Differential Geometry and Dynamics of a Lightlike Point in Lorentzian Spacetime

Heinz Krüger

Fachbereich Physik der Universität, Erwin Schrödinger Straße, Postfach 3049, 67653 Kaiserslautern, Germany
e-mail: krueger@physik.uni-kl.de

ABSTRACT. Differential geometry for isotropic, i.e., lightlike curves in Lorentzian spacetime $\mathbb{R}^{1,3}$ is developped. With respect to orbits on the group Spin(1,3), two nonequivalent comoving Frenet frames are obtained implying two independent sets of Frenet–Darboux equations. The general solution of the Frenet–Darboux equations is derived by means of a quaternion decomposition of Spin(1,3).

Based on these results, a universal form of Lagrangian dynamics for an isotropic spacetime point is constructed with the help of nonholonomic constraints. Noether's theorem is derived in order to express the equations of motion in terms of Lie derivatives with respect to the Poincaré group. This leads to an identification of momentum and angular momentum (spin). A new Lagrangian is postulated for a non–minimal coupling of an isotropic pointcharge to external fields. It implies a modification of Weyssenhoff's spin bivector and a generalization of his and Raabe's equations of motion.

RÉSUMÉ. *Avec l'intention de traiter l'électron comme un point chargé dôté de la vitesse de la lumière, on développe une théorie des courbes isotropes en espace–temps* $\mathbb{R}^{1,3}$. *On démontre que le pseudo–arc découvert par M. E. Vessiot [1] pour des courbes isotropes en trois dimensions complexes, sert tout aussi bien en* $\mathbb{R}^{1,3}$ *à déterminer les invariants différentiels relatifs au groupe de Lorentz. On obtient deux repères indépendants associés chacun à une composante connexe du groupe O(1,3) des rotations et non plus seulement à la représentation Spin(1,3) de la composante neutre* $SO^{+}(1,3)$. *Ils impliquent deux systèmes de Frenet généralisés au* $\mathbb{R}^{1,3}$ *et donc deux bivecteurs qui engendrent le mouvement des deux repères. Une décomposition du groupe Spin(1,3) en quaternions mène aux solutions générales des équations de Frenet.*

A partir de ces résultats une forme universelle d'un lagrangien avec des liaisons non holonomes est proposée pour la détermination des trajectoires isotropes d'une charge ponctuelle interagissant avec des champs extérieurs. Dans le but d'identifier les expressions pour l'impulsion–énergie linéaire et angulaire, le domaine d'application d'un théorème dû à Madame Noether est étendu aux liaisons non holonomes. On obtient des équations de mouvement qui généralisent celles de M. Weyssenhoff et de M. Raabe [7] par l'inclusion d'un couplage non minimal au champ électromagnétique.

1. Introduction

Differential geometry of complex isotropic curves has a long tradition [1], [2]. Real–valued isotropic curves in Lorentzian spacetime $\mathbb{R}^{1,3}$ however have been discussed quite rarely [3], [4]. The purpose of the first part of this article therefore is to give a selfcontained and complete treatment of real isotropic curves in $\mathbb{R}^{1,3}$ with no recourse to complex analysis. Real multivector calculus [5] will be made use of throughout.

Differential geometry of curves in $\mathbb{R}^n$ is so widespread [6] because the euclidean structure of $\mathbb{R}^n$ admits an obvious choice of a natural invariant curve parameter, namely, the arclength. Just this quantity vanishes for isotropic curves.

Section 2 of this article starts with the simple demonstration, that the natural parameter discovered by Ernest Vessiot [1] for complex curves also applies to non–straight real isotropic curves in $\mathbb{R}^{1,3}$ after a slight modification. Subsequently, higher order derivatives of the position vector in $\mathbb{R}^{1,3}$ with respect to the Vessiot parameter are formed. Inner products of these derivative vectors define two basic differential invariants which, supplemented by the Vessiot parameter, allow to express all higher order invariants in terms of linear combinations of their derivatives. One finds that in $\mathbb{R}^{1,3}$, isotropic curves of only double and triple curvature exist. This means, that the first four derivatives of the position vector with respect to the Vessiot parameter either span a 3–space $\mathbb{R}^{1,2}$ in case of double curvature, or, the full 4–space $\mathbb{R}^{1,3}$ in case of triple curvature.

In section 3, two inequivalent orthogonal Frenet frames are derived comoving on Spin(1,3) any non-straight isotropic curve in $\mathbb{R}^{1,3}$.

Nonequivalence means that **there is no unimodular spinor** on spacetime algebra which maps one frame to the other. By means of a particular Lortentz transform the two frames are reduced to normal form. Two different sets of Frenet–Darboux equations are obtained whose properties compactly are summarized in terms of two Darboux bivectors.

A quaternion decomposition of Spin(1,3) particularly adapted to the structure of the Darboux bivectors then, in section 4, leads to the general solution of the Frenet–Darboux equations. In this way, the shape of an isotropic curve in $\mathbb{R}^{1,3}$ is established in general. References [3] and [4] are restricted to particular cases.

The differential geometric properties derived thus far provide the foundation on which in section 6 a universal Lagrangian is constructed for the motion of an isotropic point in $\mathbb{R}^{1,3}$ under the influence of external fields. Noether's theorem is derived in section 7 for this universal Lagrangian involving nonholonomic constraints. Applied to the Poincaré group of spacetime translations and Lorentz transforms, Noether's theorem leads to a partition of the equations of motion into momentum– and angular momentum laws. In section 8, a particular Lagrangian is postulated for a non–minimal coupling of an isotropic pointcharge with an external field. This Lagrangian leads to equations of motion which I found three years ago via a quite laborious variation of an unimodular spinor. If the bivector part in the interaction Lagrangian is omitted (minimal coupling), the equations of motion fall back on those postulated by Jan Weyssenhoff and A. Raabe [7].

2. The natural curve parameter and differential invariants

Let the position vector of a representative curve point in $\mathbb{R}^{1,3}$ be $\lambda\!\!\!^{-} z$, **where $\lambda\!\!\!^{-}$ is a fundamental length** and the dimensionless vector $z = (z_0 + \vec{z})\gamma_0$. Now, if $\alpha \in \mathbb{R}$ is an arbitrary parameter, a curve $z = z(\alpha)$ is called **isotropic = lightlike** if the tangent vector $\dfrac{dz}{d\alpha}$ satisfies the isotropy condition

$$\left(\frac{dz}{d\alpha}\right)^2 = 0 \quad . \tag{2.1}$$

Writing $\dfrac{dz}{d\alpha} = n = (n_0 + \vec{n})\gamma_0\,,\quad n^2 = 0$ implies $n = (\pm|\vec{n}| + \vec{n})\gamma_0 = \dfrac{dz}{d\alpha}$. So,

$$\frac{d^2z}{d\alpha^2} = \left(\pm\frac{\vec{n}}{|\vec{n}|}\cdot\frac{d\vec{n}}{d\alpha} + \frac{d\vec{n}}{d\alpha}\right)\gamma_0 \quad ,$$

and hence

$$\left(\frac{d^2z}{d\alpha^2}\right)^2 = -\left(\frac{\vec{n}}{|\vec{n}|}\times\frac{d\vec{n}}{d\alpha}\right)^2 \le 0 \quad . \tag{2.2}$$

The conclusion therefore is: Except for straight isotropic lines $\dfrac{d\vec{n}}{d\alpha} = \mu\vec{n}$, $\mu \in \mathbb{R}$, the vector $\dfrac{d^2z}{d\alpha^2}$ always is spacelike, i.e.,

$$\left(\frac{d^2z}{d\alpha^2}\right)^2 < 0 \quad . \tag{2.3}$$

Thus, except for straight isotropic lines, the quantity $-\left(\dfrac{d^2z}{d\alpha^2}\right)^2$ always is positive. It is therefore ideally suited to provide a substitute for the squared tangent vector or arclength which vanishes according to (2.1). Following Ernest Vessiot [1], I define the natural invariant curve parameter β by

$$\left(\frac{d^2z(\beta)}{d\beta^2}\right)^2 = -1 \quad . \tag{2.4}$$

It is straightforward to express derivatives with respect to the **Vessiot parameter** β in terms of derivatives with respect to an arbitrary parameter α. Equation (2.1) implies $\dfrac{dz}{d\alpha}\cdot\dfrac{d^2z}{d\alpha^2} = 0$ and with

$$\frac{dz}{d\beta} = \frac{d\alpha}{d\beta}\frac{dz}{d\alpha}, \quad \frac{d^2z}{d\beta^2} = \frac{d^2\alpha}{d\beta^2}\frac{dz}{d\alpha} + \left(\frac{d\alpha}{d\beta}\right)^2\frac{d^2z}{d\alpha^2}$$

equations (2.1) and (2.4) lead to

$$\left(\frac{d\beta}{d\alpha}\right)^4 = -\left(\frac{d^2z}{d\alpha^2}\right)^2 . \tag{2.5}$$

