Extended Particles
Part I

Reformulation and Reinterpretation of the Dirac and Klein-Gordon Theories

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ABSTRACT. It is shown that, within the frame of relativistic quantum (wave) mechanics, massive elementary spin $\frac{1}{2}$ and spin 0 particles can be regarded as extended entities of characteristic size $\hbar/mc$ that are described by pertinent non-local operators, among these correct energy and momentum operators, charge operators, and size operators. Some of the aspects of such a conception are understandable in graphic terms which are in consonance with de Broglie's idea about particle structure. A number of conceptual improvements are attained as elimination of non-physical negative energies of free motion and oppositely oriented velocities and momenta, substantiation of the constancy of free-particle size in time, a physical justification of a non-divergent procedure of mass renormalization for a qualitative explanation of the anomalous magnetic moment value of electrons and muons, a physical explanation of the limits of validity of one-particle theory an the Foldy-Wouthhuysen procedure, etc. In the second part of the paper the ideology is applied to a discussion of the essence of Klein's paradox. A number of applications to elementary particle physics is left for a sequel to the work

I. Introduction

The Dirac equation (DE) and the Klein-Gordon equation (KGE) can be written in a well known "Hamiltonian" form [1, 2]

$$i\hbar\partial\psi\partial t = H\psi \quad , \tag{1}$$

where $\psi \equiv \psi'$ is a four-component spinor state for elementary spin $\frac{1}{2}$ fermions and $\psi \equiv \psi''$ is a two-component state for elementary spin 0 bosons - see Appenix 1. (Fermion and boson magnitudes will be labelled hereafter with primes and double primes, respectively, and labels will be omitted in expressions that equally apply to both kinds of particles or in specific cases ; by "particle" we always mean elementary particle below). The respective "Hamiltonians" are

$$H' = c\vec{\alpha}[\hat{\vec{p}} - \frac{e}{c}\vec{A}(t,\vec{x})] + mc^2\beta + eA^0(t,\vec{x})I_4, \tag{2}$$

$$H'' = \frac{\tau_3 + i\tau_2}{2m}[\hat{\vec{p}} - \frac{e}{c}\vec{A}(t,\vec{x})]^2 + mc^2\tau_3 + eA^0(t,\vec{x})I_2 \tag{3}$$

for massive particles ($m \neq 0$) ; here $A^\mu = (A^0, \vec{A}), \mu = 0,1,2,3,$ stands for the 4-vector potential, $\hat{\vec{p}} = -i\hbar\nabla$,

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I_4 = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} \tag{4}$$

(the zero in the expressions for β and I_4 denoting the 2×2 zero matrices), and the matrix 3-vectors $\vec{\alpha}$ has components

$$\alpha_k = \begin{pmatrix} O & \tau_k \\ \tau_k & 0 \end{pmatrix}, k = 1,2,3 \tag{5}$$

where τ_4 are the standard 2×2 Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5a}$$

The familiar anticommutation relations for the Hermitian matrices α_k, β, and τ_k read

$$\{\alpha_k, \alpha_j\} = 2\delta_{kj}I_4, \{\alpha_k, \beta\} = 0; k,j = 1,2,3 \tag{6}$$

$$\{\tau_k, \tau_j\} = 2\delta_{kj}I_2; k,j = 1,2,3 \quad . \tag{7}$$

The factor I_2 and I_4 will be dropped hereafter.

In the conventional approach to relativistic quantum mechanics (RQM) H' and gH'' are interpreted as actual Hamiltonians. This immediately entails the difficulty with the existence of negative free-motion energies $E = -E_p = -(c^2p^2 + m^2c^4)^{\frac{1}{2}} \leq -mc^2$ that appear - besides the positive eigenenergies E_p- as eigenvalues of H'_0 and H''_0. (In the field-free case $A^\mu = 0$ all operators will be labelled with subscript zero). Another strange feature of negative-energy eigenstates $\psi^{(-)}$ is that momenta and velocities in them appear to be oppositely oriented (cf. Appendix 2). Resolution of these and other difficulties within an appropriate reinterpretation of RQM, without invoking quantum field theory (which does not resolve all problems, say the one connected with Klein's paradox [3] - cf. Part II) or artificial RQM hypotheses as Dirac's negative-energy electron sea (which does not work in the boson case) would be of definite interest. We will set forth such an interpretation below that treats particles as extended entities and contains a number of non-trivial implications. In developing the interpretation we shall keep as strictly as possible to the language of QM and to the one of the interpretation itself that will be formed on this basis.

2. Actual energy and momentum operators and a charge operator in RQM

One can lawfully assume that the abovementioned difficulties derive from an inappropriate interpretation of H_0 and $\hat{\vec{p}} = -i\hbar\nabla$. This is borne out indeed by the behaviour of the said operator under charge conjugation C which interchanges positive and negative states $\psi^{(+)}$ and $\psi^{(-)}$. The familiar operators C in the above standard RQM representations are [1, 2]

$$C' = SK, C'' = \tau_1 K, \tag{8}$$

where $S = S^{-1} = S^{\dagger} = -i\beta\alpha_2, \tau_1 = \tau_1^{-1} = \tau_1^{\dagger}$, and $K = K^{-1}$ is the antilinear operator of complex conjugation. The transformed operators H_0 and $\hat{\vec{p}}$ are

$$\begin{gathered} H'_{0,C'} = C'H'_0C'^{-1} = -H'_0, H''_{0,C''} = C''H''_0C''^{-1} = -H''_0, \\ \hat{\vec{p}}_{C'} = -\hat{\vec{p}}, \hat{\vec{p}}_{C''} = -\hat{\vec{p}} \ . \end{gathered} \tag{9}$$

(In the case of $A^{\mu} \neq 0$ the C operation includes the transition $A^{\mu} \rightarrow -A^{\mu}$ and then one obtains an identity of the same kind for the respective operators H). Therefore, H_0 and $\hat{\vec{p}}$ are not the right energy and momentum operators indeed, as charge conjugation should not affect the real energy and momentum of the particle and hence the respective operators and their eigenvalues. At the same time, charge conjugation must change the sign of the actual charge $e_a = \pm e_b = \pm e$ in the problem ($e_b = e$ being the "basic" charge participating as a fixed constant $e-$ like m and $\hbar-$ in Eqs. (2, 3)), so we arrive at the idea that the appropriate magnitude are

$$\tilde{H}'_0 = H'_0\Lambda'_0, \tilde{H}''_0 = H''_0\Lambda''_0, \tilde{\vec{p}}'_0 = \hat{\vec{p}}\Lambda'_0, \tilde{\vec{p}}''_0 = \hat{\vec{p}}\Lambda''_0 \tag{10}$$

where $\tilde{H}_0'$ and $\tilde{H}_0''$ are the *actual* free operators of energy (admitting of positive energies only), $\tilde{\vec{p}}_0'$ and $\tilde{\vec{p}}_0''$ are the *actual* free momentum operators (admitting of momentum eigenvalues that are unidirectional with particle velocities), and Λ_0', Λ_0'' are the operators of the relative charge sign $\lambda = e_a/e_b$:

$$\Lambda_0' = H_0'/|H_0'|, \Lambda_0'' = H_0''/|H_0''|, \Lambda_{0,C} = -\Lambda_0, \Lambda_0^2 = 1. \tag{11}$$

(Λ_0' is known as the "energy sign operator" [2]). The operators $|H_0|$ are defined as

$$|H_0| = \tilde{H}_0 = +\sqrt{H_0^2}, \tag{12}$$

so they can be only positive indeed. The eigenvalues of Λ_0 are $\lambda = 1$ in states $\psi^{(+)}$ and $\lambda = -1$ in states $\psi^{(-)}$.

Clearly, in the theory also exist operators $\tilde{e}_0$ of the actual free electric charge, namely,

$$\tilde{e}_0' = e\Lambda_0', \tilde{e}_0'' = e\Lambda_0''; \tilde{e}_{0,C} = -\tilde{e}_0, \tag{13}$$

whose eigenvalues are indeed $e_a = e$ in states $\psi^{(+)}$ and $e_a = -e$ in states $\psi^{(-)}$. That is, our first-quantized RQM theory naturally turns the concept of actual charge into a typical QM observable possessing its own operator with two distinct eigenvalues, in contradistinction with conventional RQM in which charge operators are absent.

