Orthodox Quantum Mechanics Free from Paradoxes

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ABSTRACT. A formulation of quantum mechanics based on an operational definition of state is presented. This formulation, which includes explicitly the macroscopic systems, assumes the probabilistic interpretation and is nevertheless objective. The classical paradoxes of quantum mechanics are analyzed and their origin is found to be the fictitious properties that are usually attributed to quantum-mechanical states. The hypothesis that any mixed state can always be considered as an incoherent superposition of pure states is found to contradict quantum mechanics. A solution of EPR paradox is proposed. It is shown that entanglement of quantum states is compatible with realism and locality of events, but implies non-local encoding of information.

RÉSUMÉ. Nous présentons une formulation de la mécanique quantique basée sur une définition opérationnelle d'état. Cette formulation, qui inclut explicitement les systèmes macroscopiques, se base sur l'interprétation probabiliste et est néanmoins objective. Les paradoxes classiques de la mécanique quantique sont analysés et leur origine est liée aux propriétés fictives qui sont habituellement attribuées aux états quantiques. L'hypothèse que n'importe quel état mélangé peut être toujours considéré comme une superposition incohérente d'états purs, serait contradictoire. Une solution du paradoxe EPR est proposée. Nous montrons que l'enchevêtrement des états quantiques est compatible avec le réalisme et la localité des événements, mais qu'il implique la codification non-locale de l'information.

I. Introduction

From the very beginning of quantum mechanics a controversy about its interpretation developed. Very soon different paradoxes and problems of interpretation appeared, *e.g.* Schrödinger's cat,[1] the reduction of the wave-packet,[2] the problem posed by Einstein, Podolsky and Rosen (EPR),[3] etc.. Although most physicists do not question the Copenhagen interpretation, no general agreement exits with the solutions that have been proposed for these paradoxes as revealed by a large amount of articles on this subject that continue to be published. Extensive reviews of related topics are given by Selleri and Tarozzi,[4] de Muynck,[5] Cramer,[6] Stapp[7] and Home and Whitaker.[8] To disentangle the issues of this seventy years old controversy is not a task as simple as one may believe. For example, a very diffuse opinion [8,9] defines a whole class of interpretations of quantum mechanics (ensemble interpretations are called by Home and Whitaker) as those that accept the following statement :

($I1$) "The quantum-mechanical state represents an ensemble of similarly prepared systems".

On the contrary, we believe that such statement should not be considered an issue at all. As can be easily verified with an operational analysis, anybody that uses the standard quantum-mechanical formalism for computing probabilities assumes, explicitly or implicitly, $I1$ to be true. This fact is not related to the interpretation of quantum mechanics, but to the **empirical** meaning of probability itself. For us, ensemble interpretation and probability interpretation are synonymous.

A question which, instead, we consider a real issue is the origin of probability of quantum phenomena. The Orthodox Interpretation assumes that :

($I2'$) "In quantum mechanics probability is intrinsic."

On the other side for the Statistical Interpretation :

($I2''$) "Probability in quantum mechanics just reflects the existence of an underlying reality which is not completely described by the quantum-mechanical state."

In other words the statistical interpretation postulates the existence of a hidden state that represents the physical reality of the single system. The quantum-mechanical state represents then an ensemble of such hidden states. Note that this last ensemble is different from the one that appears in $I1$. Today it is quite clear that no statistical interpretation is possible (see for example Mermin [10]), unless one accepts that probability distributions depend on the measuring apparatus.

Another issue is the completeness of quantum mechanics. Many sustain that orthodoxy requires accepting that :

($I3'$) "The pure quantum state describes completely the physical reality of a single system."

Of course, the statistical interpretation maintains the opposite :

($I3''$) "The quantum state does not describe completely the physical reality."

The famous EPR paper was intented to prove $I3''$. As this statement is an obvious consequence of $I2''$, that paper was considered an argument in favor of the statistical interpretation. Here comes a crucial point. $I3''$ follows from $I2''$, but $I3''$ does not imply $I2''$. That is, from the fact that the quantum state does not represent completely the physical reality does not follow that there is some other kind of state that does the job. This logical relationship among the statements has the following consequences : 1) The no-go theorems that invalidate $I2''$ do not invalidate $I3''$; 2) the orthodox interpretation is not forced to assume that $I3'$ is true. Ballentine [9] has shown that the superfluous assumption $I3'$ is the origin of many absurdities that are attributed to the orthodox interpretation, but, unlike us, he considers instead valid $I2''$ which is also superfluous.

The orthodox interpretation of quantum mechanics has been labeled as subjectivistic because of the essential role played by the Observer within the theory. Actually this view is shared only by some partisans of the Copenhagen school like von Neumann, others like Bohr and Heisenberg stress the role of the measuring apparatus. Such abandonment of realism, *i.e.* the doctrine that the objects that make up the physical world have an existence and behavior independent of human mind, was attributed to the operationalist philosophy that was professed by the Copenhagen school.[11] In our opinion the operational definition of physical concepts is not in conflict with realism ; instead, by defining operationally such fundamental concepts as preparation, state and measurement the discussion of interpretation problems may be kept free from spurious assumptions.

Park and Band [12] point out that many otherwise very good texts on quantum mechanics do not give a proper presentation of the empirical significance of the formalism. Following Margenau [13] they stress the importance of the preparation-measurement format of experimental science. Taking this last approach, we present here a formulation of quantum mechanics that preserves the essence of the probabilistic interpretation, without introducing the Observer, who is replaced by macroscopic systems. Of course the idea is not new ; see, for example, Heisenberg,[14] Weisskopf [15] or Ludwig.[16] Our definitions of state and observable attribute precise empirical meanings to these concepts, independently of the mathematical formalism. From its definition it becomes evident that the quantum state does indeed represent an ensemble of similarly prepared systems ($I1$). As the generic state that results from such definitions is a mixture, the traditional axioms of quantum mechanics are not appropriate because they are expressed in terms of state-vectors. We have therefore completed the formulation with an alternative set of postulates in terms of density operators. The theory that results is objective because it refers to the objective physical reality, that is to say to the macroscopic states of macroscopic systems. Within this framework the classical paradoxes of quantum mechanics are easily resolved.

We start the paper by giving operational definitions of state and observable, then we postulate the mathematical formalism and its empirical meaning, and finally we treat some classical paradoxes of interpretation. In spite of the lack of originality of some of the ideas found in the formulation, we think it is worthwhile presenting a comprehensive treatment to establish a precise language for the discussion that follows.