The invariance of β may better be displayed by means of (vectorvalued) first and second differentials of z, i.e., $dz = d\alpha \dfrac{dz}{d\alpha}\,, \quad d^2z = d\alpha^2 \dfrac{d^2z}{d\alpha^2}\,,$ viz.,

$$(dz)^2 = 0\,, \quad (d\beta)^4 = -(d^2z)^2 \quad . \tag{2.6}$$

The advantage of employing β as a parameter for isotropic curves in $\mathbb{R}^{1,3}$ is the same as employing the propertime or arclength in the case of a timelike particle: **All quantities formed of derivatives of the vector z with respect to β then are connected with an isotropic curve in a Lorentz– and parameter invariant manner.**

Higher derivatives soon get clumsy in the traditional notation of quotients and primes. Let me therefore meet the convention

$$z_1 = \frac{dz(\beta)}{d\beta} = z'\,, \quad z_{j+1} = z_j' = \frac{dz_j}{d\beta}\,, \quad j \geq 1 \quad . \tag{2.7}$$

With this compact notation, equations (2.1) and (2.4) become

$$z_1^2 = 0, \quad z_2^2 = -1 \quad . \tag{2.8}$$

Taking successively higher order derivatives of this basic set of equations like $z_1 \cdot z_2 = 0$ and $z_2 \cdot z_2 + z_1 \cdot z_3 = 0$, which implies $z_1 \cdot z_3 = 1$, one finds the following

Table of scalar differential invariants

order	invariants
2	$z_1^2 = 0$
3	$z_1 \cdot z_2 = 0$
4	$z_2^2 = -1, \quad z_1 \cdot z_3 = 1$
5	$z_2 \cdot z_3 = 0 \quad z_1 \cdot z_4 = 0$
6	$\boxed{z_3^2 = \sigma}\,, \quad z_2 \cdot z_4 = -\sigma, \quad z_1 \cdot z_5 = \sigma$
7	$2z_3 \cdot z_4 = \sigma' = \frac{d\sigma}{d\beta}, \quad z_2 \cdot z_5 = -3z_3 \cdot z_4 = -\frac{3}{2}\sigma', \quad z_1 \cdot z_6 = \frac{5}{2}\sigma'$

Inspection of this Table shows that

$$\sigma = z_3^2 = -z_2 \cdot z_4 \tag{2.9}$$

may be selected as the next higher order invariant beyond the Vessiot parameter β. Also, one notes that

$$z_2 \cdot (z_1 \wedge z_3) = (z_2 \cdot z_1) z_3 - z_1 (z_2 \cdot z_3) = 0 \tag{2.10}$$

and hence

$$\begin{aligned} T &= z_1 \wedge z_2 \wedge z_3 = -z_2 \wedge z_1 \wedge z_3 \\ &= -z_2 (z_1 \wedge z_3) = -(z_1 \wedge z_3) z_2 \quad . \end{aligned} \tag{2.11}$$

The square of the three–vector T therefore is

$$\begin{aligned} T^2 &= z_2 (z_1 \wedge z_3) z_2 (z_1 \wedge z_3) = -(z_1 \wedge z_3)^2 \\ &= z_1^2 z_3^2 - (z_1 \cdot z_3)^2 = -1 = T^2 . \end{aligned} \tag{2.12}$$

This means, that the vectors z_1, z_2 and z_3 always span a 3–dimensional subspace of $\mathbb{R}^{1,3}$. Isotropic curves in $\mathbb{R}^{1,3}$ therefore are either straight or of double curvature at least. In the following section it will be shown, that the space of vectors v generated by the three–vector T according to $v \wedge T = 0$ is a Lorentzian space $\mathbb{R}^{1,2}$.

3. Frenet frames and their motion on Spin(1,3)

A Frenet frame comoving with an isotropic curve in $\mathbb{R}^{1,3}$ is a tetrad of orthonormalized vectors e_μ,

$$e_\mu \cdot e_\nu = \gamma_\mu \cdot \gamma_\nu \, , \quad \mu, \nu = 0, 1, 2, 3 \quad , \tag{3.1}$$

which factorize the outer product T of the first three derivatives of z into a Clifford product according to

$$T = z_1 \wedge z_2 \wedge z_3 = e_0 e_1 e_2 \quad , \tag{3.2}$$

and contains the dual of T

$$e_3 = iT, \quad i = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \quad . \tag{3.3}$$

From (2.11) one notes, that the three–vector T partially already is factorized and because of (2.8) one may define

$$e_2 = z_2 \quad . \tag{3.4}$$

So, the task of factorizing T is reduced to orthonormalize in the plane of the bivector $z_1 \wedge z_3$, which according to (2.12) is timelike. The most general way to orthonormalize z_1 and z_3 is to put as a first step

$$\varrho g_0 = \lambda z_1 + z_3 \,, \quad \nu g_1 = \mu z_1 - z_3 \quad , \tag{3.5}$$

and to determine the four scalars $\lambda, \mu, \nu, \varrho$ through the requirements

$$g_0 \cdot g_1 = 0 \,, \quad g_0^2 = 1 = -g_1^2 \quad . \tag{3.6}$$

In this way one finds $\mu = \lambda + \sigma$ and $\nu^2 = 2\lambda + \sigma = \varrho^2$. With no loss of generality one may put $\nu = \varrho$, where $\varrho^2 > 0$. Equations (3.5) then become

$$\begin{aligned} \varrho g_0 = \lambda z_1 + z_3 \,, \quad \varrho g_1 &= (\lambda + \sigma) z_1 - z_3 \,, \\ \varrho^2 &= 2\lambda + \sigma > 0 \quad . \end{aligned} \tag{3.7}$$

It is exceedingly important to recognize that in the plane spanned by $z_1 \wedge z_3$, a Lorentz transform may be performed from g_0, g_1 to an orthonormal basis e_0, e_1 such that the parameter λ disappears and only the sign of ϱ remains,

$$\begin{aligned} e_\mu &= e^{-\frac{\varphi}{2} z_1 \wedge z_3} g_\mu e^{\frac{\varphi}{2} z_1 \wedge z_3} = g_\mu e^{\varphi z_1 \wedge z_3} , \\ e^\varphi &= \eta = \sqrt{\varrho^2} > 0 \quad . \end{aligned} \tag{3.8}$$

For $\mu = 0, 1$, equation (3.7) and $(z_1 \wedge z_3)^2 = 1$ (see (2.12)) imply

$$\begin{aligned} g_\mu \quad e^{\varphi z_1 \wedge z_3} &= g_\mu \mathrm{ch}\varphi + g_\mu \cdot (z_1 \wedge z_3) \mathrm{sh}\varphi \\ &= \frac{1}{2} \left[g_\mu \left(\eta + \frac{1}{\eta} \right) + \left(\eta - \frac{1}{\eta} \right) g_\mu \cdot (z_1 \wedge z_3) \right] . \end{aligned}$$