The operators Λ_0 satisfy the conditions $\Lambda_0 = \Lambda_0^{-1} = \Lambda_0^\dagger$. The sense in which they are Hermitian will be considered below. Correspondingly, all operators $\tilde{\vec{p}}_0, \tilde{H}_0, \tilde{e}_0$ are Hermitian as well. Being function of H_0, the operators $|H_0|$ and $|H_0|^{-1}$ commute with H_0, so $\Lambda_0 = H_0|H_0|^{-1} = |H_0|^{-1}H_0$.

In the presence of fields $A^\mu \neq 0$ that are not strong enough to obliterate the gap between positive and negative pseudo-energy states and "mix" them we can introduce the respective operators of our interpretation in an analogous way :

$$\begin{gathered} \Lambda = H/|H| = H/(H^2)^{\frac{1}{2}}, \Lambda^2 = 1 \\ \tilde{H} = H\Lambda, \tilde{e} = e\Lambda, \end{gathered} \tag{14}$$

the origin for energy eigenvalues in a given frame of reference lying inside the gap, say in its middle. Therefore, the existence of a clear-cut gap is a prerequisite for the possibility to define sensible physical operators of the kind (14), etc. (In the case of $A^\mu \neq 0$ operators as $-i\hbar\nabla\Lambda$ are not self-adjoint but momentum has a well defined physical sense in QM for free motion only, when its eigenvalues can actually be determined, so the above Hermitian operator $\tilde{H}$ is to be treated as an "entity"). For such fields we shall have once again positive eigenvalues of $\tilde{H}$, eigenvalues ± 1 and $\pm e$ of Λ and $\tilde{e}$, respectively, and $\Lambda = \Lambda^{-1} = \Lambda^\dagger, \tilde{e} = \tilde{e}^\dagger$

Since we wish to preserve Eqs. [1] but treat $\tilde{H}$ rather than H as the actual energy operator, we see that the actual states of motion in our interpretation are

$$\tilde{\psi} = \Lambda\psi \tag{15}$$

and the respective equations of motion read

$$i\hbar\frac{\partial}{\partial t}(\Lambda\tilde{\psi}) = \tilde{H}\tilde{\psi}, \tag{16}$$

which is written out as

$$i\hbar\frac{\partial}{\partial t}(\Lambda'\tilde{\psi}') = c\vec{\alpha}[\hat{\vec{p}}\Lambda' - \vec{A}(t,\vec{x})\frac{\tilde{e}'}{c}]\tilde{\psi}' + mc^2\beta\Lambda'\tilde{\psi}' + A^0(t,\vec{x})\tilde{e}'\tilde{\psi}' \quad , \tag{17}$$

$$i\hbar\frac{\partial}{\partial t}(\Lambda''\tilde{\psi}'') = \frac{\tau_3 + i\tau_2}{2m}[\hat{\vec{p}}\Lambda'' - \vec{A}(t,\vec{x})\frac{\tilde{e}''}{c}]\Lambda''[\hat{\vec{p}}\Lambda'' - \vec{A}(t,\vec{x})\frac{\tilde{e}''}{c}]\tilde{\psi}'' + mc^2\tau_3\Lambda''\tilde{\psi}'' + A^0(t,\vec{x})\tilde{e}''\tilde{\psi}'' \tag{18}$$

for fermions and bosons, respectively. A next step in the development of the theory - which will not be undertaken here - should be the replacement of $\vec{x}$ in A^μ with the correct Hermitian one-particle non-local operator $\tilde{\vec{x}}$ that will commute with the appropriately redefined operators Λ. (In the field-free case such operators $\tilde{\vec{x}}_0$ will be considered below). This step would bring to a Hermitian operator $\tilde{e}A^\mu(t,\tilde{\vec{x}}) = A^\mu(t,\tilde{\vec{x}})\tilde{e}$.

A similar remark applies to the operators $\hat{\vec{p}}\Lambda$, the usual formal prescription being $\hat{\vec{p}}\Lambda \to (\hat{\vec{p}}\Lambda + \Lambda\hat{\vec{p}})/2$.

Eqs. (16) are not of a Schrödinger-like form due to the presence of Λ in them, so the original Eqs. (1) are only apparently Hamiltonian.

3. Preliminaries on particle extension

Now, the linear operator $|H_0|^{-1}$ (and hence Λ_0, Λ and so on) is non-local (cf. e.g. [4]). It represents an integral operator whose characteristic range of action is $\sim \lambda_C = \hbar/mc$, where λ_C is the Compton wavelength (cf. Appendix 3). The same range characterizes $|H_0| \equiv \tilde{H}_0 = H_0^2|H_0|^{-1} = |H_0|^{-1}H_0^2$ and more generally $\tilde{H}$ when the physical fields are not too strong in the above sense. Consequently, the eigenvalues E of energy, $e_a = \pm e$ of actual charge and - as will be evident - of all one-particle operators are obtained with the aid of non-local linear operators that test he behaviour of our states $\tilde{\psi}$ at a distance ("radius") $\sim \lambda_C$. Therefore, $2\lambda_C$ can be interpreted by order of magnitude as the unsharp objective "diameter" of the region in $\vec{x}$-space that carries the particle-like characteristics of the micro-object as energy and electric charge. (This is certainly a rest-frame dimension since the spatial oscillations of the non-zero momentum eigenstates will reduce the effective range of action of the integral kernel, at that - as can be readily demonstrated - in agreement with the relativistic formula for length contraction). We thus arrive at a model of the massive particle in which the measurable particle-like characteristics are objectively contained within a region (kernel) of rest-frame dimensions $\sim \hbar/mc$, whereas its wave-like characteristics must be contained in a much larger spatial region. This model is strongly reminiscent of the well known one proposed by de Broglie [6], with the concretization that we have now an indication of the magnitude of the kernel's "radius".

A directly recognizable feature of the non-local theory outlined above is that the states $\tilde{\psi}$ do not play in it the same role as that of the states ψ in conventional theory. In particular, the magnitude $\psi'^{\dagger}\psi'$ could no longer be regarded in all cases as the (local) probability density for pointlike electron position since now it is represented as $(\Lambda'\tilde{\psi}')^{\dagger}(\Lambda'\tilde{\psi}')$, where the non-local operator Λ' is applied to the function $\tilde{\psi}'$ that is treated as the actual state of a non-point fermion. (An interpretation of the said magnitude as the probability density of position would appear possible in only certain cases of slow variation of $|\psi'|$ over distances $\sim \hbar/mc$).

Independent theoretical facts - that are probably known to the reader in a different interpretation or context - also agree with the existence of a characteristic rest-frame kernel dimension $\sim \hbar/mc$. For example, in the case of extreme relativistic velocities de Broglie's wavelength $\psi_B = 2\pi\hbar/|\vec{p}|$ behaves as $\pi(2\lambda_C)\gamma^{-1}$ where $\gamma \equiv \gamma_v = (1-v^2/c^2)^{-\frac{1}{2}}, \vec{v} = \vec{p}/m\gamma$. This is the behaviour of a Lorentz-contracted body of rest-frame size $\sim 2\lambda_C$. One can thus say that at such velocities comes to the fore the objectively existing particle-like characteristic of the extended quantum entity that complies with the requirements of special relativity. (On the other hand, the tendency of λ_B to infinity when $v \to 0$ is not a particle-like property, so there is really more to the particle than simply a massive kernel of finite size).

For motion along a given (z) axis an analogous conclusion can be reached by examining positive states and the uncertainty relation $\triangle z \triangle p \gtrsim \hbar$ in them ($\triangle p \equiv \triangle p_z$). Really, given $\langle v \rangle \approx c$, one arrives in a straingthforward manner at a smallest possible $\triangle z_{\min} \sim \lambda_C \gamma_{\langle v \rangle}^{-1}$ when $\triangle p / \langle p \rangle \sim 1$ (the condition for a well defined motion in a given direction) which behaviour is the same as that of λ_B. The impossibility to obtain $\triangle z \lesssim \triangle z_{\min}$ without introducing states $\psi^{(-)}$ in the respective wave packet (cf. e.g. the consideration in Sec. 3.3 of ref. 7, valid for $\langle v \rangle = 0$) is readily interpreted as the impossibility to localize with such a precision the particle's kernel without disrupting its inner structure and thereby generating new particles and antiparticles, the latter corresponding to negative states.