II. Definitions of State and Observable

In the definition of state presented here, as in the one given by Jauch,[17] the preparation of the system determines the state. Here the essential role played by macroscopic systems within the theory is clearly exposed.

The main primitive concepts that we use are: system, part of a system, macroscopic observable, macroscopic measurement, classical field, interaction and time.

A system is a **macrosystem** if, associated with it, there is a set of physical quantities (macroscopic observables) with the following properties:

a) At least for limited lapses their evolution follows deterministic laws.
b) They may be measured simultaneously and without changing their values.
c) Their measurement can be performed without affecting their evolution.

The fact that the macroscopic observables may be measured, or deduced from the physical laws, by different observers independently and without affecting their evolution, allows us to consider the values of these macroscopic observables as properties of the macrosystems, regardless of the measurements that are or are not performed.

The **macrostate** of a system is the set of values of its macroscopic observables.

We will say that the macrostates of macrosystems represent the "objective physical reality". Here "objective" means that it can be empirically verified by different observers.

It is possible to state objectively that something happens only if some macrostate changes. Thus an **event** can be defined as any change of macrostate of any macrosystem.

The aim of any dynamical theory is to predict the probability of occurrence of a set of events M with the condition that another set of events P occurs. The set of events P is called preparation and the set of events M measurement. In order to avoid any misunderstanding it is convenient to note that "to predict" is devoid here of any temporal meaning. It means that the theory yields the probabilities independently of the actual occurrence of the events. The events of P and M can be spread in time. Some of the events of P could be in the future of events of M and vice-versa. Nevertheless it is assumed that there is a subset of P, the initial preparation, which is in the past of all events of M. Moreover such initial preparation should be a sufficient one, as defined below. The theory does assume the arrow of time. The theory does not require the existence of a physicist "measuring" or "preparing ;" the Observer is superfluous.

We now give more precise definitions of preparation and measurement.

An **elemental preparation** of a system consists of :

a) An interaction of the system with a macrosystem.
b) A set of conditions that must be satisfied by the macrostates of the macroscopic system during this interaction.

A **compound preparation** of a system is made of :

a) Any number of elemental preparations of the system.
b) Any number of interactions of the system with other systems previously prepared.

Measurement is any interaction between a system and a macrosystem (the measuring device) which can change its macrostate as a result of the interaction. It can happen that the macrosystem does not change its macrostate during the interaction, but it is essential that the physical situation allows the change to take place in order to consider the interaction as a measurement, otherwise there are no events associated with it.

The macrostates the measuring device may reach are the possible **results** of the measurement. Usually the results of a measurement are associated to numerical values. Any measurement is a preparation, but the converse is not true, because there are preparations in which the macrosystem cannot change its macrostate.

A preparation is **sufficient** if it determines the probability distribution of the results of any successive measurement. Intuitively one may say that a sufficient preparation disturbs so much the system that its previous history becomes irrelevant for the time evolution that follows.

Two sufficient preparations are **equivalent** if they determine the same probability distribution of the results of any measurement. The **states** of a system are the classes of equivalence of its sufficient preparations.

Two measurements are **equivalent** if they have equal probability distributions of their results for any state. The classes of equivalence of the measurements are the **observables**. These definitions of state and observable are similar to those given by Beltrametti and Casinelli.[18]

The state in quantum mechanics, as it is clear from its definition, is not a property of a single system. It is a property of the preparation, or, what is the same thing, of the statistical ensemble of systems with equivalent preparations. The state represents the information that is available about the system, that is, the conditions of the probabilities predicted by the theory. That quantum-mechanical state does not represent the "physical reality" of the single system is better seen with an example: Consider two preparations which have in common the macrosystem but that differ by the conditions that the macrostates must fulfil (*e.g.* in one case some event related to the macrosystem is assumed to occur, in the other such assumption is not made); assume that a system is prepared and that it satisfies both sets of conditions (which are supposed to be non-equivalent); in such case one may assign two different states to the same system depending on which set of conditions one wants to consider. In other words the single system may belong to different statistical ensembles. All this is very nicely explained by Newton,[19] by Park [20] and by Tschudi.[21]

III. Postulates

We can now state the postulates of the theory. In order not to complicate unnecessarily the exposition we give a formulation without superselection rules. Postulates I and II assure the existence of states. Postulate III is equivalent to say that once the history previous to the preparation becomes irrelevant, nothing can be done afterwards to make it relevant again. Postulates IV–VIII are equivalent to the usual postulates of quantum mechanics as given for example by von Neumann ;[22] postulate IV introduces the Hilbert space and its relation with the states ; postulate V introduces the observables ; postulate VI gives the connection between the formalism and the empirical reality ; postulate VII consider compound systems and postulate VIII gives the time evolution of states. Finally postulate IX is a modified version of the reduction postulate that takes into account the fact that not all measurements are ideal.

Postulate I. *There are macrosystems.*

Postulate II. *There are sufficient preparations.*

Postulate III. *A preparation composed of a sufficient preparation followed by another arbitrary preparation is also a sufficient one.*

Postulate IV. *Any system has an associated Hilbert space such that there is a one-to-one correspondence between the states of the system and the density operators* ρ *with the properties :*

$$\rho = \rho^\dagger, \qquad \rho \geq 0, \qquad \mathrm{Tr}\,\rho = 1. \tag{3.1}$$

The mathematical properties satisfied by the density operators, Eq. (3.1), imply that they have a bounded discrete spectrum.

A state is **pure** if its density operator is idempotent. In this case the operator is a projection operator that projects into a subspace of dimension 1. A state is **mixed** or a **mixture** if it is not pure.