After a little algebra one finds with

$$\frac{\varrho}{\eta} = \pm 1 = \varepsilon \tag{3.9}$$

<u>two</u> orthonormal frames

$$2\varepsilon e_0 = z_1(1 - \sigma) + 2z_3 \,, \quad 2\varepsilon e_1 = z_1(1 + \sigma) - 2z_3 \quad , \tag{3.10}$$

which can **not** mapped onto each other by means of an **unimodular** spinor R,

$$R\tilde{R} = 1 \,, \quad iR = Ri \quad . \tag{3.11}$$

A change of sign, i.e., $\varepsilon \rightarrow -\varepsilon$, however can be effected within the Clifford group $\Gamma(1,3)$, [8], with $\Gamma = z_1 \wedge z_3 = \Gamma^{-1}$, viz.,

$$-e_\mu = \Gamma e_\mu \Gamma\,, \quad \mu = 0,1 \quad . \tag{3.12}$$

The construction of Frenet frames is accomplished when the vector e_3 in (3.3) also is expressed in terms of the derivatives of z. To that end I define the further differential invariant χ beyond the Vessiot parameter β and $\sigma = z_3^2$ as the dual of the following vector of grade four:

$$z_1 \wedge z_2 \wedge z_3 \wedge z_4 = \chi i = T \wedge z_4 = -i(e_3 \cdot z_4), \tag{3.13}$$

or,

$$\chi = -e_3 \cdot z_4 \quad . \tag{3.14}$$

From $e_\mu \cdot e_3 = i(T \wedge e_\mu)$ and equations (3.10) and (3.4) one notes

$$e_\mu \cdot e_3 = 0, \quad \mu = 0,1,2 \quad , \tag{3.15}$$

and from (2.12), (3.3)

$$e_3^2 = -1 \quad . \tag{3.16}$$

The set $\{e_\mu\}, \mu = 0,1,2,3$ therefore is a complete orthonormal frame in $\mathbb{R}^{1,3}$ in which z_4 may be decomposed because of (3.14) in the form $z_4 = e_0(e_0 \cdot z_4) - e_1(e_1 \cdot z_4) - e_2(e_2 \cdot z_4) + e_3\chi$. The inner products $e_\mu \cdot z_4$ easily may be expressed in terms of σ and its derivative with the help of (3.10), (3.4) and the Table of invariants. After elimination of e_0, e_1, e_2 in favour of z_1, z_2 and z_3, the desired expression for $e_3\chi$ in terms of derivatives of z results. Let me summarize the final formulas for the two Frenet frames:

$$\begin{aligned} \varepsilon e_0 &= z_1 \frac{1-\sigma}{2} + z_3, \quad \varepsilon e_1 = z_1 \frac{1+\sigma}{2} - z_3, \\ \varepsilon &= \pm 1, \quad e_2 = z_2, \end{aligned} \tag{3.17}$$

$$\begin{aligned} \chi e_3 &= z_4 - \sigma z_2 - \frac{\sigma'}{2} z_1, \quad \sigma = z_3^2, \\ i\chi &= z_1 \wedge z_2 \wedge z_3 \wedge z_4, \end{aligned} \tag{3.18}$$

$$e_\mu = R\gamma_\mu \tilde{R}, \quad R\tilde{R} = 1, \quad iR = Ri, \tag{3.19}$$

$$\begin{aligned} \varepsilon z_1 &= e_0 + e_1, \quad z_2 = e_2, \\ 2\varepsilon z_3 &= e_0(\sigma+1) + e_1(\sigma-1) \end{aligned} \tag{3.20}$$

$$z_4 = \varepsilon(e_0 + e_1)\frac{\sigma'}{2} + e_2\sigma + e_3\chi \quad . \tag{3.21}$$

From these formulas it is straightforward to derive equations for the derivatives e'_μ for $\mu = 0, 1, 2$, which generalize the classical Frenet equations from euclidean $\mathbb{R}^3$ to isotropic curves in the non-euclidean $\mathbb{R}^{1,3}$. It is sufficient to illustrate this derivation in detail for e'_0 only. Equation (3.17) leads to $\varepsilon e'_0 = z_2\dfrac{1-\sigma}{2} - z_1\dfrac{\sigma'}{2} + z_4$ and elimination of the derivatives of z with the help of (3.20), (3.21) yields the generalized Frenet formula

$$\varepsilon e'_0 = e_2\frac{1+\sigma}{2} + e_3\chi \quad . \tag{3.22}$$

In the same way one finds generalized Frenet equations for e'_1 and e'_2. A calculation of e'_3 starting with equation (3.18) however would turn out to be quite awkward because $z'_4 = z_5$ had to be decomposed in the Frenet frame. It is much simpler to start from the definition of e_3, equation (3.3), and to exploit (3.20), (3.21) afterwards, viz.,

$$\begin{aligned} e'_3 &= i(z_1 \wedge z_2 \wedge z_3)' = i\,(z_1 \wedge z_2 \wedge z_4) \\ &= i\varepsilon\chi(e_0 + e_1)e_2e_3 = \varepsilon\chi(e_0 + e_1) \quad . \end{aligned} \tag{3.23}$$

Two independent sets of generalized Frenet equations result which may be combined to one system with $\varepsilon = \pm 1$,

$$\begin{aligned} \varepsilon e'_0 &= e_2\frac{1+\sigma}{2} + e_3\chi \\ \varepsilon e'_1 &= e_2\frac{1-\sigma}{2} - e_3\chi \\ \varepsilon e'_2 &= e_0\frac{\sigma+1}{2} + e_1\frac{\sigma-1}{2} \\ \varepsilon e'_3 &= e_0\chi + e_1\chi \quad . \end{aligned} \tag{3.24}$$

Equation (3.19) relates the comoving Frenet frame e_μ to the standard frame γ_μ which is fixed in $\mathbb{R}^{1,3}$, i.e., $\dfrac{d\gamma_\mu}{d\beta} = \gamma'_\mu = 0$ by means of the variable unimodular spinor $R = R(\beta)$. The derivative of the condition of unimodularity

$$R\tilde{R} = 1 (= \tilde{R}R) \quad , \tag{3.25}$$

$$R'\tilde{R} + R\tilde{R}' = 0 = R'\tilde{R} + (R'\tilde{R})^{\sim} \tag{3.26}$$

implies that

$$\Omega = 2R'\tilde{R} = -\tilde{\Omega} \tag{3.27}$$

is a bivector, called **Darboux bivector**, whence (3.19) leads to the Frenet equations

$$e'_\mu = \frac{de_\mu}{d\beta} = \frac{1}{2}(\Omega e_\mu - e_\mu \Omega) = \Omega \cdot e_\mu \quad . \tag{3.28}$$

That conversely a given Frenet system like (3.24) **uniquely** determines a Darboux bivector Ω rests on the theorem, that for every bivector Ω

$$\sum_\mu (\Omega \cdot g_\mu) \wedge g^\mu = 2\Omega \quad , \tag{3.29}$$

where $\{g_\mu\}$ is an **arbitrary** (non–orthogonal) basis in $\mathbb{R}^{1,3}$ and $\{g^\mu\}$ the corresponding reciprocal basis defined by

$$g^\mu \cdot g_\nu = \begin{cases} 1 & \text{for } \mu = \nu \\ 0 & \text{for } \mu \neq \nu \end{cases} \quad . \tag{3.30}$$

Choosing in (3.29) the Frenet frame e_μ and making use of (3.28), the two Darboux bivectors

$$2\Omega = \sum_{\mu=0}^{3} e'_\mu \wedge e^\mu = e'_0 \wedge e_0 + \sum_{k=1}^{3} e_k \wedge e'_k \tag{3.31}$$

determined by the two sets of Frenet equations become

$$\begin{aligned} \varepsilon\Omega &= \left(e_2 \frac{1+\sigma}{2} + e_3\chi\right) e_0 + \left(e_3\chi + e_2 \frac{\sigma - 1}{2}\right) e_1, \\ \varepsilon &= \pm 1 \quad . \end{aligned} \tag{3.32}$$

In order to integrate the spinor equation (3.27)

$$2R' = \Omega R \quad , \tag{3.33}$$

the transformed Darboux bivector

$$\omega = \tilde{R}\Omega R \tag{3.34}$$

is more convenient because according to $e_\mu = R\gamma_\mu \tilde{R}$ and (3.32) it is of the form

$$\varepsilon\omega = \left(\gamma_2 \frac{1+\sigma}{2} + \gamma_3\chi\right)\gamma_0 + \left(\gamma_3\chi + \gamma_2\frac{\sigma-1}{2}\right)\gamma_1 \tag{3.35}$$

$$\varepsilon\omega = \vec{\sigma}_2\frac{1+\sigma}{2} + \vec{\sigma}_3\chi - i\left(\vec{\sigma}_2\chi + \vec{\sigma}_3\frac{1-\sigma}{2}\right), \tag{3.36}$$

involving the non-moving standard basis γ_μ of $\mathbb{R}^{1,3}$. Inserting (3.34) into (3.33), one notes that ω determines R according to

$$2R' = R\omega \quad . \tag{3.37}$$

The square of ω qualitatively indicates the geometric shape of the motion generated by R according to (3.19). From (3.34) and (3.35) one obtains

$$\omega^2 = \Omega^2 = \sigma - 2i\chi \quad . \tag{3.38}$$

For $\chi = 0$ the bivectors Ω and ω are **simple**, i.e., they are an outer product of two vectors. In fact, by means of equations (3.17) and (3.18) the vectors e_μ may be eliminated from (3.22) for $\mu = 0, 1, 2$, with the result

$$\Omega = z_2 \wedge z_3 + \chi e_3 \wedge z_1 = R\omega\tilde{R} \quad . \tag{3.39}$$