Other arguments on the $(\hbar/mc)$-size idea will be given in the subsequent sections.

Here is the right place to mention that $\hbar/mc$ is the smallest possible (unsharp) kernel radius characterizing particles that are in some sense elementary. Indeed, the size of, say, atomic nuclei exhibits increase - rather than decrease - with the increase of nuclear mass. So there appears the problem to decide which particles may be treated as elementary enough to admit kernel size $\sim \hbar/mc$. (The proton and neutron radii, for example, are correctly predicted by order of magnitude as $\hbar/m_p c$ and $\hbar/m_n c$; at the same time, p and n are envisoned nowadays as composite particles made of quarks). A possible answer to this question based on our interpretation will be given in a future paper.

It is natural as well to pose the question whether particle dimensions $\sim \lambda_C$ would lead to contradiction with experiment ? We are unaware at present about such facts. Indeed, our reinterpretation of RQM preserves its substantial features. For example, we obtain the same energy spectra, with the exception that the unphysical negative energies are now expelled from the theory. It can thus be asserted that with respect to numerical precision the interpretation, in its present status, is equivalent to the physically sensible part of RQM, so one could hardly anticipate there invalidations of the interpretation. Outside RQM there do appear to exist phenomena which indicate a much smaller (and de facto pointlike) size of e.g. the electron, as is the case of scattering of highly relativistic electrons from protons, etc [8]. Inspection of the respective argumentation shows, however that the pointlike-electron conception essentially rests on the fact that the electron is no seen to be composed of other "sub-particles" and on the interpretation of the expression for the electron 4-current j'^{μ} as the one of material points. With respect to the former conception we can say that a particle need not necessarily consist of other particles in order to be non-pointlike. As for the latter conception, it should be stated that one and the same expression, generating a definite result, coudl be interpreted in different ways depending on the particular viewpoints as demonstrated above with the consideration of the sense of $\psi'^{\dagger}\psi'$.

Therefore, a disproof of the present interpretation, if possible indeed, could hardly rely on trivial arguments.

It must be mentioned that Foldy and Wouthhuysen [9] noted too features of the behaviour of the electron as an extended particle of size $\hbar/mc$ but they do not seem to be definitive about whether this is an actual size or just a mathematical property of the specific DE representation used by them. The approach of Barut (cf. e.g. [10, 11]), to quantum electrodynamics rests on the extended-electron conception as well. In particular, Barut guessed that the particle's characteristic size might be λ_C [11]. A newer work on the non-point electron idea is ref. [12], whereas Rosen's paper [13] treats the possible extended character of spin 0 particles, attributing to them a size quite different from $\hbar/mc$.

In what follows we shall dispense with concepts as charge and mass density within extended particles, etc, used by some of the named authors, since RQM, within whose frame we wish to reside, will be seen to contain no evidence that these are applicable to an elementary particle with a quantized charge and mass, so their use could prove restrictive or even misleading from a conceptual viewpoint.

4. One-particle operators

We shall now define a number of new one-particle physical operators within our interpretation. To this end we note that the charge operator $\tilde{e}$ in the present theory imposes the requirement that all the admissible one-particle operators should commute with it since the charge value must coexist with the eigenvalue of each physical magnitude. Correspondingly, even operators only [commuting with Λ and having zero matrix elements of the kind $(\tilde{\psi}_1^{(\pm)}, \tilde{O}\tilde{\psi}_2^{(\mp)})$] can be the correct one-particle operators in question. This was felt long ago by some authors, cf. e.g. [14]. $\tilde{H}_0$ and $\tilde{\vec{p}}_0$ are certainly such operators ($\tilde{H}$ and $\tilde{\vec{p}}$ in the general case too).

Eqs. (10, 11) make possible the definition of a free one-particle macro-velocity operators via the relativistic formula

$$\tilde{\vec{v}}_0 = c^2 \tilde{\vec{p}}_0 \tilde{H}_0^{-1} \quad , \tag{19}$$

with the aid of which we can also write

$$\tilde{H}_0 = m\tilde{\gamma}_0 c^2, \tilde{\vec{p}}_0 = m\tilde{\gamma}_0 \tilde{\vec{v}}_0, \tilde{\gamma}_0 = (1 - \tilde{\vec{v}}_0^2/c^2)^{-\frac{1}{2}}, \tag{20}$$

where $\tilde{\gamma}_0^{-1}$ can be termed the operator of Lorentz contraction.

The foregoing discussion shows that we can also introduce for both free fermions and bosons the operator

$$\tilde{D}_0 = 2c\hbar/\tilde{H}_0 = (2\hbar/mc)\tilde{\gamma}_0^{-1} \quad , \tag{21}$$

which describes up to a dimensionless constant factor ~ 1 the longitudinal size (along the direction of motion) of the particle's kernel. Indeed, the said size is a variable magnitude in our interpretation and in keeping with the QM principles it must have an operator counterpart. Eq. (21) establishes a link between the eignevalues D_0 of $\tilde{D}_0$ and E_p of $\tilde{H}_0$:

$$D_0 E_p = 2c\hbar. \tag{22}$$

The inverse proportionality $D_0 \propto E_p^{-1}$ and the universality of the relation (22) for all massive spin 0 and $\frac{1}{2}$ particles appears strange at a first sight but it is to be regarded by order of magnitude as a consequence of the union of relativity ($c \neq \infty$) with the quantum nature of the micro-entities ($\hbar \neq 0$). In fact, this universality goes beyond the case of $m \neq 0$: if we assign after (22) a longitudinal size $D_f = 2c\hbar/\varepsilon_{pf}$ to the photon of energy $E_{pf} = cp = c\hbar k$, we arrive at the reasonable result $D_f = \lambda_f/\pi, \lambda_f$ being the usual photon wavelength.

It is worth noting in this connection that, as demonstrated by Moller [15] (Sec. 6.3.), a classical-type system of mass m, internal momentum $\vec{\mu}$, and positively defined internal energy density must have in special relativity a finite size $R \geq |\vec{\mu}|/mc$. Replacing $|\vec{\mu}|$ by the electron spin $\hbar/2$, we arrive at $R \geq \hbar/2mc$ and it could be conjectured that classical relativistic mechanics by itself is capable of explaining particle size. However, the mentioned generality of the formula (22), which also applies to spin 0 particles (that possess no internal angular momentum and hence satisfy the trivial classical inequality $R \geq 0$) and photons (that are massless and hence respect no meaningful inequality), and the very fact that the electron spin $\hbar/2$ is a consequence of RQM make it evident that RQM cannont possibly be dispensed with in the discussion of elementary particle size at the present status of classifical-type and quantum theories.

The introduction of $\tilde{D}_0$ resolves an old difficulty of the conventional (Copenhagen) interpretation. According to the familiar approach of the latter to the Heisenberg : uncertainty relation $\triangle p \triangle z \gtrsim \hbar$, an uncertainty $\triangle p$ of the measured momentum magnitude somehow smears the particle over a distance $\gtrsim \hbar/\triangle p$, so high-precision p-measurement could smear the particle over say the entire Milky Way. At the same time each expreimenter is certain that the measured particle is always inside his apparatus and in particular an atomic nucleus is always a compact object rather than a cloud of variable density. The resolution of the conundrum now is that the particle's kernel is objectively contained all the time inside a volume $\lesssim (4/3)\pi(\hbar/mc)^3$, the wave-like precesses being due to factors outside the massive kernel, in consonance with de Broglie's idea. More formally, $\tilde{D}_0$ is a constant of free motion (cf. Sec. 5).