Postulate V. *There is a one-to-one correspondence between self-adjoint operators and observables. The correspondence is such that if* G *is the operator corresponding to the observable* g *and if* f *is a real function, then* $f(G)$ *is the operator corresponding to the observable* $f(g)$*.*

Postulate VI. *If a system has been prepared in a state with a density operator* ρ *the expectation value of the measurements of an observable*

g with corresponding operator G is given by

$$\langle g \rangle = \mathrm{Tr}(\rho G). \tag{3.2}$$

Postulate VII. *If a system S is made of two parts A and B with corresponding Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ then the Hilbert space corresponding to S must be a subspace of $\mathcal{H}_A \otimes \mathcal{H}_B$. If an observable g is related only to the subsystem A and if its operator of $\mathcal{H}_A$ is G_A then its operator as an observable of S is given by $G_S = G_A \otimes I$.*

As a corollary of postulates IV–VII one gets that the density operator ρ_A of subsystem A is obtained from ρ, the density operator of S, by contracting all indices relative to B, that is, by performing a partial trace

$$\rho_A = \mathrm{Tr}^{(B)} \rho. \tag{3.3}$$

Consider a preparation composed of : I) the preparation of a state ρ_0, II) no other interaction with the environment until time t. Postulate III ensures that this preparation is sufficient. The system will be therefore in a state, which, because of its internal dynamics, is in general different from ρ_0. In other words, after a system has been prepared in a state it continues to have a time-dependent state $\rho(t)$.

Postulate VIII. *If a system is isolated or if it interacts only with fields determined by macrosystems which are insensitive to the reaction of the system, there is a self-adjoint operator $H(t)$ (Hamiltonian operator) that determines the time evolution of the system ; the time-dependent density operator satisfies the von Neumann-Schrödinger equation*

$$i\hbar\, d\rho/dt = [H, \rho]. \tag{3.4}$$

We define **filter** as an observable that can assume only two values : 0 and 1. The operator of a filter is a projection operator. The measurement of any observable can be reduced to the measurement of the set of filters corresponding to its spectral measure.

The measurement of a filter F on a state ρ is said to be **ideal** or **minimal** if the density operator after the measurement is

$$\rho' = F\rho F/\mathrm{Tr}(F\rho) \tag{3.5}$$

if the result were 1, or

$$\rho' = (I - F)\rho(I - F)/\mathrm{Tr}((I - F)\rho) \tag{3.6}$$

if the result were 0.

When the result of an ideal measurement of F is disregarded the new density operator is given by

$$\rho' = F\rho F + (I - F)\rho(I - F). \tag{3.7}$$

Ideal measurements produce the minimum disturbance of the state. The immediate repetition of an ideal measurement gives the same result and does not modify further the state.

Postulate IX. *There are ideal measurements of any filter.*

Suppose there is a state ρ and a device that performs an ideal measurement of an observable which commutes with the operator ρ. If the result is disregarded the interaction with the device leaves the state unchanged. On the other hand it is possible to use the different results to classify the states after the measurement. It is obvious that in this way the information about the system is increased. It should be noted that a pure state cannot be reduced further by this procedure ; one may say that the pure states have a maximum content of information. This statement, as any other one of this paper referring to amount of information, can be made quantitative if one defines the content of information of a state as

$$I(\rho) = \mathrm{Tr}(\rho \ln(\rho)). \tag{3.8}$$

It is obvious that $I(\rho) \leq 0$ and that the equality holds only for pure states.

IV. Comments

1) A still open problem is whether quantum mechanics applies to macrosystems and whether the laws of macroscopic physics can be deduced as suitable limits of the quantum-mechanical laws or if, instead, such a program is impossible and there is a sharp boundary between microscopic and macroscopic. The present formulation is consistent with

both alternatives, although macrosystems appear explicitly in it. To give a solution to this problem is outside the scope of this paper ; we nevertheless believe that the first point of view is the correct one and we will assume it is so when discussing the paradoxes, where it does make a difference.

A substantial step toward the solution of the problem is the formulation of R. Omnès [23] that does not assume the existence of macroscopic systems from the beginning. He first introduces the mathematical formalism, then he shows that some systems behave classically and finally he gives an empirical meaning to the formalism. We consider his formulation compatible with our more traditional one.

2) The formulation is objective because the intervention of the observer has been replaced by the interaction with macrosystems.

3) Postulate I does not mean that macrosystems are an essential part of physical reality ; macrosystems are only a prerequisite for acquiring any knowledge about that reality.

4) Our definition of state does not distinguish between pure and mixed states ; we define pure state *a posteriori* by using the properties of Hilbert space. Nor are the empirical determinations of pure and mixed states different : in both cases one has to measure frequency distributions of different observables on a statistical ensemble of equally prepared systems. Only after the state has been determined does one know how pure it was. The formulation in terms of density operators, and not that in terms of state-vectors, is therefore the one that follows naturally from this definition of state. This point of view is, for example, shared by Fano [24] and by Park and Band.[12,20,25] Although it is possible to define the pure states before the introduction of Hilbert space by using the property of convexity,[26] we consider the formulation based on density operators better, because, as we will see in Sec. IX, there are non-pure states which are not statistical mixtures of pure states.

5) Many formulations of quantum mechanics include a reduction (or projection) postulate that implies that any measurement is ideal. Many authors have indicated that in general it is impossible to determine the new state prepared by a measurement without considering the details of the experimental arrangements ; indeed, in practice, most measurements are not ideal (*e.g.* one measures a photon by absorbing it). The weaker postulate IX only asserts the theoretical possibility of performing an ideal measurement. If a particular measurement is ideal or not should be decided by analyzing the physical situation. Margenau [13] advocates

the rejection of the reduction postulate, partially because of the previous reasons, and mainly because he envisages the possibility that without the reduction postulate the EPR paradox does not arise. We will show that it is not so. It is maybe true that the postulate IX is superfluous, but it is useful and it seems to do no harm.

V. Non-Factorizability of Quantum States

The state of a system composed of two parts A and B is not determined by the states of its parts ρ_A and ρ_B alone. Only when the states of A and B are statistically independent the state of the whole system equals $\rho_A \otimes \rho_B$. This property has been called entanglement or non-separability of quantum states. d'Espagnat [27] uses the term separability for a different, but related concept implying a space separation. We prefer to call it non-factorizability. Actually there is nothing specially quantum-mechanical in this property, it is a common feature of statistical ensembles, *i.e.* the marginal probability distributions do not determine the joint probability distribution. For example a similar situation arises also in classical statistical mechanics. The non-factorizability means that the measurements of observables of A are correlated to the measurements of observables of B. **The state of the whole contains more information than the states of its parts** (see appendix).

It is easy to prove that if the state of one subsystem is pure then it is statistically independent of the rest of the system.