The vanishing of $\chi = -i(z_1 \wedge z_2 \wedge z_3 \wedge z_4)$, cf. (3.13), means that according to (3.18), z_4 is a linear combination of the vectors z_1 and z_2, whence the isotropic line is of **double curvature** only. On the other hand, provided that $\chi \neq 0$, the first four derivatives of z span the full $\mathbb{R}^{1,3}$ and the curvature is triple. The conclusion therefore is: **Isotropic lines in $\mathbb{R}^{1,3}$ either are straight, of double curvature (class 2), or, of triple curvature (class 3).**

4. General solution of Frenet–Darboux equations

Equations (3.19), (3.35) and (3.36) completely enclose the differential geometry of isotropic curves in $\mathbb{R}^{1,3}$. The purpose of this section is to construct the general solution of (3.37). The Darboux bivector (3.35) does not depend on $\gamma_1\gamma_0 = \vec{\sigma}_1$. I exploit this independence by defining the projectors

$$P_\pm = \frac{1}{2}(1 \pm \vec{\sigma}_1) \tag{4.1}$$

with the properties

$$\begin{aligned} \vec{\sigma}_1 P_\pm &= \pm P_\pm = P_\pm \vec{\sigma}_1, \\ P_\pm P_\mp &= 0, \quad P_+ + P_- = 1 \end{aligned} \tag{4.2}$$

$$\vec{\sigma}_3 P_\pm = P_\mp \vec{\sigma}_3, \quad \tilde{P}_\pm = P_\mp \quad . \tag{4.3}$$

The spinor R according to (3.19) commutes with the pseudoscalar $i = \gamma_0\gamma_1\gamma_2\gamma_3$. Therefore it is composed of all **even–graded** elements of the Clifford algebra of spacetime $\mathbb{R}^{1,3}$,

$$R = \alpha + \vec{a} + i(\vec{b} + \beta), \quad \alpha, \beta \in \mathbb{R}, \quad \vec{a}, \vec{b} \in \mathbb{R}^3 \ , \tag{4.4}$$

where the condition of unimodularity

$$R\tilde{R} = (\alpha + i\beta)^2 - \vec{a}^2 + \vec{b}^2 - 2i(\vec{a} \cdot \vec{b}) = 1 \tag{4.5}$$

implies that R depends on $8 - 2 = 6$ real parameters only. Equivalently, R in (4.4) may be decomposed in two quaternions Q_1 and Q_2 ,

$$R = \alpha + i\vec{b} + i(\beta - i\vec{a}) = Q_1 + iQ_2 \quad , \tag{4.6}$$

$$iQ_k = Q_k i \, , \ Q_k^* \equiv \gamma_0 Q_k \gamma_0 = Q_k \, , \ k = 1, 2 \ . \tag{4.7}$$

From equation (4.6) it is immediate to infer a quaternion reduction of (3.36) upon applying the decomposition of unity $1 = P_+ + P_-$ from the right hand side to R,

$$\begin{aligned} R &= R(P_+ + P_-) = Q_1(P_+ + P_-) + Q_2 i(P_+ + P_-) \\ &= Q_1(P_+ + P_-) + Q_2 i(\vec{\sigma}_1 P_+ - \vec{\sigma}_1 P_-) \\ &= (Q_1 + Q_2 i_1)P_+ + (Q_1 - Q_2 i_1)P_- \\ &= \phi P_+ + \theta P_- = R \quad , \end{aligned} \tag{4.8}$$

where the definition

$$i_k = i\vec{\sigma}_k, \quad \vec{\sigma}_k = \gamma_k\gamma_0, \quad k = 1, 2, 3 \tag{4.9}$$

for the unit quaternions is made. Multiplication of (4.8) from the right hand side by $P_\pm$ entails because of (4.2) formulae to project from R onto the quaternions ϕ and θ,

$$\phi P_+ = RP_+, \quad \theta P_- = RP_- \quad . \tag{4.10}$$

So, equation (3.37)

$$2R' = R\omega \quad , \tag{4.11}$$

by application of (4.10) may be decomposed in two coupled quaternion equations as follows,

$$2\phi' P_+ = R\omega P_+ = RP_-\omega = \theta P_-\omega = \theta\omega P_+ \tag{4.12}$$

$$2\theta' P_- = R\omega P_- = RP_+\omega = \phi P_+\omega = \phi\omega P_- \quad . \tag{4.13}$$

Equation (3.36) yields

$$\varepsilon\omega P_+ = -i_3 P_+, \quad \varepsilon\omega P_- = (i_3\sigma - 2i_2\chi)P_- \quad , \tag{4.14}$$

whence (4.12) and (4.13) lead to

$$2\varepsilon\phi' = -\theta i_3, \quad \theta = 2\varepsilon\phi' i_3 \tag{4.15}$$

and

$$2\varepsilon\theta' = \phi(i_3\sigma - 2i_2\chi), \quad \varepsilon = \pm 1 \quad . \tag{4.16}$$

From (4.15) and (4.16) one finds

$$4\phi'' = \phi(\sigma - 2\chi i_1) \tag{4.17}$$

and from (4.15), (4.8),

$$R = \phi P_+ + 2\varepsilon\phi' i_3 P_- \quad . \tag{4.18}$$

Now, the condition of unimodularity $R\tilde{R} = 1$, according to (4.18), implies for the quaternion ϕ the constraint

$$2\varepsilon(\phi' i_3 P_- \tilde{\phi} - \phi P_+ i_3 \tilde{\phi}') = 1 \quad . \tag{4.19}$$

The structure of this constraint suggests to split the quaternion ϕ in the form

$$\phi = \varphi_1 + i_3\varphi_2 \quad , \tag{4.20}$$

where the $\varphi_{\frac{1}{2}}$ defined by

$$\begin{aligned} \varphi_1 &= \varphi_{11} + i_1\varphi_{12}, \quad \varphi_2 = \varphi_{21} + i_1\varphi_{22}, \\ \varphi_{kl} &\in \mathbb{R}, \quad k,l = 1,2, \end{aligned} \tag{4.21}$$

are Gaußian complex numbers in $\mathbb{C}(i_1)$ with the imaginary unit $i_1 = i\vec{\sigma}_1 = \gamma_3\gamma_2$. A quaternion in the form (4.20) is a solution of (4.17) if and only if each of its i_1–complex components φ_1 and φ_2 separately solve (4.17),

$$\begin{aligned} 4\varphi_k'' &= \varphi_k(\sigma - 2\chi i_1) \\ &= (\sigma - 2\chi i_1)\varphi_k, \quad k = 1,2 \quad . \end{aligned} \tag{4.22}$$

For the Wronskian

$$\begin{aligned} W &= \varphi_1\varphi_2' - \varphi_1'\varphi_2 \\ &= w_1 + i_1 w_2, \quad w_k \in \mathbb{R}, \quad k = 1,2, \end{aligned} \tag{4.23}$$

condition (4.19) then implies

$$P_+W - i_3P_+Wi_3 = -\frac{\varepsilon}{2} = P_+W + p_-\tilde{W} \quad , \tag{4.24}$$

which recalling (4.1) leads to

$$2(\varphi_1\varphi_2' - \varphi_1'\varphi_2) = -\varepsilon = \mp 1 \quad . \tag{4.25}$$

Assuming that the particular pair φ_1 and φ_2 satisfies (4.22) and (4.25), one may form with four i_1–complex constants c_k, the linear combinations

$$\psi_1\sqrt{\Delta} = c_1\varphi_1 + c_2\varphi_2, \; \psi_2\sqrt{\Delta} = c_3\varphi_1 + c_4\varphi_2, \tag{4.26}$$

where Δ is the determinant

$$\Delta = c_1c_4 - c_2c_3 \neq 0 \quad . \tag{4.27}$$

Again ψ_1 and ψ_2 fulfill equations (4.22) and (4.25). As is wellknown from the theory of linear second order differential equations, the pair