We pass now to the definition of the operators of microscopic position in RQM. As is well known, such a definition is connected with great difficulties in the conventional interpretation : each of the numerous proposals fails to satisfy some apparently natural requirements [16]. Examine however the one-particle operators written in our notations as

$$\tilde{\vec{x}}_0' = \vec{x} + \frac{1}{4}[i\beta\vec{\alpha} - \frac{i\beta(\vec{\alpha}c\hat{\vec{p}})c\tilde{\vec{p}}}{|H_0'|(|H_0'| + mc^2)} - \frac{\vec{\Sigma} \times c\hat{\vec{p}}}{|H_0'| + mc^2}]\tilde{D}_0' \quad , \tag{23}$$

$$\tilde{\vec{x}}_0'' = \vec{x} + \frac{i\tau_1 c\hat{\vec{p}}}{4|H_0''|}\tilde{D}_0'', \tag{24}$$

which are sometimes called the Newton-Wigner position operators ; here $\vec{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3), \Sigma_k = -i\alpha_1\alpha_j, i, j, k$ cyclic, k=1, 2, 3, and $|H_0| \equiv \tilde{H}_0$, eq. (12). (The N-W operator [23] [17] was also obtained earlier in ref. [14] ; the commutativity of $\tilde{D}_0'$ with the operator in the square brackets deserves noticing).

Both $\tilde{\vec{x}}_0'$ and $\tilde{\vec{x}}_0''$ are evidently non-local operators of range $\hbar/mc$. The only conventional objection against these operators is that they lack in some sense Lorentz invariance [16, 18]. (N-W themselves point to the fact that localizability of bosons, i.e. a definite eigenvalue of $\tilde{\vec{x}}_0''$, is achievable only in a fixed frame of reference). The said objection however is no longer effectual in an extended-particle interpretation.

Indeed, essential part of a high precision measurement of the location of an extended body at a given moment t in frame K should consist in determining its dimensions at that moment. But due to the non-invariance of the relativistic conception of simultaneity, an observer in frame $K' \neq K$ would not perceive this act as strict location measurment since measurement of dimensions at moment t in K will be spread over a non-zero time interval in K', whereas "actual" location measurement in K' should be accomplished at a fixed moment t' only. We thus have to do with relativistically non-invariant measurement conceptions ant it appears natural that the operators $\tilde{\vec{x}}_0'$ and $\tilde{\vec{x}}_0''$ share this feature.

An additional argument on the apt character of $\tilde{\vec{x}}_0'$ and $\tilde{\vec{x}}_0''$ is offered by their remarkable form : their components are commutative and, besides, they represent the sum of a point operator $\vec{x}$ plus a non-local operator in agreement with the idea that the position of an extended object can be determined by determining both the (approximate) position of some characteristic point within its volume and the way in which this volume (kernel) is located around the said point. At that, examining the action of the non-local part of $\tilde{\vec{x}}_0$ (containing the factor $\tilde{D}_0$) on highly relativistic eigenstates of momentum $\vec{p}$ (when $E_p \approx cp$ due to $mc^2 \ll cp$), we see that it is consistent with the Lorentz contraction of a massive body having a rest frame "radius" of the correct order of magnitude (here $\hbar/2mc$).

The free non-local one-particle operators of orbital angular momentum $\tilde{\vec{L}}_0$, spin $\tilde{\vec{S}}$, and total angular momentum $\tilde{\vec{J}}_0'$ can now be defined as

$$\tilde{\vec{L}}_0 = \tilde{\vec{x}}_0 \times \tilde{\vec{p}}_0, \tilde{\vec{S}} = (\hbar/2)\tilde{\vec{\Sigma}}, \tilde{\vec{J}}_0' = \tilde{\vec{L}}_0' + \tilde{\vec{S}}_0 \quad , \tag{25}$$

where $\tilde{\vec{\Sigma}} = (1/2)(\vec{\Sigma}\Lambda' + \Lambda'\vec{\Sigma})$. (Indeed, $\vec{\Sigma}$ is not the necessary spin operator since $\vec{\Sigma}\Lambda_0' \neq \Lambda_0'\vec{\Sigma}$ and $\vec{\Sigma}_{C'} = -\vec{\Sigma}$ (the spin operators proposed in [9,14] and [16] change sign too under C'), whereas we need a one-particle operator $\tilde{\vec{\Sigma}}$ satisfying $\tilde{\vec{\Sigma}}_{C'} = \tilde{\vec{\Sigma}}$ in analogy with what we had for $\tilde{H}_0$ and $\tilde{\vec{p}}_0$). Correspondingly,

$$\tilde{h} = \tilde{\vec{S}}_0\tilde{\vec{p}}_0'/|\tilde{\vec{p}}_0'| = (\hbar/2)\vec{\Sigma}\hat{\vec{p}}/|\hat{\vec{p}}| \tag{26}$$

will be the spin $\frac{1}{2}$ helicity operator which turns out the same as the conventional one.

It should be noted that the commutation relations

$$\tilde{\vec{\Sigma}}_0 \times \tilde{\vec{\Sigma}}_0 = -2mc^3 H_0^{'-2}[(\vec{\alpha} \times \hat{\vec{p}})\beta - imc\vec{\Sigma}]$$

for the components of the free operator $\tilde{\vec{\Sigma}}_0$ will have approximately the same operator effect as the well known ones $\vec{\Sigma} \times \vec{\Sigma} = 2i\vec{\Sigma}$ only when applied to free states that correspond to sufficiently small velocities. Nevertheless, the components of $\tilde{\vec{\Sigma}}_0$ will have the same eigenvalues as those of $\vec{\Sigma}$ as it follows e.g. from Eq. (26).

We conclude our operator definitions with that of the spin $\frac{1}{2}$ dipole magnetic moment $\tilde{\vec{\mu}}$. In non-relativistic QM we have $\tilde{\vec{\mu}} = e\hbar\vec{\tau}/2mc = ec\tilde{\vec{s}}/mc^2$, where $\tilde{\vec{s}} = \hbar\vec{\tau}/2$ is the non-relativistic spin operator. Clearly, in our theory the factor e in $\tilde{\vec{\mu}}$ should be generated by the operator $\tilde{e}'$ whereas mc^2 can be regarded as the eigenvalue of energy in the non-relativistic limit $v \ll c$, so we can write

$$\tilde{\vec{\mu}} = \frac{c\tilde{e}'}{2}(\tilde{\vec{S}}\tilde{H}^{'-1} + \tilde{H}^{'-1}\tilde{\vec{S}}) \quad , \tag{27}$$

$\tilde{\vec{\mu}}$ being - save a small anomalous term - the operator of interest. In the field-free (27) reduces to the simpler expression

$$\tilde{\vec{\mu}}_0 = c\tilde{e}_0'\tilde{\vec{S}}_0\tilde{H}_0^{'-1} \tag{27a}$$

The expression in the right-hand side of (27a) is equal to $(1/2\hbar)\tilde{e}_0'\tilde{\vec{S}}_0\tilde{D}_0'$, which shows that a non-zero magnetic moment can exist only within particles of non-zero extension in agreement with the internal character of such moments.

The Eqs. (27) yield the same energy dependence $\propto E_p^{-1}$ with correct numerical factor as the equation of Bargmann et al. [19] for spin polarization in the limit of quasiclassical motion in a sufficiently weak and smooth magnetic field $\vec{H}$, when the evolution of the rest-frame average value $\langle\tilde{\vec{s}}\rangle$ is given by $d\langle\tilde{\vec{s}}\rangle/dt = (ec/E_p)\langle\tilde{\vec{s}}\rangle \times \vec{H}$ in the absence of an an anomalous addition to $\tilde{\vec{\mu}}$. Indeed, this is the moment of force acting on a magnetic moment with the correct magnitude at rest.

We shall return to the definition of $\tilde{\vec{\mu}}_0$ in Sec. 7.