An interesting consequence of non-factorizability is the fact that the measurement of an observable of part A prepares a new state of part B. The correlation of the measurements yields information about one system when measurements are performed on the other one. For example consider the ideal measurement of a filter F of A, $F = F_A \otimes I$; the measurement of F prepares the following state of B when the result is 1 (Eq. 3.5)

$$\rho_B^{(1)} = \mathrm{Tr}^{(A)}(F\rho F)/\mathrm{Tr}(\rho_A F_A) \tag{5.1}$$

and the state

$$\rho_B^{(0)} = \mathrm{Tr}^{(A)}((I-F)\rho(I-F))/\mathrm{Tr}(\rho_A(I-F_A)) \tag{5.2}$$

when the result is 0 (Eq. 3.6) . Note that ρ_B is not modified if only the fact that there has been a measurement, but not the result, is included into the conditions that define the new state. The reason is that

$$\mathrm{Tr}^{(A)}(F\rho(I-F)) = \mathrm{Tr}^{(A)}((I-F)\rho F) = 0 \tag{5.3}$$

and therefore

$$\mathrm{Tr}^{(A)}[F\rho F + (I-F)\rho(I-F)] = \rho_B. \tag{5.4}$$

If there is statistical independence, the state of B is not modified even by considering the results. In that case

$$\mathrm{Tr}^{(A)}(F\rho F) = \mathrm{Tr}^{(A)}(F_A\rho_A F_A \otimes \rho_B) = \mathrm{Tr}^{(A)}(\rho_A F_A)\rho_B \tag{5.5}$$

and

$$\begin{aligned}\mathrm{Tr}^{(A)}[(I-F)\rho(I-F)] &= \mathrm{Tr}^{(A)}[(I-F_A)\rho_A(I-F_A)\otimes\rho_B] \\ &= \mathrm{Tr}^{(A)}[\rho_A(I-F_A)]\rho_B\end{aligned} \tag{5.6}$$

and therefore it follows from equations (5.1) and (5.2) that $\rho_B^{(1)} = \rho_B^{(0)} = \rho_B$.

VI. The Reduction of the Wave-Packet

Consider a photon in a state represented by a wave-packet which falls upon a semireflecting mirror and splits into two divergent pieces. The photon is measured to be in one of the beams ; as a consequence the wave-packet in the other beam disappears. Two questions arise :

1) Is this discontinuous change of the wave-packet in contradiction with the continuous evolution described by the Schrödinger equation which may also include the measuring device ?

2) Is not the instantaneous disappearance of a piece of the wave-packet, when a measurement is performed in another place arbitrarily far away, in contradiction with relativity ?

The origin of the paradoxes is the attempt to give a physical reality to the quantum-mechanical state ; to consider that the wave-function were a classical field. The paradoxes disappear once one realizes that the statements about the wave-packet are not about the physical reality but about the predictions one can make about it. The theory predicts conditional probabilities, the conditions being represented by the initial state. The predictions pertaining to the time after a measurement has been performed must include as a condition the occurrence of that measurement. That is what is meant when one says that a measurement prepares a new state. Contrary to what has been asserted by many authors (*e.g.* von Neumann,[22] Wigner,[28] Burgos [29]), the reduction postulate does not imply any kind of different evolution of the quantum-mechanical state ; it is just a prescription for assigning the operator density to the new state that results when the fact that there has been an ideal measurement is included into the conditions that define the state. The state changes because its definition changes. No wonder that for von Neumann the reduction happens in the mind of some observer ; indeed to change a definition is a mental process, but this has nothing to do with any kind of physical evolution of the state.

Schrödinger's equation rules the time evolution of the predictions that can be made ; it is valid only between the preparation of the state and the measurement, because then what is being predicted happens. Of course one can consider a bigger system that includes the measuring apparatus ; in such case the Schrödinger's equation is valid until a different measurement happens. But this is the argument of the next section.

VII. Schrödinger's Cat

What has been called "the measurement problem" of quantum mechanics is dramatically illustrated by the Schrödinger's cat paradox. There is no better presentation of this paradox than in the words of Schrödinger himself.[1,30]

"A cat is placed in a steel chamber together with the following hellish contraption (which must be protected against direct interference by the cat) : in a Geiger counter there is a tiny amount of radioactive substance, so tiny that maybe within an hour one of the atoms decays, but equally probably none of them decays. If one decays then the counter triggers and via a relay activates a little hammer which breaks a container of cyanide. If one has left this entire system for an hour, then one would say that the cat is still living if no atom has decayed. The first decay would have poisoned it. The ψ-function of the entire system would express this by containing equal parts of the living and dead cat.

The typical feature of these cases is that an indeterminacy is transferred from the atomic to the crude macroscopic level, which then can be decided by direct observation. . . . By itself it is not at all unclear or contradictory."

One must agree with what Schrödinger says. This example is not paradoxical at all unless one insists on considering that the quantum-mechanical state represents the physical reality of the single system. Then one gets the absurd conclusion that the cat is neither dead nor alive until someone looks at it ! The truth is instead that the death of the cat is completely independent on whether we look at it or not. The cat might be already dead, but if one has not looked inside the chamber one has to continue to predict a probability for it being alive. It is so because the whole apparatus, including the counter and the cat, is a macroscopic system and, therefore, always has a determined macrostate. On the other hand quantum mechanics only yields the probabilities of the various macrostates, and hence **it cannot predict which is the actual macrostate of the cat**. Actually, this is the essence of the probabilistic nature of quantum phenomena. In other words, **the macrostate of a macrosystem is not a function of its quantum-mechanical state**. What the quantum-mechanical state determines is the distribution of macrostates. In some cases such distribution is a sharp peak around a macrostate, in others like in this example of Schrödinger the distribution is multimodal or broad.

One can use the macrostate of the cat to decide (measure) whether the radioactive substance has decayed. So two different states can be assigned to the system depending if one imposes the macrostate of the cat as a condition. None of the two states is "better" or more "real" than the other one, they just correspond to different conditions. This shows, once again, that the quantum-mechanical state does not represent the physical reality of the single system.

The real open question about measurement is to determine under which conditions does a system behave as a macrosystem. A puzzling question : How can the macrostate, which is a property of the macrosystem, emerge from the quantum-mechanical state, which is not ?

VIII. Quantum Mechanics and Determinism

It has been stated many times (see for example von Neumann [22]) that the time evolution of a system is deterministic except during measurements, because the evolution of the state is governed by a differential equation. We have also here a confusion between the evolution of the **system** and the evolution of the **predictions** (state) that one can make about it. The question of whether the evolution of a non-macroscopic system is deterministic or not, cannot even be posed because in this case there are no events between the preparation and the measurement, while the evolution of a system is a succession of events. If the system has a macroscopic part it is clear that in general the evolution of events is not deterministic (think of Schrödinger's cat). The probabilities of events are completely determined by the initial state, not so the events that actually happen.