ψ_1, ψ_2 is the most general solution to (4.22) and (4.25). It should be noted that the transformation (4.26) from (φ_1, φ_2) to (ψ_1, ψ_2) defines the matrix group SL(2,C), $C = \mathbb{C}(i_1)$. This group depends on three complex–, or, on **six real parameters** just as the group Spin(1,3) of unimodular spinors R does. Insertion of the particular pair φ_1, φ_2 into (4.20) and (4.18) supplies a particular solution of (4.11) or (3.37),

$$2R' = R\omega \quad . \tag{4.28}$$

Equation (4.28) remains unchanged if it is multiplied from the left hand side by an arbitrary constant unimodular spinor R_0,

$$iR_0 = R_0 i, \quad R_0\tilde{R}_0 = 1, \quad R_0' = 0 \quad . \tag{4.29}$$

This leads to the conjecture that

$$\mathcal{R} = R_0 R \tag{4.30}$$

corresponds with the general solution of (4.28) obtained from (4.18) by insertion of (4.26) instead of the particular pair φ_1, φ_2. That this correspondence between the group SL(2,C) and Spin(1,3) is an isomorphism will be shown in the following section.

5. An isomorphism between the left–multiplicative group of unimodular biquaternions and SL(2,C)

The stated isomorphism now will be given in an explicit algebraic form. Let (cf. (4.1) – (4.3), (4.8), (4.27))

$$R_0 = \mathcal{C}_+ P_+ + \mathcal{C}_- P_-, \quad 2P_\pm = 1 \pm \vec{\sigma}_1 \quad , \tag{5.1}$$

$$\mathcal{C}_+\sqrt{\Delta} = c_1 + i_3 c_3, \quad \sqrt{\Delta}\tilde{\mathcal{C}}_- = c_4 + c_2 i_3 \quad , \tag{5.2}$$

$$\Delta = c_1 c_4 - c_2 c_3 \neq 0 \quad , \tag{5.3}$$

where the c_k are i_1–**complex constants**, i.e.,

$$\begin{aligned} c_k = c_{k1} + i_1 c_{k2}, \quad c_{k1} &\in \mathbb{R}, \quad c_{k2} \in \mathbb{R}, \\ k &= 1,2,3,4, \end{aligned} \tag{5.4}$$

then, R_0 is an unimodular spinor, i.e., $R_0 \in \mathrm{Spin}(1,3)$,

$$R_0\tilde{R}_0 = 1 \quad . \tag{5.5}$$

The proof proceeds by calculation of $R_0\tilde{R}_0$, viz.,

$$R_0\tilde{R}_0 = \mathcal{C}_+P_+\tilde{\mathcal{C}}_- + (\mathcal{C}_+P_+\tilde{\mathcal{C}}_-)^\sim \quad , \tag{5.6}$$

$$\begin{aligned} \mathcal{C}_+P_+\tilde{\mathcal{C}}_- \quad &= \quad P_+\frac{c_1}{\Delta}(c_4 + c_2 i_3) \\ &\quad\; -P_-\left(\frac{c_3}{\Delta}\right)^\sim (c_2 + i_3 c_4)^\sim \ , \end{aligned} \tag{5.7}$$

which after insertion in (5.6) proves the statement (5.5). On the other hand, when equations (4.8), (4.20) and (4.21) correspondingly are applied on an arbitrary unimodular spinor R_0 instead of R, one concludes that **the quaternion decomposition (5.1) – (5.4) holds completely general!** Therefore, according to equations (4.18) and (4.28) – (4.30), the most general solution of the generalized Darboux equation (4.28) is of the form

$$\mathcal{R} = R_0R = \Psi P_+ + 2\varepsilon(\Psi P_+)' i_3, \quad R_0' = 0 \ , \tag{5.8}$$

with

$$\Psi P_+ = R_0\phi P_+ = \mathcal{R}P_+ \quad . \tag{5.9}$$

In order to establish the stated isomorphism completely, the quaternion Ψ is shown to have the decomposition

$$\Psi = \psi_1 + i_3\psi_2 \quad , \tag{5.10}$$

where the pair ψ_1, ψ_2 is defined by equations (4.26) and (4.27). In fact, from (4.20) and (5.1) one finds by making use of (4.2) and (4.3),

$$\begin{aligned} R_0\phi P_+ \quad &= \quad (\mathcal{C}_+P_+ + \mathcal{C}_-P_-)(\varphi_1 + i_3\varphi_2)P_+ \\ &= \quad (\mathcal{C}_+\varphi_1 + \mathcal{C}_- i_3\varphi_2)P_+ = \Psi P_+ \quad , \end{aligned} \tag{5.11}$$

whence (5.2) – (5.4) lead to (5.10), (4.26) and (4.27).

Summary: The general solution of the Darboux equation (4.28) always may be obtained from a particular solution of (4.22), (4.25), (4.20) and (4.18) by left–multiplication with an arbitrary constant unimodular spinor.

6. A universal Lagrangian for the motion of an isotropic spacetime point in external fields

As an introduction to the principle on which my construction of a universal Lagrangian for an isotropic spacetime point $\lambda\!\!\!^{-} z$ rests, let me recall the variational principle of K. Schwarzschild [9] for the **timelike** motion of a pointcharge in an external electromagnetic field.

A motion, or, a curve in $\mathbb{R}^{1,3}$ is timelike if its arclength is positive. Choosing this arclength as the (invariant) curve parameter (proportional to the propertime), the curve tangent or velocity vector has to fulfill the **kinematical constraint**, that its square be a positive constant. Schwarzschild's Lagrangian involves just this kinematical constraint by means of a multiplier [10], plus the inner product product of the velocity with the charge–weighted potential vector of the external electromagnetic field (minimal coupling). In short, **the kinematical constraints, i.e., differential geometry marks the scope within external fields can act!** This is the principle, which now is applied to the motion of a lightlike = **isotropic** point.

In section 2, I have shown that differential geometry of triply curved isotropic lines is based on the two constraints (2.8), namely, the condition of isotropy

$$(z')^2 = 0, \quad z' = \frac{dz}{d\beta} \quad , \tag{6.1}$$

and, as a substitute for the euclidean arclength, the definition of the Vessiot parameter β,

$$(z'')^2 = -1 \quad . \tag{6.2}$$

On these **two** kinematical constraints, the free motion part $\mathcal{L}_0$ of the Lagrangian has to be built. This rises the question how to avoid derivative–orders higher than the first in the Lagrangian. **Leonhard Euler** [11] already solved that problem by introducing velocities as suitable auxiliary variables with the help of multipliers [10].

In terms of two scalar multipliers ζ, η and four vector–valued multipliers f, g, h and p, I define

$$\begin{aligned}\mathcal{L}_0 &= p\cdot(z' - z_1) + \frac{\zeta}{2}z_1^2 + f\cdot(z_2 - z_1') \\ &\quad -\frac{\eta}{2}(z_2^2+1) + g\cdot(z_3 - z_2') \\ &\quad +h\cdot(z_4 - z_3')\,. \end{aligned} \tag{6.3}$$

The structure of $\mathcal{L}_0$ is best elucidated by applying variational derivatives to $\mathcal{L}_0$, in order to obtain equations of motion. With respect to their algebraic grade, there are only two kinds of variational derivatives [12] which need to be formed here from a **scalar** Lagrangian: scalar derivatives

$$\delta_\eta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \eta} - \left(\frac{\partial \mathcal{L}}{\partial \eta'}\right)', \quad \eta \in \mathbb{R} \quad , \tag{6.4}$$

and vector-valued derivatives

$$\delta_v \mathcal{L} = \partial_v \mathcal{L} - (\partial_{v'}\mathcal{L})', \quad v = < v >_1 \quad . \tag{6.5}$$

The equations of motion then, as usually, are given by the kernel on which the corresponding derivatives vanish. Application of (6.5) and (6.4) to (6.3) yields

$$\begin{aligned}\delta_p\mathcal{L}_0 &= z' - z_1 = 0, \quad \delta_f\mathcal{L}_0 = z_2 - z_1' = 0, \\ \delta_g\mathcal{L}_0 &= z_3 - z_2' = 0, \quad \delta_h\mathcal{L}_0 = z_4 - z_3' \end{aligned} \tag{6.6}$$

$$2\delta_\zeta\mathcal{L}_0 = z_1^2 = 0, \quad -2\delta_\eta\mathcal{L}_0 = z_2^2 + 1 = 0 \quad , \tag{6.7}$$

$$\begin{gathered}\delta_{z_1}\mathcal{L}_0 = -p + \zeta z_1 + f' = 0, \\ \delta_{z_2}\mathcal{L}_0 = f - \eta z_2 + g' = 0 \quad , \\ \delta_{z_3}\mathcal{L}_0 = g + h' = 0, \quad \delta_{z_4}\mathcal{L}_0 = h = 0 \quad , \end{gathered} \tag{6.8}$$

$$\delta_z\mathcal{L}_0 = -p' = 0 \quad . \tag{6.9}$$

One notes that (6.6) coincides with (2.7). Equations (6.7) reproduce (6.1)–(6.2), or, (2.8). The system (6.8) may be decoupled to provide a definition of the vector p,

$$p = \zeta z_1 + (\eta z_2)' \quad , \tag{6.10}$$

which according to (6.9) is constant. So much about free isotropic motion.