5. Scalar products and matrix elements

The above theory will be complete in the presence of pertinent scalar products that would generate probabilities and operator averages and, specifically, demonstrate that the operator eigenvalues coincide with the operator average values in the respective eigenstates. (This feature is generally absent in the conventional approach [1, 2]). For the spin $\frac{1}{2}$ case we adopt the definitions

$$(\tilde{\psi}', \tilde{\varphi}') = \langle \tilde{\psi}' | \tilde{\varphi}' \rangle, (\tilde{\psi}', \tilde{O}', \tilde{\varphi}') = \langle \tilde{\psi}' | \tilde{O}' | \tilde{\varphi}' \rangle, \tag{28}$$

which are of the same form as the conventional ones. For the spin 0 case we introduce the definitions

$$(\tilde{\psi}'', \tilde{\varphi}'') = \langle \tilde{\psi}'' | \tau_3 \Lambda'' \tilde{\varphi}'' \rangle, (\tilde{\psi}'', \tilde{O}'' \tilde{\varphi}'') = \langle \tilde{\psi}'' | \tau_3 \Lambda'' \tilde{O}'' \tilde{\varphi}'' \rangle \quad , \tag{29}$$

where $\langle - | - \rangle''$ is of the customary form as applied to two-component functions, namely, $\langle \tilde{\psi}'' | \tilde{\varphi}'' \rangle = \int d^3x (\tilde{\psi}_1''^* \tilde{\varphi}_1'' + \tilde{\psi}_2''^* \tilde{\varphi}_2'')$ correspondingly, in the latter case one has to make a distinction between the Hermitian conjugate $\tilde{O}''^\sharp$ of the operator $\tilde{O}''$ defined via $\langle \tilde{\psi}'' | \tilde{O}'' \tilde{\varphi}'' \rangle = \langle \tilde{O}''^\sharp \tilde{\psi}'' | \tilde{\varphi}'' \rangle$ with respect to the auxiliary scalar product $\langle - | - \rangle''$ and the Hermitian conjugate $\tilde{O}''^\dagger$ of actual interest with respect to $(-,-)''$, defined via $(\tilde{\psi}'', \tilde{O}'' \tilde{\varphi}'') = (\tilde{O}''^\dagger \tilde{\psi}'', \tilde{\varphi}'')$. The definitions (29) are justified by their properties and implications. They differ from the respective definitions adopted in [1] (and marked here with the subscript FV) : $(\tilde{\psi}'', \tilde{\varphi}'') = (\psi'', \Lambda'' \varphi'')_{FV}, (\tilde{\psi}'', \tilde{O}'' \tilde{\varphi}'') = (\psi'', \Lambda'' \tilde{O}'' \varphi'')_{FV}$. On the other hand, we obtain the same Hermitian conjugates of one-particle operators in both theories (see Appendix 4) :

$$(\tilde{O}''^\dagger)_{FV} = \tau_3 \tilde{O}''^\sharp \tau_3 = \tilde{O}''^\dagger = \tilde{O}'' \quad . \tag{30}$$

The last Eq. (30) follows from the usual requirement for Hermiticity of the pertinent physical operators. All the boson operators defined above satisfy this requirement. In particular, Λ'' turns out both Hermitian and unitary with respect to $(-|-)''$: $\Lambda''^\dagger = \Lambda'' = \Lambda''^{-1}$ as mentioned in the beginning.

Within the range of applicability of one-particle RQM the definitions (28) and (29) ensure positive mean energies. In the field-free case they lead to unidirectional momenta and (macro)velocities. The scalar product (29), in particular, has all the features of the customary Hilbert-space definition in the stated range. Namely, it is positive definite and linear in the second member ; it satisfies the requirement $(\tilde{\psi}'', \tilde{\varphi}'') = (\tilde{\varphi}'', \tilde{\psi}'')^*$ and guarantees the orthogonality of positive and negative pseudo-energy states since the pseudo-Hamiltonian H'' is Hermitian with respect to $(-,-)''$, $H''^\dagger = H''$. But in contradistinction with $(-,-)_{FV}$ the scalar product (29) varies with time in the presence of non-stationary fields. In order to see what this means we pass to the definition of operator time derivatives.

This is done in the usual manner via the equation $\frac{\partial}{\partial t}(\tilde{\psi}, \tilde{O}\tilde{\varphi}) \equiv (\tilde{\psi}, \frac{d\tilde{O}}{dt}\tilde{\psi})$, which gives

$$\frac{d\tilde{O}'}{dt} = (i\hbar)^{-1}[\tilde{O}', H'] + \Lambda' \frac{\partial \tilde{O}'}{\partial t} \Lambda' \tag{31}$$

$$\frac{d\tilde{O}''}{dt} = (i\hbar)^{-1}[\tilde{O}'', H''] + \{\frac{\partial}{\partial t}(\Lambda'' \tilde{O}'')\}\Lambda'' \quad . \tag{32}$$

Eqs. (31) and (32) yield constancy with time of the operators $\tilde{h}, \tilde{\vec{\Sigma}}_0$ and $\tilde{\vec{S}}_0, \tilde{\vec{\mu}}_0, \tilde{\vec{L}}'_0, \tilde{\vec{L}}''_0, \tilde{\vec{J}}'_0$, and certainly $H_0, \Lambda_0, \tilde{H}_0, \tilde{\vec{p}}_0$, and $\tilde{D}_0$.

The fact that $\tilde{D}_0$ is a constant of motion represents the above-mentioned formal demonstration within the present theory that the free particle exhibits no tendency of changing its size with time. Thus the fact that the wave packets tend to get increasingly smeared with time has no relevance to what happens with the particle : its kernel retains a constant size.

Besides, for both operators (23) and (24) we obtain

$$\tilde{\vec{v}}_0 \equiv \frac{d\tilde{\vec{x}}_0}{dt} = \frac{c^2 \tilde{\vec{p}}_0}{|H_0|} \quad , \tag{33}$$

which reproduces the formula (19) and is yet another evidence for the already familiar fact that in our approach $\tilde{\vec{p}}_0$ and $\tilde{\vec{v}}_0$ are unidirectional. On the other hand, the approach in [1] leads in the respective free-particle momentum representations to velocity operators $(\vec{v}''_0)_{FV} = \tau_3 c^2 \vec{p}/E_p$ and $(\vec{v}'_0)_{FV} = \beta c^2 \vec{p} E_p$ [their formulas (2.62) and (3.43), respectively] which bespeak opposite directions of velocities and momenta in states $\psi^{(-)}$.

Taking $\tilde{O}'' = \Lambda''$ in Eqs. (32), we obtain $d\Lambda''/dt = 0$ and hence $d\tilde{e}''/dt = 0$. We thus have charge conservation for bosons. In conjunction with the time-variability of $(\tilde{\psi}'', \tilde{\psi}'')$ this implies possibility for particle number variation under the restriction of constant overall charge, i.e. creation and annihilation of charged boson pairs. (The resultant states will represent then superpositions of $\tilde{\psi}^{(+)}$ and $\tilde{\psi}^{(-)}$). On the other hand, inserting Λ' for $\tilde{O}'$ in Eq. (31), we obtain $d\Lambda'/dt = \Lambda'(\partial\Lambda'/\partial t)\Lambda' \neq 0$ in the general case of time-dependent operators H'. Correspondingly, $d\tilde{e}'/dt \neq 0$ in such cases whereas $\langle\tilde{\psi}'|\tilde{\psi}'\rangle$ is time-independent, so the DE attains particle number conservation at the expense of charge conservation. (These properties of the DE and the KGE will also be evident in the particular case discussed in Part II). The KGE theory thus appears to possess a conceptual advantage compared to DE theory contrary to the initial attitude to these theories. But we have de facto arrived here at a certain defect of both theories which poses the question about their range of applicability.

6. A superselection rule for RQM

Indeed, the requirement $\tilde{O}\tilde{e} = \tilde{e}\tilde{O}$ imposed on every one-particle operator $\tilde{O}$ in the present approach demands the introduction in RQM of a superselection rule [20] with respect to electric charge. The particular rule forbids superpositions of the kind $\psi_1^{(+)}+\psi_2^{(-)}$ in the original theories, as these would produce states $\tilde{\psi}_1^{(+)} + \tilde{\psi}_2^{(-)} = \Lambda(\psi_1^{(+)} + \psi_2^{(-)})$ that represent superposition of $(+e)$ and $(-e)$ one-particle eigenstates in the present ideology. The formal veto therefore has a clear physical justification : in its absence one could arrive at incongruous results as existence of one-particle states of zero average charge for charged spin $\frac{1}{2}$ and 0 particles.

The specific RQM superselection rule formally rests on the existence of two “coherent” subspaces. One of these consists of (and is complete with respect to) the states of charges and the other plays the same role for anticharges. The one-particle operators have zero matrix elements between these subspaces as mentioned in Sec. 4.

Superpositions of + and - states always appear in the case of time-dependent fields, so a general feature of non-stationary RQM is the generation of forbidden superpositions. But eigenstates of H with an indefinite charge behaviour of a somewhat different kind can also exist, as exemplified by Klein’s paradox, in intense static fields that strongly reduce or completely suppress the pseudo-energy gap. [Since we can always write $\Lambda = \Lambda_0 + \Delta\Lambda$ and in such fields $\Delta\Lambda$ will no longer be a small operator in comparison with Λ_0, a specific effect of strong fields on our definition (29) may be the loss of the positive definiteness of $(\tilde{\psi}'', \tilde{\psi}'')$.]