IX. EPR Paradox I : Are Density Operators Superfluous ?

The famous paper by Einstein, Podolsky and Rosen [3] is without doubt one of the most interesting and discussed works about the interpretation of quantum mechanics. The problem they have proposed has become known as the EPR paradox, in spite of the fact that in no place of their paper the authors claim to have found a paradox. On the other hand, several quite different paradoxes have been based on the same physical situation presented in this piece of work. As a result there is always an ambiguity when someone refers to the "well known EPR paradox". The core of EPR's argument is the following example, presented here in a simplified form due to Bohm.[31]

Consider a system composed of two particles of spin 1/2. Assume that the Hamiltonian commutes with the total spin $\vec{S} = \vec{S}_A + \vec{S}_B$. The system is prepared initially in a metastable bound state of total spin $S = 0$. After some time the system disintegrates, the two particles separate and no longer interact, but the total spin maintains its initial value $S = 0$. The state of both spins is pure, thus it can be factorized from the space part and it is represented by the state vector

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(|\tfrac{1}{2}, -\tfrac{1}{2}\rangle - |-\tfrac{1}{2}, \tfrac{1}{2}\rangle\big). \tag{9.1}$$

The density matrix of each spin is obtained by the partial trace :

$$\rho_A = \mathrm{Tr}^{(B)}|\psi\rangle\langle\psi| = \frac{1}{2}I; \qquad \rho_B = \mathrm{Tr}^{(A)}|\psi\rangle\langle\psi| = \frac{1}{2}I. \tag{9.2}$$

So both spins are completely unpolarized. There is, though, a complete correlation between the measurements of the components of $\vec{S}_A$ and $\vec{S}_B$ in the same arbitrary direction $\hat{n}$. In fact the results are always opposite because $(\vec{S}_A \cdot \hat{n} + \vec{S}_B \cdot \hat{n})|\psi\rangle = 0$. Then by measuring $\vec{S}_A \cdot \hat{n}$ one prepares the spin B in the pure state with eigenvalue of $\vec{S}_B \cdot \hat{n}$ opposite to the result of the measurement of spin A.

Many versions of the EPR paradox are based on the wrong assumption, sometimes implicit, that the use of density operators in the formalism of quantum mechanics is superfluous [29,32]. More precisely the following statement, that has been called microrealism,[5] is assumed to be true:

(F) "In a mixed state the density operator describes an ensemble of systems prepared with the same procedure, but it is always possible to assume that each particular element of the ensemble is in its own pure state."

By the way, the belief that F was true is the origin of the name "mixture" given to the non-pure states. Actually EPR did not make this wrong assumption. The proof of the falsity of F is as follows:

Assume that the statement F were true. Then the spin B will be in a pure state r. Now, by measuring $\vec{S}_A \cdot \hat{n}$, an action that in no way can modify the **pure** state r (see sec. V), one prepares spin B in an eigenstate of $\vec{S}_B \cdot \hat{n}$. It follows that r is such eigenstate. But since $\hat{n}$ is arbitrary, r must be eigenstate of any component of $\vec{S}_B$, in contradiction with the commutation relations.

Different authors [33,34] analyzing this same example, also conclude that there are situations that cannot be described with wave-functions but only with density matrices. d'Espagnat [27] finds the same thing and calls these cases improper mixtures (or mixtures of the second kind), but insists in considering that only pure states are "true" quantum states; this very common point of view has the unpleasant consequence that there would be systems that have no state or, alternatively that very well defined pieces of physical reality (as particle B) could not be called systems.

X. EPR Paradox II: Is Quantum Mechanics Complete?

The aim of EPR in their original paper was to prove that "the quantum-mechanical description of physical reality given by wave functions is not complete". By complete EPR meant that "every element of the physical reality must have a counterpart in the physical theory." Apart from the argumentation that EPR gave in their paper it is clear that what they wanted to prove is **true**. We have seen in Sec. VII that in general the quantum state of a macroscopic system, even if it is pure, does not determine its macrostate, but only a probability distribution of macrostates. As macrostates compose the objective physical reality the description given by wave-functions (or density operators) does not agree with EPR's definition of complete. Now return to the EPR's argumentation. They start by giving the following criterion of reality :

(R) "If, without in any way disturbing a system, we can predict with certainty (*i.e.* with probability equal to unity) the values of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity".

Then they show that the impossibility of predicting simultaneously the values of observables that do not commute implies that the following two statements cannot be both false :

(1) The quantum-mechanical description of reality given by wave-functions is not complete.

(2) When the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality.

EPR give then an example from which they pretend to conclude that statement (2) is false (*i.e.* non commuting observables may have simultaneous physical reality), and that therefore (1) must be true (QM is not complete). EPR's argumentation is composed of three parts. First they ascertain that it is possible to prepare a pure state of system B by measuring system A. In the simplified version introduced in section IX this part of the argument will read as :

($A1$) Suppose that $\vec{S}_A \cdot \hat{a}$ is measured and that the result is α ($\alpha = \pm 1/2$). Then the system B is left in the spin state $|\vec{S}_B \cdot \hat{a} = -\alpha\rangle$; take instead another direction $\hat{b}$ and let the result be β, then the system B is left in the spin state $|\vec{S}_B \cdot \hat{b} = -\beta\rangle$.

Here comes the central point in their argument. They say :

($A2$) "...On the other hand, since at the time of measurement the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to the first system. This is of course, merely a statement of what is meant by the absence of interaction between the two systems."

($A3$) "Thus it is possible to assign two different wave functions (...) to the same physical reality (the second system after the interaction with the first)".