The most general way to describe the influence of external fields on the motion of a free isotropic point is to add to $\mathcal{L}_0$ an arbitrary $\mathcal{L}_i$,

$$\mathcal{L}_i = \mathcal{L}_i(z, z', z_2, z_3, z_4) \quad , \tag{6.11}$$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_i \quad . \tag{6.12}$$

Equations (6.6) and (6.7) remain unchanged, whereas (6.8) – (6.9) now become,

$$\begin{aligned} &f = \eta z_2 - \partial_{z_2}\mathcal{L}_i + (\partial_{z_3}\mathcal{L}_i - (\partial_{z_4}\mathcal{L}_i)')', \\ &g = -\partial_{z_3}\mathcal{L}_i + (\partial_{z_4}\mathcal{L}_i)' \\ &h = -\partial_{z_4}\mathcal{L}_i, \quad p' = \partial_z\mathcal{L}_i - (\partial_{z'}\mathcal{L}_i)' \quad . \end{aligned} \tag{6.13}$$

The resulting equations of motion may be summarized as follows,

$$\begin{aligned} z_1 &= z' = \frac{dz}{d\beta}, \quad z_{j+1} = z_j', \quad j = 1, 2, 3, \\ z_1^2 &= 0, \quad z_2^2 = -1 \quad , \end{aligned} \tag{6.14}$$

$$\begin{aligned} p = \zeta z_1 \quad + \quad &\big[\eta z_2 - \partial_{z_2}\mathcal{L}_i \\ &+(\partial_{z_3}\mathcal{L}_i - (\partial_{z_4}\mathcal{L}_i)')'\big]' \end{aligned} \tag{6.15}$$

$$p' = \delta_z\mathcal{L}_i = \partial_z\mathcal{L}_i - (\partial_{z'}\mathcal{L}_i)' \quad . \tag{6.16}$$

Note, that (6.15) still contains the two scalar multipliers ζ, η of (6.3) which are to be eliminated by exploiting the kinematical constraints (6.14). Of course the multiplier ζ trivially may be eliminated by exterior multiplication of (6.15) with the vector z_1,

$$\begin{aligned} z_1 \wedge p = z_1 \quad \wedge \quad &\big[\eta z_2 - \partial_{z_2}\mathcal{L}_i \\ &+(\partial_{z_3}\mathcal{L}_i - (\partial_{z_4}\mathcal{L}_i)')'\big]' \; . \end{aligned} \tag{6.17}$$

In the following section, this bivector equation is seen to be identical with the angular momentum law (7.27).

7. The theorem of A. E. Noether: momentum and angular momentum

The essence of the theorem of Lady Amalie Emmy Noether is to express multivector–valued variational derivatives like (6.4) and (6.5) in terms of directional derivatives in parameter spaces of Lie groups, which act on the configuration space of a Lagrangian. If in particular a Lagrangian possesses a symmetry, i.e., it does not depend on some parameters of a Lie group, the derivatives along these parameters vanish and the coadjoint momenta are constants of motion. But irrespective of symmetries, the theorem may be applied to rewrite the Euler–Lagrange equations in terms of derivatives with respect to group parameters (Lie derivatives).

The purpose of this section is to rewrite equations (6.15)–(6.17) in terms of Lie derivatives with respect to the Poincaré group since its generators define momenta and angular momenta.

Let me begin with the inhomogeneous part of the Poincaré group, i.e., the spacetime translations,

$$\begin{aligned} \bar{z} &= z + \varepsilon b, \quad b' = 0, \quad \varepsilon' = 0, \quad \varepsilon \in \mathbb{R}, \\ \bar{z}_j &= z_j, \quad j = 1,2,3,4 \quad , \end{aligned} \tag{7.1}$$

$$\begin{aligned} \bar{p} &= p, \quad \bar{f} = f, \quad \bar{g} = g, \quad \bar{h} = h, \\ \bar{\zeta} &= \zeta, \quad \bar{\eta} = \eta \quad , \end{aligned} \tag{7.2}$$

implying

$$\bar{z}' = \bar{z}, \quad \bar{z}'_j = z_j \quad , \tag{7.3}$$

and

$$\begin{aligned} \frac{\partial \bar{z}}{\partial \varepsilon} = b, \quad \frac{\partial \bar{z}_j}{\partial \varepsilon} &= 0, \quad \frac{\partial \bar{p}}{\partial \varepsilon} = 0 = \frac{\partial \bar{f}}{\partial \varepsilon} = \frac{\partial \bar{g}}{\partial \varepsilon} = \frac{\partial \bar{h}}{\partial \varepsilon}, \\ \frac{\partial \bar{\zeta}}{\partial \varepsilon} &= 0 = \frac{\partial \bar{\eta}}{\partial \varepsilon} \quad . \end{aligned} \tag{7.4}$$

Before inserting (7.1) and (7.2) in the Lagrangian (6.12), it is worthwhile to regroup the set of (multivector–valued) variables into spacetime variables

$$\mathcal{Z} = \{z, z_1, z_2, z_3, z_4\}, \quad \mathcal{Z}' = \{z', z'_1, z'_2, z'_3\} \quad , \tag{7.5}$$

and into multipliers

$$\mathcal{M} = \{\zeta, \eta, f, g, h, p\} \quad , \tag{7.6}$$

whose velocities $\mathcal{M}'$ do not occur. The mapping (7.1)–(7.2) then is a very particular case of the general commutative family of transformations

$$\mathcal{M} \to \bar{\mathcal{M}}(\varepsilon, \mathcal{M}, \mathcal{Z}), \quad \mathcal{Z} \to \bar{\mathcal{Z}}(\varepsilon, \mathcal{M}, \mathcal{Z}) \quad , \tag{7.7}$$

assumed to depend analytically on the single real parameter ε,

$$\varepsilon' = 0 \quad , \tag{7.8}$$

and assumed to be sufficiently smooth with respect to $\mathcal{M}$ and $\mathcal{Z}$. Defining instead of $\mathcal{L} = \mathcal{L}(\mathcal{M}, \mathcal{Z}, \mathcal{Z}')$ the transformed Lagrangian

$$\bar{\mathcal{L}} = \mathcal{L}(\bar{\mathcal{M}}, \bar{\mathcal{Z}}, \bar{\mathcal{Z}}') \quad , \tag{7.9}$$

which implies the equations of motion

$$\partial_{\bar{\mathcal{M}}} \bar{\mathcal{L}} = 0, \quad \partial_{\bar{\mathcal{Z}}} \bar{\mathcal{L}} - (\partial_{\bar{\mathcal{Z}}'} \bar{\mathcal{L}})' = 0 \quad , \tag{7.10}$$

the derivative of $\bar{\mathcal{L}}$ with respect to ε,

$$\begin{aligned} \frac{\partial \bar{\mathcal{L}}}{\partial \varepsilon} \equiv \partial_\varepsilon \bar{\mathcal{L}} \quad = \quad & \big[(\partial_\varepsilon \bar{\mathcal{M}}) \bullet \partial_{\bar{\mathcal{M}}} + (\partial_\varepsilon \bar{\mathcal{Z}}) \bullet \partial_{\bar{\mathcal{Z}}} \\ & + (\partial_\varepsilon \bar{\mathcal{Z}})' \bullet \partial_{\bar{\mathcal{Z}}'} \big] \bar{\mathcal{L}} \end{aligned} \tag{7.11}$$