We shall therefore have a one-particle RQM theory containing no unacceptable features only as long as the particular static or time-dependent fields have no strong tendency to produce wave functions with a mixed charge behaviour. Considerations as that in ref. 9 for fermions with characteristic velocities $v \ll c$ determine the necessary restrictions on the fields: $(\hbar/mc)\nabla(eA^{\mu})$ and $(\hbar/mc^2)\partial(eA^{\mu}/\partial t)$ must be small magnitudes in comparison with mc^2.

Our approach offers a simple physical picture of the essence of these restrictions and further extends their validity as necessary conditions for charged bosons as well. But prior to considering it we shall have to examine in some detail the implication of our model about the existence of

7. Subquantum motions and structures

The old idea about the existence of subquantum physics will be discussed here in the specific language of our extended-particle interpretation.

Going back to Eq. (16) we see that the evolution of the states of motion $\tilde{\psi}$ is determined by the non-local operators Λ and $\tilde{H}$ (of characteristic range λ_C in zero or - by continuity - sufficiently weak external (classical) fields). We can therefore say that with the aid of Λ and $\tilde{H}$ it is possible to obtain information about certain properties of particle motion in any state $\tilde{\psi}$, the same applying in one way or another to all operators whose definition contains Λ and $\tilde{H}$.

Since the heaviest constituent of the particle according to the present interpretation is its extended kernel, we anticipate to obtain via Λ and $\tilde{H}$ information about certain properties of the motions of matter within the kernel, about the translational motion of the kernel as a whole or a combination of these motions, etc. Therefore, if we have e.g. a free particle at rest, then the eigenvalue equation $\tilde{H}_0\tilde{\psi} = mc^2\tilde{\psi}$ is to be interpreted in the sense that mc^2 is the energy of the motions of matter within the particle's kernel (since $\tilde{H}_0$ acts within its range). The eigenvalue equation $\tilde{e}\tilde{\psi} = e_a\tilde{\psi} \equiv e_b\Lambda\tilde{\psi}$, on its turn admits the interpretation that charge generating motions within the kernel are responsible for the existence there of an electric charge e_a of magnitude $|e_a| = |e|$. In the case of $\tilde{p}_0\tilde{\psi} \equiv -i\hbar\nabla\Lambda\tilde{\psi} = \vec{p}\tilde{\psi}$ we see that the multiplicative combination of $-i\hbar\nabla$ and Λ determines both the magnitude of momentum and its (correct)direction in a $\vec{p}$-eigenstate, whereas with the aid of $\tilde{\vec{x}}_0$ we can extract information about the place and volume in space in which all internal motions are performed, etc.

In such a way, within the present interpretation of RQM, all the individual particle properties (rest mass, charge, spin, momentum, etc) represent the result of certain motions of matter, some of which are subquantum, e.g. those performed inside the particle's kernel. The conclusion that motions determine all physical properties remains unchanged in the presence of external fields as well since Λ and $\tilde{H}$ preserve their role of information-generating machines about motions in this case too. Therefore, particle interactions with classical fields can also be regarded as the result of motions (perhaps some form of matter exchange between the kernel and the field). RQM will have to stop at this point since it remains a one-particle theory but what we saw above can readily be formulated as a general hypothesis (or postulate) applying to the case of interactions among particles too, namely :

Physical properties of matter are the result of matter motions.

This hypothesis will not be used by us in its full scope but its generality is worth noting since it well fits the spirit of present-day physics.

We can now obtain certain non-trivial consequences of the above general consideration.

(i) Returning to the eigenvalue equation $\tilde{e}_0\tilde{\psi} \equiv e_b\Lambda\tilde{\psi} = e_a\tilde{\psi}$ and having in mind that Λ is an integro-differential operator of range $\sim \hbar/mc$, we see that in RQM the value of the particle's charge is obtained by a certain differentiation of the state $\tilde{\psi}$ and its subsequent integration over a volume in space of the same order of magnitude as that of the particle's kernel. In such a way one obtains with the aid of Λ the total electric charge value as a certain overall property of internal motions, without integrating any charge density function whose possible existence is not borne out by the theory. What the theory thus actually says is that one would be able to see a definite charge and - as a logical consequence - a definite Coulomb electrostatic field when one is outside the kernel (and then it could even seem from there that the charge is concentrated at a mathematical point at the kernel's centre). The theory does not say, however, that when one is inside the kernel one would see little "subcharges" in the infinitesimal elements of its volume, so classical-type problems as the hypothetical presence of internal forces of electrostatic repulsion that could tear the kernel apart are now nonexistent.

One could expect of course that in some cases (e.g. hadrons that are subject to both electromagnetic and strong interactions) the concept of charge "density" could sometimes prove useful in a certain particular non-classical sense. But in the case of leptons no reasons are seen to employ the density concept in even such a restricted sense.

It is a curious coincidence that after carrying out radiation corrections to the field of a point electron charge one can demonstrate within quantum electrodynamics that the Coulomb law is valid only at distances $\gtrsim \hbar/mc$ for the charge ([19b] § 114). None the less, we are not going to regard this fact as an argument in favour of our theory due to the conceptual defects of the former theory (one of these being treatment of "bare" charges as mathematical points that is most probably at the root of at least some of the divergent results of that theory).

An analogous consideration of the kernel's mass shows that the theory gives its total value and offers no mass density. Results of this kind are already known in physical: Einstein's general relativity does not admit a non-contradictory definition of gravitational energy in infinitesimal volumes ([15], Sec. 11.9).

(ii) Since outside the particle's kernel one anticipates the existence of a well defined classical Coulomb electric field in our picture, one should also anticipate the existence of a well defined density function $f(\vec{x}) = \vec{E}^2(\vec{x})/8\pi$ of the electrostatic energy in the kernel's exterior. (We employ here the CGSE system of units). The total energy E_{Coul} connected with such an electrostatic field ($|\vec{E}| = |e|/r^2$) by order of magitude is

$$E_{Coul} = \int\int\int_{r \geq \hbar/mc} \frac{E^2}{8\pi} r^2 sin\Theta dr d\Theta d\varphi = \frac{1}{2}\alpha mc^2 \tag{34}$$

where $\alpha = e^2/\hbar c \approx 1/137$ is the fine structure constant. Eq. (34) corroborates de Broglie's idea that practically all of the particle's mass is contained within the kernel: even in the case of electrically charged particles (carrying a long-range electric field) only $\sim 1/274$ part of their mass is contained outside the kernel. Note that, according to this logic, kernel dimensions say $\sim 10^{-2}\hbar/mc$ would not bear out the above conclusion.

The complex of kernel plus outer electric field may be termed *subquantum structure* since it is not a direct mathematical result of the RQM equations of motion.

(iii) Eq. (34) shows that the rest mass of the electrically charged particle can be represented as the sum

$$m = m_0 + \delta m_0, \tag{35}$$

where m_0 is the effective mass of the charge-carrying kernel whereas $\delta m_0 \sim \alpha m/2$ is the effective mass of the electric field enveloping the kernel. Concentrating on the spin $\frac{1}{2}$ Hamiltonian $\tilde{H}'_0$ we see that this subdivision of m entails the following problem. $\tilde{H}'_0$ acts inside the lepton's kernel, so $\tilde{H}'_0$ should be held responsible (at rest) for the existence of just the kernel's mass m_0 and could not possibly explain the existence of the long-range-field part $\delta m_0 = m - m_0$ of the total mass m. On the other hand, $\tilde{H}'_0$ depends only on m which does not presuppose a consistent resolution of the problem. Which are then the pertinent operators ?