Then they use the reality criterion to conclude that both $\vec{S}_B \cdot \hat{a}$ and $\vec{S}_B \cdot \hat{b}$ simultaneously have physical reality, and therefore as $\vec{S}_B \cdot \hat{a}$ does not commute with $\vec{S}_B \cdot \hat{b}$ statement (2) is contradicted and (1) must be true. Up to here EPR's argument. But there is more; given the arbitrariness of $\hat{b}$, with the same reasoning one gets that any component of $\vec{S}_A$ and $\vec{S}_B$ must have physical reality and therefore must be predetermined since the time when the two systems were interacting. Here an unavoidable difficulty appears: since this last statement implies that Bell's inequalities are satisfied, it contradicts quantum mechanics.[35,36,7] As different authors [37,38,39] claim that the proof of Bell's inequalities is based on the probability theory of Kolmogorov that perhaps is not valid in this case, I present here a proof of one of Bell's inequalities that does not make use of probabilities. Consider a number N of pairs of spins prepared as above. Define $A(\hat{a}, i)$ and $B(\hat{b}, i)$ as the predetermined results of $2\vec{S}_A \cdot \hat{a}$ and $2\vec{S}_B \cdot \hat{b}$ of the pair labelled by i. Define also $P(\hat{a}, \hat{b})$ as

$$P(\hat{a}, \hat{b}) = 1/N \sum_i A(\hat{a}, i)B(\hat{b}, i). \tag{10.1}$$

Given that $|A| = |B| = 1$ and that $B(\hat{b}, i) = -A(\hat{b}, i)$ it immediately follows that

$$|P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{c})| \leq 1 + P(\hat{b}, \hat{c}). \tag{10.2}$$

But, if quantum predictions are of any use, as N goes to ∞ the quantity $P(\hat{a}, \hat{b})$ must approach $\langle 2\vec{S}_A \cdot \hat{a}\ 2\vec{S}_B \cdot \hat{b} \rangle = -\hat{a} \cdot \hat{b}$ and this last expression does not satisfy the inequality (10.2). Therefore there is something wrong in the argument of EPR because it contradicts quantum mechanics; so EPR did not prove that quantum mechanics was incomplete. After all the EPR argument was indeed a paradox.

This last argument reveals a main difference between classical probabilistic theories and quantum mechanics. In classical probability for any two random variables a joint probability distribution exists. In quantum mechanics it is not so. Only **compatible** observables may have a joint probability distribution.[36]

XI. EPR Paradox III : What Went Wrong ?

The interesting question now is : what is wrong in the argument of EPR ? They themselves state the possibility that that their reality criterion were not valid, and indeed the problems have been usually attributed to such criterion.[40,41] It is our opinion that there is nothing wrong with EPR's reality criterion. In fact the statement $A3$ of EPR's argumentation, which appears before the criterion is used, already contains an absurdity : what sense may have to assign two different **predictions** to the same physical reality ? Since the statement $A1$ is just a consequence of standard quantum mechanical rules the troubles must be attributed to $A2$. If $A2$ were right the only alternative left would be to assume that the hypothesis of isolated systems cannot be made ! But this has immense consequences as relativity implies that two systems can be isolated from each other during some time interval by separating them in space (locality).

For d'Espagnat [27,42] the non-factorizability of quantum states indeed implies non-locality and supraluminal influences.

Let us analyze $A2$ carefully. Of course $A2$ implies the isolation of the systems, but the converse is not true. $A2$ is too strong. What is really needed by the absence of interaction between two systems is :

($A2'$) Nothing that may be done to the first system can produce a change in the second system.

And indeed no measurement performed on system A can change the state $\rho_B = \frac{1}{2}I$ of spin B, see Eq. (5.4).

But then, what is the meaning of the reduction of the state ρ_B to the states $|\vec{S}_B \cdot \hat{a} = -\alpha\rangle$? This change is not due to the interaction of spin A with the measuring device but to the fact that the result of such a measurement has been **included** into the **definition** of the state of B. Therefore with this new condition the statistical ensemble which is represented by the state of spin B is different from the original one (half of the spins have been excluded from it !), as it is different from the ensemble which is obtained when the (incompatible) condition that specifies the result of the measurement of spin A in another direction is included. It is therefore wrong to claim that those three situations are the "same physical reality". So $A3$ does not follow from $A1$ and $A2'$. There is no contradiction with locality. It is the reduction of the wave-packet paradox again.

In the EPR paper the paradox arises because it was overlooked that to specify a result of the measurement of system A implies the inclusion of a new condition into the definition of the state of system B ; that, of course, "disturbs" the system (*i.e.* changes its state) and hence the reality criterion cannot be applied.

In summary, if the measurement of system A is performed and the result is disregarded then the state of system B does not change (in accord with $A2'$), but the reality criterion cannot be applied because there is no prediction with certainty. In order to have probability equal to unity the result of the measurement of A has to be included as a condition, but then, as the state of B changes, the reality criterion cannot be applied to the original physical situation. Of course it can be applied to the new state that results. That is to say, a component of spin B acquires physical reality only if the measurement of the same component of spin A is actually performed and the result is taken into account. Therefore it is impossible to simultaneously give physical reality to a different component of spin B, because the corresponding two components of spin A are not compatible.

To claim that non-factorizability implies supraluminal influences is to confuse a logical fact (the inclusion of a condition into the definition of a state) with a physical interaction : the hypothetical influence of measurement of system A on system B.

A three particle extension of the EPR problem, due to Greenberger, Horne and Zeilinger has shown more directly that to apply the reality criterion the way EPR did, contradicts quantum mechanics.[43,41] This new case can be also analyzed with the arguments given above.

What did lead to the mistake ? In $A1$ EPR tell us that the state of B changes, in $A2$ they say that such change is not *real*, that is they are implicitly assuming that there is some underlying reality which is not described by the quantum state, *i.e.* that there is some kind of different, objective state, the "real one", representing the physical situation of system B ; of course, such an objective state could not be modified by the measurement of system A. The EPR paradox shows that similar assumptions contradict quantum mechanics because they lead to Bell's inequalities. In this sense quantum mechanics cannot be completed.

XII. EPR Paradox IV : Some Replies

In the light of the previous discussion we will try to analize Bohr's reply to the EPR paper which appeared shortly after.[40] Few words about Bohr's philosophy before considering the central point of the paper. As Bohr was very well aware of the ambiguous empirical content that the wave-function had in the traditional formulation, he always followed the recommendation of analyzing the whole experimental arrangement when discussing the paradoxes of interpretation. But what surely is a good measure of caution was elevated by Bohr to the category of a postulate, denying even the possibility of giving a precise empirical meaning to quantum states. For example, he writes in the cited reply : "In accordance with this situation there can be no question of any unambiguous interpretation of the symbols of quantum mechanics other than that embodied in the well-known rules which allow to predict the results to be obtained by a given experimental arrangement described in a totally classical way,...". Our formulation of quantum mechanics shows that this extreme instrumentalist point of view was unjustified.

