomic level. To confine our attention to the problem in this section, however, we will assume for simplicity that the background fields are chosen to satisfy the above constraints.

We take $\square^2(34)$, and replace with appropriate densities from the table to obtain

$$\partial_t \mathbf{p} - \rho_e \mathbf{u} \times \mathbf{B}_b - \rho_e \mathbf{E}_b = 0. \tag{35}$$

Integrating over the particle volume, we obtain

$$\partial_t (m\mathbf{u}) - q\mathbf{u} \times \mathbf{B}_b - q\mathbf{E}_b = 0 \tag{36}$$

which agrees with the conventional expression for the equation of motion of a particle in a background electric and magnetic field.

6 Discussion

In this paper, we have proposed that dynamics of interacting charge and matter can be described by a governing equation for the 'potential flux' 4-vector A_μ. If this idea is to be validated, it will have to be tested against a wide range of examples because it purports to explain a wide range of phenomena. In this paper, we have made only the most basic tests.

Specifically, in one case where we assumed mass is either not present or dynamically unimportant, the governing equation (2) together with relationships given in the table were used to produce Maxwell's equations.

In the contrasting limit where the electrodynamic aspects become unimportant (as can happen when averaging over neutral material media) the correct description of inertia is obtained (which, in the examples given, for the case of a fluid is described by the unforced momentum equation and for the case of a non-interacting dust is given by an energy tensor).

We also examined perhaps the simplest case where both electromagnetic and inertial effects are important. We considered a moving charged particle in a background electric and magnetic field and successfully (subject to an important caveat mentioned) reproduced the conventional forced equation of motion for the particle. In the terminology of the theory presented, this would be better stated as the following: We considered a potential flux A_μ, the magnitude and uniform gradients of which were dominated by prescribed background values. The curvature in the fields was, however, dominated by the presence of the particle because the background field gradients were uniform. In applying the D'Alembertian curvature operator to the governing equation for the A_μ flux, we were then able to find the equation describing the particle motion subject to the 'forces' due to the background components of A_μ. The particle and background field are, though, simply aspects of A_μ.

To discuss now some of the short-comings in validating the proposed theory, we first note that even to obtain the simple validations given, we have had to make various approximations which while they seem reasonable to us might be further scrutinized. Probably, the biggest two short-comings are that there remain questions about the interpretation of field rest mass, and that gravity has not been included. We discuss these two points further:

The fourth component of A_μ is unequivocally related to the square root of the total field energy for all cases. In the case of radiation, the space components of A_μ are the flux of this root total field energy and we have $|A_i| = |A_4|$. More generally though, when field rest energy is present $|A_i| \neq |A_4|$. Our preferred view is that rest energy arises due to a correlation of the A_i field at small unresolved scales. That is, when we say, 'rest energy is present,' we mean that A_i is implicitly an averaged macroscopic variable. If all scales were resolved then the equality for radiation would always hold and the use of rest energy would not be needed. We have not, however, validated this idea.

The open question of whether rest energy is distinct is not just a philosophical question but can effect the way calculations are performed. If rest energy is distinct then it only contributes to the fourth component of A_μ. If, it arises from averaging as speculated then it also effects our expectations for certain correlations in A_i products when considering larger-scale averages. Such correlations were considered in speculations concerning how gravity might appear from the theory proposed.

It is not simply an added convenience if gravity can be derived from the proposed theory but rather that the theory would be largely inconsistent without this. Matter and fields interact electromagnetically and traditionally it is known that an energy tensor describing the electromagnetic field must be included to conserve energy and momentum. In the formulation proposed here, this interaction is more direct since fields and particles are treated simultaneously as aspects of one 'thing', A_μ.) (In general relativity it is the sum of the energy tensors of matter and electromagnetic fields that acts directly as the source of the gravitational field. Hence, it would be inconsistent if the gravitational effects were not also contained in the formulation of A_μ.

Clearly, work that is much needed is to test the formulation presented for subatomic scales. As a preliminary, we point out some potential sources of confusion.

It should be kept in mind that the theory we have presented is most naturally expressed and manipulated in terms of the field quantities. The curvature (obtained by applying the D'Alembertian) of field quantities produces another set of quantities (see the table) which are described as 'densities' in this paper. Hence, there is both a field energy and an an energy density. This distinction does not necessarily appear in conventional work and making comparisons can become confusing. For example, conventionally both the squared electric field and squares or products of the electric potential may be called field energy (or energy density) even though one involves derivatives of the other.

Consider some further effects of the distinctions between fields and densities as we have defined them: The field energy ϕ^2 is always positive (we also prefer to view the potential ϕ as positive). The energy density however is $\rho_\mathcal{E} = -\epsilon_o \Box^2\phi^2$ and can have either sign. If we consider applying this definition to a charged point particle with potential $\phi \propto 1/r$ we find that while the energy density at the point (whatever that means) is likely positive reflecting the mass of the particle, outside the point in the $1/r$ field the energy density is everywhere negative. While the concept of negative energy is not new (e.g.[2]) it should be remembered when making comparisons that in the theory presented here only the energy density can be negative; the field energy is always positive. Also, care must be used in determining whether conventional results describing 'energy' correspond here to field energy or energy density.

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