The problem is resolved by noting first that free motion of the overall particle (with rest mass m) with a given velocity $\vec{v}$ also means free motion with the same $\vec{v}$ of its kernel (rest mass m_0) and introducing then a pseudo-energy operator $H'_0(m_0) = c\vec{\alpha}\hat{\vec{p}} + \beta m_0 c^2$ that determines the internal and external motions of the kernel via

$$i\hbar \frac{\partial}{\partial t}(\Lambda'_0(m_0)\tilde{\psi}'(m_0)) = \tilde{H}'_0(m_0)\tilde{\psi}'(m_0), \tag{36}$$

where $\Lambda'_0(m_0) = H'_0(m_0)/|H'_0(m_0)|, \tilde{H}'_0(m_0) = H'_0(m_0)\Lambda'_0(m_0)$ and $\tilde{\psi}'(m_0) = \Lambda'_0(m_0)\psi'(m_0)$ is the "renormalized" state corresponding to mass m_0 and not m (a similar remark applies to $\hat{\vec{p}} = -i\hbar\nabla$; modifications of the above kind are to be understood in the definitions of operators and scalar products in all cases in which the particle carries an electric charge). Under the above interpretation of $\hat{\vec{p}}$ as corresponding to m_0 the overall energy operator $\tilde{H}'_0 \equiv \tilde{H}'_0(m)$ should no longer be employed for describing the kernel's evolution per se and is defined as $\tilde{H}'_0(m) = +(c^2(m/m_0)^2\hat{\vec{p}}^2 + m^2c^4)^{\frac{1}{2}}$ yielding total free energy eigenvalue $E(m) = +c^2(m/m_0)^2p^2 + m^2c^4)^{\frac{1}{2}}$ in the states $\tilde{\psi}'_0(m_0)$ that correspond to eigenvalues $p^2 = m_0^2v^2/(1 - v^2/c^2)$ exactly as it should be for motion of m and m_0 with equal velocities. (In such a way, the states $\tilde{\psi}'_0(m_0)$ are eigenstates of two different energy and momentum operators that participate in the same problem!) m is determined by experiment and the same applies to m_0 if one would measure independeltly the eigenvalues of the particle's magnetic moment. Indeed, the electric field outside the kernel could play no role in forming the lepton's magnetic moment that is generated by matter motions within the kernel (of rest mass m_0), so (27a) should evidently be modified as

$$\tilde{\vec{\mu}}_0(m_0) = c\tilde{e}'_0(m_0)\tilde{\tilde{\vec{S}}}_0(m_0)\tilde{H}'_0(m_0)^{-1} \tag{37}$$

in obvious notations (say, $\tilde{e}'_0(m_0) = e\Lambda'_0(m_0)$), the operators in [37] being commutative.

In the rest frame this will give an eigenvalue $|\vec{\mu}_0| = |e|\hbar/2(m - \delta m_0)c = |e|\hbar/2m_0c$ for the magnitude of the magnetic moment. This is nothing else than mass renormalization and having in mind the s-mall magnitude of δm_0 we obtain in a first approximation an "anomalous" correction to $|e|\hbar/2mc$ of magnitude $(\alpha/2)|e|\hbar/2mc$. By order of magnitude and sign the correction agrees with the well known value $(\alpha/2\pi)|e|\hbar/2mc$ given by quantum electrodynamics [19b] and confirmed by experiment. By necessity, our estimate is qualitative at this stage but it has the advantage of giving a graphic physical reason for the renormalization and is free of divergences that mar the former theory.

(iv) We return now to the essence of the restrictions on the Foldy-Wouthhuysen procedure mentioned at the end of Sec. 6. In order to see their physical reason we concentrate for a moment on positive pseudo-states and note that factors as $eA^\mu(t,\vec{x})$ in the original equations only apparently define interactions in a local, "external" fashion as being due to the presence of a point charge e at the 4-point $(t,\vec{x})$. If charges have finite size indeed (equivalently , if $\vec{x}$ is replaced in A^μ with the pertinent non-local operator $\tilde{\vec{x}}$), part of the overall interaction energy will have to be assigned an internal sense as being due to the energy introduced by the classical fields $\vec{E}$ and/or $\vec{H}$ in the interior of the charge. The charge-field entity so formed has internal energy of its own and is of basic interest for us.

For concrete estimates we assume that we have fields and (non-relativistic) velocities that do not affect considerably the charge's dimensions, so that our charge can be visualized as a compact "ball" of radius λ_C. We also assume that the fields $\vec{E}$ and/or $\vec{H}$ - denoted below with the commun symbol $\vec{M}$ - are of approximately uniform magnitude throughout the charge's volume W (satisfying $\langle \vec{M}^2 \rangle \sim \langle |\vec{M}| \rangle^2$ upon averaging over W) in the regions within which the charge resides all the time or which the moving charge is bound to traverse as are certain conditions generating Klein's paradox for both fermions [3, 21] and bonsons [22].

We anticipate that the one-particle theory – both conventional and its non-local variant considered in this work – will be physically meaningful only as long as the additional energy due to $\vec{E}$ and/or $\vec{H}$ is $\lesssim mc^2$ under the specified conditions. (Really, otherwise the kernel's matter carrying energy $\sim mc^2$ will merge with field matter carrying energy $\gtrsim mc^2$ and whatever the concrete dynamics, the conduct of the new charge-field formation will have little to do with that of a material point with well defined parameters as rest mass m moving in well defined external fields A^μ, as are the conditions necessary for the validity of one-particle RQM ; one obvious possible effect in such strong fields is birth of new massive particles which means that the energy gap between + and - states will no longer exist). We therefore define the respective critical field limit by order of magnitude via $mc^2 = (4\pi/3)(\hbar/mc)^3 \langle M^2 \rangle_{crit}/8\pi$. Writing then $M_{crit} \equiv \langle |\vec{M}| \rangle_{crit} \sim \langle M^2 \rangle_{crit}^{\frac{1}{2}}$ we obtain under our assumptions

$$M_{crit} = \kappa m^2 c^3 / |e| \hbar \quad , \tag{38}$$

where $\kappa = |e|(6/c\hbar)^{\frac{1}{2}} = (6\alpha)^{\frac{1}{2}} \approx 0.21$. Eq. (38) applies to both static and non-static fields. A static field $H_{crit} \sim m^2c^3/|e|\hbar$ is a well known limit of applicability of classical electrodynamics that turns out to apply to RQM too. A static field $E_{crit} \sim m^2c^3/|e|\hbar$ corresponds to a variation $\sim mc^2$ of the point-potential energy eA^0 over a distance of λ_C, which is a limit beyond which one arrives at Klein's paradox in RQM [21, 22] (to be examined in detail in Part II) and hence a limit of applicability of classical electrodynamics too. The case of a 4-potential $A^\mu \propto e^{i\omega t}$ is just a non-static combination of the first two cases. It is not difficult to see that these limits, defined here by internal factors, fully agree with the mentioned restrictions in ref. [9], so these have a common physical reason.

We could certainly invert the above logic, considering $\frac{m^2c^3}{|e|\hbar}$ as a known common value for E_{crit} and H_{crit}, and ask what is the kernel's free "radius". The answer is $R_{kernel} \approx 0.35\hbar/mc$.

8. Miscellaneous remarks

We close Part I by examining other features of the model.

(i) The problem of the Lorentz and gauge invariance of the present theory is essentially the problem of the respective invariance of the conventional DE and KGE theories, with obvious exceptions for operators for which non-invariance appears to be justified by the very physics of extended objects (as was the case with the position operators $\tilde{\vec{x}}_0$). It is also worth noting that, by definition, the operators Λ are gauge invariant and Λ_0 is explicitly Lorentz invariant.

(ii) As noted in Sec. 7, the small mass of the electric field outside the kernel agrees with de Broglie's model of the micro-particle. The concept of de Broglie wavelength, however, should not be envisioned as being somehow due to the existence of that field. Really, it was mentioned in Sec. 3 that for ultrarelativistic velocities de Broglie's wavelength is to be regarded as a certain property of the kernel's size. In the general case our consideration gives no information about the nature of the wave.

(iii) The present extended-particle interpretation represents a negation of the Stückelberg-Feynman interpretation treating antiparticles as negative-energy particles spreading backwards in time (cf. [7] and references therein). Indeed, negative-energy particles are just non-existent in our case and this makes unnecessary the use of transcendental concepts as simultaneous existence of motions in both time directions in the presence of a single time parameter in the QM equations of motion, namely, that of ordinary forward-evolving time.

The same applies to conceptions as pointlike virtual particles that are instantaneously born at a given point in space and absorbed at another point, as pointlike objects do not exist in our case. Concepts of this kind are now assigned the role of certain heuristic metaphors.