Most of Bohr's reply is devoted to obtain the following result that is given here in a form appropriate for Bohm's two spins example. By measuring a component of spin A the same component of spin B can be predicted with certainty, but an orthogonal component of spin B is completely indeterminated. If one chooses to measure an orthogonal component of spin A then the same component of spin B can now be predicted, but the previous component becomes indeterminate. Afterwards follows the central point of Bohr's argument. We prefer to reproduce it literally as it has been interpreted in different ways.[5,7,44] Bohr writes :

"From our point of view we now see that the wording of the abovementioned criterion of physical reality proposed by Einstein, Podolsky and Rosen contains an ambiguity as regards the meaning of the expression "without in any way disturbing a system." Of course there is in a case like that just considered no question of a mechanical disturbance of the system under investigation during the last critical stage of the measuring procedure. But even at this stage there is essentially the question of *an influence on the very conditions which define the possible types of predictions regarding the future behavior of the system.* Since these conditions constitute an inherent element of the description of any phenomenon to which the term "physical reality" can be properly attached, we see that the argumentation of the mentioned authors does not justify their conclusion that quantum-mechanical description is essentially incomplete."

Our interpretation of Bohr's argument is the following. Bohr states that the conditions that define the possible types of predictions that can be made are part of the "physical reality" of system B, which therefore is different when incompatible observables of system A are measured. So the argumentation of EPR does not go through. Within our interpretation of quantum mechanics the above argument would be incorrect because of a subtle but essential point. What is determined by the "physical reality" of system B is not the *type* of predictions that *could* be made, but the predictions that *can* be made. In the first case the *type* of measurement that is performed on system A (*i.e.* what kind of measuring apparatus is acting upon A) would be part of the "physical reality" of system B. If that were the case it would be very hard to sustain that there is no mechanical connection between systems A and B, as we would have an explicit contradiction of $A2'$. Instead in our interpretation what makes the change of "physical reality" (state) of system B is not the physical process of measurement on system A, but to include as a condition the result of such measurement. Bohr's argumentation could be made valid within our interpretation giving a different wording to the phrase in italics : "*. . . conditions which define the predictions . . .* ", but still an explanation of why the change of "physical reality" of system B is not a "mechanical disturbance" would be missing.

A final comment about Bohr's reply ; he explicitly rejects the criterion of physical reality of EPR, but what his argument actually shows is that the criterion cannot be applied because system B has been "disturbed" by the measurement on system A.

We see that what is very difficult to explain with the traditional formulation becomes simple and clear with ours.

Following Margenau,[13] de Muynck [5] states that an instrumentalist interpretation of quantum mechanics which is compatible with locality is possible, provided the projection postulate is rejected, because, they claim, then the EPR paradox cannot be formulated. We have two comments : 1) It is an illusion to believe that by rejecting the projection postulate the EPR problem cannot be posed. In the Bohm version, the very existence of the perfect correlation, precisely because both particles are not interacting, implies that any measurement of spin A, ideal or not, provided the measuring device does not interact with spin B, prepares spin B in a pure state if the result of the measurement of spin A is specified. This is completely independent of the projection postulate. 2) There is no need to reject the weak version of the projection postulate, because the origin of the problems is not in the postulate itself but in its objectivist interpretation ; that is to assume that the collapse is something that happens at the physical level and not at the logical level as it is in our interpretation.

Considering the EPR situation Costa de Beauregard [45] correctly points out that the statement "the first of the two measurements instantaneously collapses the other substate" is unacceptable because it is manifestly noncovariant. He proposes to eliminate from the theory the intermediate concept of state and to use only transition amplitudes. Actually, once again, it is not the concept of state in itself but its objectivist interpretation the origin of the problems. The "reduction" of the state of B (*i.e.* the inclusion of the result of a measurement of A in the definition of the state of B) is an *a*-temporal fact. It is perfectly valid to consider the state of B "collapsed" even before the measurement of A has taken place, provided we impose the condition that this measurement will eventually be performed and its result taken into account. Which measurement does "collapse" the other substate is arbitrary : it depends on which one of the two measurements one wants to consider as a condition for the probability of the other one.

XIII. EPR Paradox V : Does Non-Factorizability Imply Non-Locality ?

The state in quantum mechanics represents the preparation, *i.e.* the information that is available about the system. As the state determines the probabilities of the measurements the information should be considered encoded into the system itself. Now imagine a system with state ρ composed by two parts A and B, with states ρ_A and ρ_B respectively. There is no problem in assuming that the information represented by ρ_A is encoded in subsystem A, while the information of ρ_B is encoded in part B, but where is the information pertaining to ρ that is not included in $\rho_A \otimes \rho_B$ encoded ? **In both parts taken together**. Therefore the complete system encodes more information than its parts (see appendix).

By analyzing the EPR situation d'Espagnat [46] obtains that quantum mechanics is incompatible with the simultaneous assumption of realism, inductive inference and Einstein locality. From the assumption of realism and the existence of a perfect correlation, even when the systems are isolated, he concludes that some property must be already present in each subsystem A and B before the measurement were performed. Using inductive inference the presence of the property can be extrapolated to those systems that have not been measured. That is all what is needed in order to derive the validity of one of Bell's inequalities, which contradicts the predictions of quantum mechanics. Let us analyze carefully the previous argument. In order to deduce the existence of a preexistent property in each subsystem two hypothesis are needed : a) matter has encoded into itself the information that together with the physical laws determines its behavior (realism), b) such information is encoded locally. Alone, the first hypothesis allows to deduce the existence of a property in the composed system (the correlation), but not in each part separately. As discussed in the beginning of this section the second hypothesis seems to be in direct contradiction with the non-factorizability of quantum states, therefore it should be dropped instead of such fundamental assumptions as realism, inductive inference or Einstein locality.

The EPR and GHZ [43] paradoxes show that quantum mechanics is incompatible with the assumption of locally encoded information. In this sense we may say that non-factorizability of quantum states does imply some kind of non-locality.

On the other hand quantum mechanics is consistent with the locality of **events**. Consider two events a and b happening respectively in two systems A and B which contain macroscopic parts. In order to verify that event a is cause of event b (or b effect of a) one has to : (1) prepare both systems independently and preserve the independence of states up to the occurrence of event a, (2) verify that the occurrences of the events are correlated. The statistical independence is essential otherwise the correlation could be attributed to a common cause of both events.