may be written by means of (7.10) in the form

$$\partial_\varepsilon \bar{\mathcal{L}} = (\bar{\mathcal{Z}}_\varepsilon \bullet \bar{\mathcal{P}})', \quad \bar{\mathcal{Z}}_\varepsilon = \partial_\varepsilon \bar{\mathcal{Z}}, \quad \bar{\mathcal{P}} = \partial_{\bar{\mathcal{Z}}'} \bar{\mathcal{L}} \quad , \tag{7.12}$$

where $\bar{\mathcal{Z}}_\varepsilon \bullet \bar{\mathcal{P}}$ is the momentum coadjoint to the transformation group (7.7). Equation (7.12) ist the theorem of Noether, generalized to non-holonomic constraints and to higher derivative orders! Note, that a symmetry $\partial_\varepsilon \bar{\mathcal{L}} = 0$ implies the constancy of the coadjoint momentum. In any case, be $\partial_\varepsilon \bar{\mathcal{L}} = 0$ or not, formula (7.12) is the projection of equations of motion (7.10) on the tangent vector $\bar{\mathcal{Z}}_\varepsilon$ of the transformation (7.7).

Let me now proceed by applying the general theorem (7.12) to the particular case (7.1) – (7.4), (6.3), (6.11) and (6.12). One finds

$$\bar{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_i(z + \varepsilon b, z', z_2, z_3, z_4) \tag{7.13}$$

and hence

$$\partial_\varepsilon \bar{\mathcal{L}} = b \cdot \partial_{\bar{z}} \mathcal{L}_i(\bar{z}, z', z_2, z_3, z_4) = b \cdot \partial_{\bar{z}} \bar{\mathcal{L}}_i \quad . \tag{7.14}$$

According to (7.4), the coadjoint momentum is

$$\bar{\mathcal{Z}} \bullet \bar{\mathcal{P}} = b \cdot \partial_{\bar{z}'} \bar{\mathcal{L}} = b \cdot \partial_{z'} \mathcal{L}_0 + b \cdot \partial_{z'} \bar{\mathcal{L}}_i \quad , \tag{7.15}$$

whence theorem (7.12) becomes

$$b \cdot p' = b \cdot \left[\partial_{\bar{z}} \bar{\mathcal{L}}_i - (\partial_{\bar{z}'} \bar{\mathcal{L}}_i)'\right] = b \cdot \delta_{\bar{z}} \bar{\mathcal{L}}_i \quad . \tag{7.16}$$

This equation holds for an arbitrary constant vector $b \in \mathbb{R}^{1,3}$ and hence it implies the vector equation (6.16) for $\varepsilon = 0$. The equation of motion (6.16) therefore may be called the **momentum law**, and **the vector** p **is to be identified as the total momentum of the isotropic point.**

There still remains the task to investigate the homogeneous part of the Poincaré group, namely, that part of the Lorentz group SO(1,3), which continuously is connected with the identity. It is convenient to describe that part in terms of one branch of the 2:1 covering Spin(1,3).

With the constant bivector Γ, I define

$$S = e^{\frac{\varepsilon}{2}\Gamma}, \quad \varepsilon \in \mathbb{R}, \quad \varepsilon' = 0, \quad S' = 0 \quad , \tag{7.17}$$

and instead of (7.1)–(7.2), the transformation group

$$\begin{gathered} \bar{z} = S z \tilde{S}, \quad \bar{z}_j = S z_j \tilde{S}, \quad j = 1,2,3,4 \quad , \\ \bar{p} = S p \tilde{S}, \quad \bar{f} = S f \tilde{S}, \quad \bar{g} = S g \tilde{S}, \quad \bar{h} = S h \tilde{S} \ , \\ \bar{\zeta} = \zeta, \quad \bar{\eta} = \eta \ . \end{gathered} \tag{7.18}$$

The derivative with respect to ε is

$$\partial_\varepsilon \bar{z} = \bar{z}_\varepsilon = \Gamma \cdot \bar{z} \quad , \tag{7.19}$$

and correspondingly for the other variables. The transformation (7.18) is a symmetry of the free part $\mathcal{L}_0$ of Lagrangian (6.12),

$$\bar{\mathcal{L}} = \bar{\mathcal{L}}_0 + \bar{\mathcal{L}}_i = \mathcal{L}_0 + \mathcal{L}_i(\bar{z}, \bar{z}', \bar{z}_2, \bar{z}_3, \bar{z}_4) \quad , \tag{7.20}$$

which leads to the Lie derivative

$$
\begin{aligned}
\partial_\varepsilon \bar{\mathcal{L}} &= \partial_\varepsilon \bar{\mathcal{L}}_i = \partial_\varepsilon \mathcal{L}_i(\bar{z}, \bar{z}', \bar{z}_2, \bar{z}_3, \bar{z}_4) = \qquad (7.21)\\
&= \Gamma \cdot (\bar{z} \wedge \partial_{\bar{z}} + \bar{z}' \wedge \partial_{\bar{z}'} + \sum_{j=2}^{4} \bar{z}_j \wedge \partial_{\bar{z}_j})\\
&\quad \bar{\mathcal{L}}_i(\bar{z}, \bar{z}', \bar{z}_2, \bar{z}_3, \bar{z}_4) \quad . \qquad (7.22)
\end{aligned}
$$

The momentum coadjoint to (7.18) becomes

$$
\begin{aligned}
\bar{\mathcal{Z}}_\varepsilon \bullet \bar{\mathcal{P}} &= (\bar{z}_\varepsilon \cdot \partial_{\bar{z}'} + \sum_{j=1}^{3} \bar{z}_{j\varepsilon} \cdot \partial_{\bar{z}'_j})\bar{\mathcal{L}} = \\
&= \Gamma \cdot (\bar{z} \wedge \partial_{\bar{z}'} + \sum_{j=1}^{3} \bar{z}_j \wedge \partial_{\bar{z}'_j})\\
&\quad [\mathcal{L}_0(\bar{z}', \bar{z}'_1, \bar{z}'_2, \bar{z}'_3) + \mathcal{L}_i(\bar{z}', \bar{z}_2, \bar{z}_3, \bar{z}_4)] \quad .
\end{aligned} \qquad (7.23)
$$

With the help of the chain rules

$$
\partial_{Sz\tilde{S}} = S\partial_z \tilde{S}, \ (Sz\tilde{S}) \wedge \partial_{Sz'\tilde{S}} = S(z \wedge \partial_{z'})\tilde{S} \ , \qquad (7.24)
$$

the coadjoint momentum (7.22) may be pulled back to the untransformed, original variables z and z_j,

$$
\begin{aligned}
\bar{\mathcal{Z}}_\varepsilon \bullet \bar{\mathcal{P}} &= \Gamma \cdot \left(z \wedge \partial_{z'} + \sum_{j=1}^{3} z_j \wedge \partial_{z'_j} \right)\\
&\quad [\mathcal{L}_0(\bar{z}', \bar{z}'_1, \bar{z}'_2, \bar{z}'_3) + \mathcal{L}_i(\bar{z}', \bar{z}_2, \bar{z}_3, \bar{z}_4)] \ . \qquad (7.25)
\end{aligned}
$$

For $\varepsilon = 0$, the theorem (7.12) then implies, according to (7.22), when pulled back to original variables,

$$
\begin{aligned}
\Gamma \quad &\cdot \quad \left(z \wedge \partial_z + z' \wedge \partial_{z'} + \sum_{j=2}^{4} z_j \wedge \partial_{z_j} \right) \mathcal{L}_i = \\
&= \Gamma \cdot \left[\left(z \wedge \partial_{z'} + \sum_{j=1}^{3} z_j \wedge \partial_{z'_j} \right) \mathcal{L} \right]' \quad . \qquad (7.26)
\end{aligned}
$$

Since this equation holds for any (constant) bivector $\Gamma \in \bigwedge^2(\mathbb{R}^{1,3})$, it is equivalent with **the angular momentum law**

$$\begin{aligned}\mathcal{J}' &= \left[\left(z \wedge \partial_{z'} + \sum_{j=1}^{3} z_j \wedge \partial_{z'_j}\right)\mathcal{L}\right]' \\ &= \left(z \wedge \partial_z + z' \wedge \partial_{z'} + \sum_{j=2}^{4} z_j \wedge \partial_{z_j}\right)\mathcal{L}_i \ . \end{aligned} \tag{7.27}$$

With (6.3) and (6.11) – (6.13) it leads to the following expression for the total angular momentum bivector $\mathcal{J}$,

$$\begin{aligned}\mathcal{J} &= z \wedge (p + \partial_{z'}\mathcal{L}_i) + \Sigma, \\ \Sigma &= f \wedge z_1 + g \wedge z_2 + h \wedge z_3 \ . \end{aligned} \tag{7.28}$$

One recognizes as the first summand the orbital angular momentum, whereas the bivector Σ is to be identified as the spin.