(iv) Our approach to RQM led to the recognition of certain general properties of motion as overall (internal) energy, electric charge, spin, the existence of a compact kernel, non-divergent renormalization procedures, etc. At the same time, the door is left open for the possible existence of divergence-free, quantitative theories of subquantum motion that could eventually give a detailed description of what happens in the kernel and so on. Still, the interpretation proved capable in itself of producing a somewhat surprising amount of physical consequences thus turning into a theory as well. A number of these apply to elementary particle physics and will be examined elsewhere, together with a straightforward generalization of the charge-operator concept that will cover charge-like parameters as baryon and lepton numbers, introduction of the concept of pseudo-charge, a demonstration of the consistency of the superselection rule of Sec. 6 as applied to certain facts of kaon physics, an argument on the non-existence of purely neutral spin $\frac{1}{2}$ fermions and so on. Although the present status of the theory made possible the assessment of certain quantities just by order of magnitude, it will be seen that the theoretical predictions can be sharp enough.

(v) We did not examine in this paper Kemmer's approach [23] to spin 0 and spin 1 bosons. It admits the introduction of a pseudo-Hamiltonian H_K too and, in principle, our approach can be applied in this case too by defining a relative charge-sign operators $\Lambda_K = H_K/|H_K|$. The algebra of Kemmer's matrices, however, is more involved that the one in ref. 1.

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Critical remarks by Professor Lochak that induced a considerable extension of the physical motivations of the present approach are gratefully acknowledged.

Appendix 1

The 4-spinor states $\psi'(t,\vec{x})$ are familiar enough, whereas the less familiar states $\psi''(t,\vec{x})$ [1] are defined as

$$\psi'' = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{A.1}$$

where

$$\begin{aligned} \varphi &= [1 - \frac{\hbar}{mc^2}(-i\frac{\partial}{\partial t} + (e/\hbar)A^0)]\eta/\sqrt{2} \quad , \\ \chi &= [1 + \frac{\hbar}{mc^2}(-i\frac{\partial}{\partial t} + (e/\hbar)A^0)]\eta/\sqrt{2} \end{aligned} \tag{A.2}$$

$\eta = \eta(t,\vec{x})$ standing for an arbitrary (scalar) solution of the KGE in its usual form

$$(i\hbar\frac{\partial}{\partial t} - eA^0)^2\eta = c^2[(-i\hbar\nabla - \frac{e}{c}\vec{A})^2 + m^2c^2]\eta \quad . \tag{A.3}$$

The two-component form for ψ'' is thus a consequence of the fact that the KGE is of the second order in the time derivative, so the initial conditions at $t = t_0$ can be arbitrary with the respect to both η and $\partial\eta/\partial t$, the latter functions determining the components (A.1, A.2) of ψ''.

Appendix 2

Examine e.g. a (non-normalized)state of one-dimensional motion of spin $\frac{1}{2}$ particles along z of the form

$$\psi'(z) = w_p e^{ipz/\hbar} \quad , \tag{A.4}$$

where the transpose of the spinor w_p is given by

$$w_p^T = (1, 0, cp/(mc^2 - E_p), 0) \quad .$$

This is an eigenstate of $\hat{p}_z = -i\hbar\partial/\partial z$ and H'_0, corresponding to eigenvalues $p > 0$ and $-E_p < 0$ respectively. The general formula $\vec{j}' = c\psi'^{\dagger}\vec{\alpha}\psi'$ for particle flux density [2] gives in our case $j'_z = 2c^2p/(mc^2 - E_p) < 0$, and defining in a straightforward manner the particle's velocity along z via $v = j'_z/\psi'^{\dagger}\psi'(\psi'^{\dagger}\psi'$ being the position probability density distribution in conventional DE theory), we obtain that positive eigenvalues of p correspond to negative velocities in the case of interest. An analogous inference applies to spin 0 particles too, as it follows from a similar use of the field-free electric charge and current probability density distributions (cf. e.g. [2] for definitions ; the use of electric rather than particle densities in the latter case is necessitated by the fact that generally there do not exist positively defined particle position density distributions in KGE theory).

Appendix 3

The non-local character of $\tilde{H}_0^{-1} = +(m^2 - \nabla^2)^{-\frac{1}{2}}$ and hence of $\tilde{H}_0 = +(m^2 - \nabla^2)(m^2 - \nabla^2)^{-\frac{1}{2}}$ as well (we employ here the units $\hbar = c = 1$) follows from the fact that, given any Fourier-transformable function $\psi(t, \vec{x})$, we obtain after performing first its Fourier transformation and then its inverse

$$+(m^2 - \nabla^2)^{-\frac{1}{2}}\psi(t, \vec{x}) = (m^2 - \nabla^2)^{-\frac{1}{2}} \int \frac{d^3k}{(2\pi)^3} ei\vec{k}\vec{x}\bar{\psi}(t, \vec{k})$$

$$= (2\pi)^{-3} \int \frac{d^3k}{(m^2 + k^2)^{\frac{1}{2}}} \bar{\psi}(t, \vec{k})$$

$$= \int d^3x' F(\vec{x} - \vec{x}')\psi(t, \vec{x}'),$$

where $\bar{\psi}(t, \vec{k})$ is the Fourier transform of $\psi(t, \vec{x})$. The integral kernel of interest therefore is

$$F(\vec{x} - \vec{x}') = (2\pi)^{-3} \int d^3k \frac{e^{i\vec{k}(\vec{x}-\vec{x}')}}{(m^2 + k^2)^{\frac{1}{2}}} \tag{A.5}$$

Integrating over the angular variables and employing then formula (2.5.9.11) of ref. [5], we obtain

$$f(r) \propto \frac{1}{r} K_1(rm) \quad , \tag{A.6}$$

where K_1 is one of the McDonald functions and r stands for $|\vec{x} - \vec{x}'|$. In the usual units we have $r/\lambda_C, \lambda_C = \hbar/mc$, for the argument of K_1, and having in mind the asymtotic form of $K_1(z)$ when $z \to \infty$, we arrive at

$$F(r) \propto r^{-3/2} e^{-r/\lambda_C} \tag{A.7}$$

when $r/\lambda_C \gg 1$.

Appendix 4

We demonstrate here that the Hermitian conjugate $\tilde{O}''^{\dagger}$ of the one-particle operator $\tilde{O}''$ statisfies in our theory the first Eq. (30), i.e. $\tilde{O}''^{\dagger} = \tau_3 \tilde{O}''^{\sharp} \tau_3$. The computation may serve as a model for the typical manipulations in our variant of KGE theory. We omit below the double primes and obtain the following chain of identities:

$$(\tilde{\varphi}, \tilde{O}, \tilde{\psi}) \equiv \langle \tilde{\varphi} | \tau_3 \Lambda \tilde{O} \psi \rangle = \langle \tilde{O}^{\sharp} \Lambda^{\sharp} \tau_3^{\sharp} \tilde{\varphi} | \tilde{\psi} \rangle$$

(by the customary Hermitian conjugation) $= \langle \Lambda^{\sharp} \tilde{O}^{\sharp} \tau_3^{\sharp} \tilde{\varphi} | \tilde{\psi} \rangle$($\tilde{O}$ is one-particle, so $\tilde{O}^{\sharp} \Lambda^{\sharp} = \Lambda^{\sharp} \tilde{O}^{\sharp}$) $= \langle \Lambda^{\sharp} \tau_3^{\sharp 2} \tilde{O}^{\sharp} \tau_3^{\sharp} \tilde{\varphi} | \tilde{\psi} \rangle$ (since $\tau_3^{2\sharp} = \tau_3^{\sharp 2} = 1$) $=$ $\langle \Lambda^{\sharp} \tau_3^{\sharp} \tau_3 \tilde{O}^{\sharp} \tau_3 \tilde{\varphi} | \tilde{\psi} \rangle$ (since $\tau_3^{\sharp} = \tau_3$) $= \langle \tau_3 \tilde{O}^{\sharp} \tau_3 \tilde{\varphi} | \tau_3 \Lambda \tilde{\psi} \rangle$ (by the customary operation) $= (\tilde{O}^{\dagger} \tilde{\varphi}, \tilde{\psi})$. Q.E.D.

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