How do these last concepts apply to the example of EPR ? The correlation between measurements of $\vec{S}_A \cdot \hat{n}$ and $\vec{S}_B \cdot \hat{n}$ does not imply a causal connection because the preparations of the states of spins A and B are not independent. Moreover, if the same states of spins A and B were prepared independently there would be no correlation between the measurements. Therefore the correlation should be attributed to the common preparation of both systems. This is also revealed by the impossibility of using the correlation between measurements to send messages. In fact the probabilities of all measurements on spin B are completely determined by its state ρ_B which is not affected at all by the measurements performed on spin A when their results are ignored. In other words it is impossible from the sole observation of spin B to infer if the spin A has been measured or not. In conclusion, there is no contradiction between the correlation and the isolation of the spins.

XIV. Conclusion

The three paradoxes that were analyzed in this paper have a common origin : the attribution to quantum-mechanical states of properties that they do not actually have, such as being a representation of the physical reality, being a property of the single system or being an ensemble of hidden realistic states. Our definition of state keeps that concept free from superfluous assumptions and at the same time allows a probabilistic interpretation of quantum mechanics that is also objective. We show that quantum mechanics, notwithstanding it cannot be completed with realistic states, is not complete in the sense of EPR, because macroscopic observables always have determined values and quantum mechanics only yields probabilities for such values. We also show that quantum-mechanical correlations are not in conflict with realism or with Einstein locality, but that they imply non-local encoding of information.

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Appendix

In this appendix we prove some results about the content of information of composed systems. Consider a system composed of two parts A and B with states ρ_A and ρ_B. First we show that if the two systems are statistically independent then the content of information is additive.

$$\begin{aligned} I(\rho_A \otimes \rho_B) &= \mathrm{Tr}[\ln(\rho_A \otimes \rho_B)\rho_A \otimes \rho_B] \\ &= \mathrm{Tr}[(\ln \rho_A \otimes I + I \otimes \ln \rho_B)\rho_A \otimes \rho_B] \\ &= \mathrm{Tr}(\ln \rho_A \rho_A \otimes \rho_B) + \mathrm{Tr}(\rho_A \otimes \ln \rho_B \rho_B) \\ &= \mathrm{Tr}^{(A)}(\ln \rho_A \rho_A) + \mathrm{Tr}^{(B)}(\ln \rho_B \rho_B) \\ &= I(\rho_A) + I(\rho_B) \end{aligned} \tag{A.1}$$

Now we will show that the state of the composed system that has the minimum content of information is $\rho_A \otimes \rho_B$. Let

$$\rho_A = \sum_i p_i^A |u_i\rangle\langle u_i| \tag{A.2}$$

$$\rho_B = \sum_i p_i^B |w_i\rangle\langle w_i|. \tag{A.3}$$

A generic state of the composed system is

$$\rho = \sum_i p_\alpha |e_\alpha\rangle\langle e_\alpha| \tag{A.4}$$

where

$$e_\alpha = \sum_{ij} U_{\alpha,ij} u_i \otimes w_j \tag{A.5}$$

and $U_{\alpha,ij}$ is some unitary matrix.

$$\sum_\alpha U_{\alpha,ij} U^*_{\alpha,kl} = \delta_{ik}\delta_{jl} \tag{A.6}$$

The conditions $\mathrm{Tr}^{(B)}\rho = \rho_A$ and $\mathrm{Tr}^{(A)}\rho = \rho_B$ become

$$\sum_{\alpha j} p_\alpha U_{\alpha,ij} U^*_{\alpha,kj} = \delta_{ik} p_i^A \tag{A.7}$$

and

$$\sum_{\alpha i} p_\alpha U_{\alpha,ij} U^*_{\alpha,il} = \delta_{jl} p^B_j \tag{A.8}$$

In order to obtain the state of minimum content of information one has to minimize $I(\rho) = \sum_\alpha p_\alpha \ln p_\alpha$ with the conditions (A.6), (A.7) and (A.8). Using the method of Lagrange one has to minimize the quantity S

$$\begin{aligned} S = &\sum_\alpha p_\alpha \ln p_\alpha \\ &+ \sum_{ik} a_{ik}(\delta_{ik} p^A_i - \sum_{\alpha j} p_\alpha U_{\alpha,ij} U^*_{\alpha,kj}) \\ &+ \sum_{jl} b_{jl}(\delta_{jl} p^B_j - \sum_{\alpha i} p_\alpha U_{\alpha,ij} U^*_{\alpha,il}) \\ &+ \sum_{ijkl} c_{ijkl}(\delta_{ik}\delta_{jl} - \sum_\alpha U_{\alpha,ij} U^*_{\alpha,kl}). \end{aligned} \tag{A.9}$$

This yields the equations

$$0 = 1 + \ln p_\alpha - \sum_{ijk} a_{ik} U_{\alpha,ij} U^*_{\alpha,kj} - \sum_{ijl} b_{jl} U_{\alpha,ij} U^*_{\alpha,il}, \tag{A.10}$$

$$0 = -\sum_k a_{ik} p_\alpha U^*_{\alpha,kj} - \sum_l b_{jl} p_\alpha U^*_{\alpha,il} - \sum_{kl} c_{ijkl} U^*_{\alpha,kl} \tag{A.11}$$

and

$$0 = -\sum_i a_{ik} p_\alpha U_{\alpha,il} - \sum_j b_{jl} p_\alpha U_{\alpha,kj} - \sum_{ij} c_{ijkl} U_{\alpha,ij}. \tag{A.12}$$

Now one has only to verify that $\rho = \rho_A \otimes \rho_B$ fulfils equations (A.6), (A.7), (A.8), (A.10), (A.11) and (A.12). The state $\rho_A \otimes \rho_B$ is obtained setting $\alpha = (m, n)$, $p_{mn} = p^A_m p^B_n$ and $U_{mn,ij} = \delta_{mi}\delta_{nj}$. Equations (A.6), (A.7) and (A.8) are obviously satisfied. The other three equations are also fulfilled if one sets

$$a_{mn} = (\ln p^A_m + 1/2)\delta_{mn}, \tag{A.13}$$

$$b_{mn} = (\ln p^B_m + 1/2)\delta_{mn} \tag{A.14}$$

and

$$c_{ijkl} = -p^A_i p^B_j (\ln p^A_i + \ln p^B_j + 1)\delta_{ik}\delta_{jl}. \tag{A.15}$$

Therefore for any state ρ of the composed system

$$I(\rho) \geq I(\rho_A) + I(\rho_B). \tag{A.16}$$

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