By making use of the momentum law (6.16), it is now straightforward to show, that the angular momentum law (7.27) in fact is identical with the equations of motion (6.17) or (6.15).

8. Generalized Weyssenhoff equations

As a particular example for the general formalism in the preceding two sections, let me now derive the equations of motion, which I found three years ago by means of a quite laborious spinor variation. These equations now are obtained from the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_i \quad , \tag{8.1}$$

where $\mathcal{L}_0$ is given by (6.3) and

$$\mathcal{L}_i = z' \cdot \mathcal{A}(z, z_2) \tag{8.2}$$

$$\mathcal{A} = \mathcal{A}(z, z_2) = a(z) + z_2 \cdot \mathcal{K}(z), \quad \mathcal{K} =< \mathcal{K} >_2 \quad . \tag{8.3}$$

The dimensionless vector potential $a(z) \in \mathbb{R}^{1,3}$ may be related to the electromagnetic potential vector $A(\lambda z)$ at the spacetime point λz by

$$a(z) = \frac{q}{mc^2} A(\lambda z) \quad , \tag{8.4}$$

where the constant q is some elementary charge, the constant m defines a mass scale, c is the speed of light in vacuo and $\lambda\!\!\!^-$ is the fundamental constant which defines the scale of length (cf. begin of section 2). According to Maxwell's theory, the bivector of the electromagnetic fieldstrengthes at the point $\lambda\!\!\!^- z$, $F(\lambda\!\!\!^- z) = \vec{E}(\lambda\!\!\!^- z) + i\vec{B}(\lambda\!\!\!^- z)$ then is determined by the exterior vector derivative of $a(z)$ in the form

$$\partial \wedge a(z) = \mathcal{F}(z) = \frac{q\lambda\!\!\!^-}{mc^2} F(\lambda\!\!\!^- z) \quad . \tag{8.5}$$

For the time being, the dimensionless bivector field $\mathcal{K}(z)$ in (8.3) is kept arbitrary. Later it may be restricted for instance by $\partial \wedge \mathcal{K} = 0$ or even by putting $\mathcal{K}(z) = \mathcal{F}(z)$.

The equations of motion (6.14) – (6.16) now are

$$\begin{aligned} z_1 &= z', \quad z_{j+1} = z_j', \quad j = 1,2,3, \\ z_1^2 &= 0, \quad z_2^2 = -1 \quad , \end{aligned} \tag{8.6}$$

$$p = \zeta z_1 + (\eta z_2 - \mathcal{K} \cdot z_1)' \quad , \tag{8.7}$$

$$p' = \partial_z \mathcal{L}_i - \mathcal{A}' \quad . \tag{8.8}$$

Defining the directional derivative of the multivector field $M(z)$ along any vector $v \in \mathbb{R}^{1,3}$ according to

$$M_v = \lim_{\mathbb{R} \ni \varepsilon \to 0} \frac{\partial}{\partial \varepsilon} M(z + \varepsilon v) \equiv \partial_\varepsilon|_0 M(z + \varepsilon v) \quad , \tag{8.9}$$

which implies (because of (8.6))

$$M_{z_1} = M_{z'} = (M(z))' \equiv M' \quad , \tag{8.10}$$

one finds after a little multivector analysis

$$\begin{aligned} \partial_z \mathcal{L}_i &= \mathcal{A}_{z_1} + (\partial_z \wedge \mathcal{A}) \cdot z_1, \\ \mathcal{A}_{z_1} &= \mathcal{A}' + \mathcal{K} \cdot z_3 \quad , \end{aligned} \tag{8.11}$$

$$\partial_z \wedge (z_2 \cdot \mathcal{K}) = \mathcal{K}_{z_2} - z_2 \cdot (\partial \wedge \mathcal{K}) \quad . \tag{8.12}$$

With the help of (8.5), (8.11) and (8.12), **the momentum law** (8.8) then finally may be brought into the form

$$p' = [\mathcal{F} + \mathcal{K}_{z_2} - z_2 \cdot (\partial \wedge \mathcal{K})] \cdot z_1 + \mathcal{K} \cdot z_3 \ . \tag{8.13}$$

Equations (7.28) and (8.2) – (8.3) lead to the following expression for **the total angular momentum**

$$\mathcal{J} = z \wedge (p + \mathcal{A}) + \Sigma \quad , \tag{8.14}$$

where

$$\Sigma = (\eta z_2 - \mathcal{K} \cdot z_1) \wedge z_1 \tag{8.15}$$

is the **spin bivector**.

There still remains the task to eliminate the multipliers ζ and η in (8.7). Note, that equation (8.15) in the case $\mathcal{K} = 0$ distinguishes the multiplier η as a generalized gyromagnetic factor. In order to determine η, I evaluate by use of (8.6) and the Table of invariants in section 2 the inner products of (8.7) with z_1 and z_2,

$$z_1 \cdot p = \eta - z_1 \cdot \mathcal{K} \cdot z_2 \quad , \tag{8.16}$$

$$z_2 \cdot p = -\eta' - z_2 \cdot \mathcal{K}' \cdot z_1 \quad . \tag{8.17}$$

The momentum law (8.13) implies $z_1 \cdot p' = z_1 \cdot \mathcal{K} \cdot z_3$, which with (8.17) yields

$$(z_1 \cdot p)' = (z_1 \cdot \mathcal{K} \cdot z_2 - \eta)' = (\eta - z_1 \cdot \mathcal{K} \cdot z_2)' \ , \tag{8.18}$$

according to (8.16), or, the result

$$\eta = \eta_0 + z_1 \cdot \mathcal{K} \cdot z_2, \quad \eta_0 = z_1 \cdot p \quad , \tag{8.19}$$

$$\eta_0' = 0 = (z_1 \cdot p)' \quad . \tag{8.20}$$

Two cases are to be distinguished,

1. $\underline{\eta_0 = z_1 \cdot p = 0}$: Since $z_1^2 = 0$, the momentum p is spacelike (tachyon),

2. $\underline{\eta_0 \neq 0}$: The quantities $\eta, p, \mathcal{F}$ and $\mathcal{K}$ in (8.7), (8.8) and (8.13) may be renormalized such that finally

$$p \cdot z_1 = \eta \quad , \tag{8.21}$$

$$\eta = 1 + z_1 \cdot \mathcal{K} \cdot z_2 \quad , \tag{8.22}$$

and, according to (8.17),

$$p \cdot z_2 = -z_1 \cdot \mathcal{K} \cdot z_3 \quad . \tag{8.23}$$

The conservation law (8.21) is a distinction of the Vessiot parameter in the particular dynamical model defined by (8.2). The calculation of the multiplier ζ involves both differential invariants (2.9), (3.14) and is less simple as the determination of η. It is not presented here.

This publication ends with a few remarks on the variational principle investigated by Jan Weyssenhoff [13]. In this reference, equation 4.21, he proposes a **Lagrangian for a free isotropic point.** This Lagrangian however is **not** parameter invariant! **Parameter invariance in his case holds only if the velocity vector is constraint to be isotropic.** Because of this error, he always has to refer to timelike motion and approach the isotropic case in an (undefined) limit. See his comment in [13] below equation 4.56.

On the other hand, his and Raabe's equations of motion (29) – (31) in [7] are correct, as is seen from (8.7), (8.15) and

$$\Sigma' = p \wedge z_1 + z_2 \wedge (\mathcal{K} \cdot z_1) \tag{8.24}$$

for minimal coupling $\mathcal{K} = 0$. Ironically, in his later publication [14], section 10, Weyssenhoff discovered the Vessiot parameter, but without recognizing its geometrical significance.

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