Two cluster stability of an ensemble of interacting resonators

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ABSTRACT. We study the behaviour of two clusters of an ensemble of N interacting resonators, and show the existence of two periodical solutions with different stability and different values for the period. In one of this periodical orbits all the oscillators (one cluster) have the same phase ϕ, and period depending on N This is a (doubly) unstable state : For two clusters, that is, if the former one is perturbed in such a way that $N_2(< N/2)$ of them take a different value ϕ_2 of the phase (arbitrarily close to the initial phase ϕ_1 of the other $N_1 = N - N_2$), the system will be found in the limit $t \to \infty$, over a doubly stable periodical orbit. Unless $N_2 = N_1 = N/2$, in this periodical orbit the asymptotical value of the phase difference oscillates around a certain fixed value. Only when $N_2 = N_1$ (two clusters with the same weight) is this phase difference constant and equal to π : the ensemble is then split into two sub-ensembles in opposition of phases, with the same number of resonators, and oscillating with the "unperturbed" value 2π of the period. The behaviour of the ensemble in the neighbourhood of these two periodical orbits is studied by means of the Poincaré sequence functions. It is seen that the system leaves very slowly the neighbourhood of the unstable limit cylce, while the stable one seems to be attained more quickly (at least when $N_1 = N_2$).

RÉSUMÉ. On étudie les propriétés d'un ensemble de N résonnateurs en interaction et on démontre l'existence de deux solutions périodiques, lesquelles diffèrent par le type de stabilité et par la valeur de leurs périodes. Dans une de ces solutions, tous les oscillateurs se trouvent avoir la même phase ϕ, et leur période est une fonction de N. Cette solution est doublement stable en ce sens que si un certain nombre $N_2(< N/2)$ d'oscillateurs, possédant tous la même phase ϕ_2, se détache du reste de l'ensemble ($N_1 = N - N_2$ oscillateurs avec phase ϕ_1), les deux populations se trouveront, à la limite $t \to \infty$, *sur une orbite périodique doublement stable. Dans ces conditions (et sauf si $N_1 = N_2$) la valeur limite de la différence de phase oscille autour d'une certaine valeur fixe. Dans le cas des deux populations égales ($N_1 = N_2$) cette différence de phase est constante et égale à π, c'est-à-dire que l'ensemble est également distribué en deux sous-ensembles oscillant en opposition de phase et avec la valeur 2π de période "non perturbée". On calcule des fonctions de séquence de Poincaré pour étudier la stabilité de ses solutions périodiques et on trouve que le système s'écarte très lentement du voisinage du cycle limite instable, alors que dans le cas du cycle limite stable il s'en approche plus rapidement (au moins lorsque $N_1 = N_2$).*

The behaviour and self-organization of an ensemble of N oscillators in interaction has been treated by the author by first making use of a pulsatile model (oscillators as Andronov cloks (1). See also (2) for the case $N = 2$, the so-called Huyghens effect, treated in (3) in a more generalized way) and then with resonators, that is, oscillators generating stationary harmonic waves (4). The equations of evolution, the bifurcational behaviour and some other essential properties were presented. Yet no attention was given (out of the case $N = 2$) to periodical states and stability, a problem which is tackled in the present note.

In the phase plane $x, y \equiv \dot{x}$ the evolution of the representative point (RP) of a perturbed harmonic oscillator

$$\dot{x} = y \quad , \quad \dot{y} = -x + f(t)$$

is given (for small $f(t)$) by a slightly modified unit circle with center in the origin described in clockwise sense. In polar coordinates $x = r\cos\phi \quad y = r\sin\phi$, we have

$$\dot{r} = f(t) \cdot \sin\phi \quad , \quad \dot{\phi} = -1 + \frac{\cos\phi}{r} \cdot f(t)$$

Now it is well known that even for $f(t)$ very small the RP may show some instability, in the sense that in the course of time it may get out of an arbitrarily small neighbourhood of the circle $x^2 + y^2 = 1$. In order to avoid such possibility we assume that the oscillator is coupled with some automatic mechanism (viz., similar to the escapment of the clocks - see (5)) acting whenever the RP goes through a certain fixed value of the phase (for instance, $\phi = 0$) and that will instantaneously bring the RP on the corresponding unperturbed value of the radius ($r = 1$)

without modification of the corresponding phase $\phi = 0$. For this kind of oscillator (employed in (4)) we have $r(t) \cong 1 (\forall t)$, and its state is then merely given by the value of the phase according to

$$\dot{r} = 0 \quad (r = 1) \Rightarrow \quad \dot{\phi} = -1 + \cos\phi \cdot f(t)$$

We also assume that the oscillator is a resonator, in the sense that along its oscillating motion a stationary harmonic wave is generated, with very weak amplitude κ and phase equal to that of the oscillator itself. If we now consider a great number $N \gg 1$ of identical resonators of this kind, each one generating its own stationary wave (with the same value of $\kappa \ll 1$ for all the waves), any oscillator of the ensemble will then be perturbed by the superposition of these $N - 1 \cong N$ small waves. In order to keep the above mentioned hypothesis of the stability of each individual oscillator we need to assume that the amplitude of this perturbing remains small. In other words, we take $\kappa N \ll 1$.

Needless to say, such a model is particularly useful for the description of the creation of a coherent, high intensity wave in a medium in which a great number of identical (apparently) random incoherent, low intensity waves coexist in weak interaction. Such phenomena are frequent in biophysics (where the small identical waves are the electrical waves generated by each nerve cell) and geophysics (where the small waves occur suddenly in a very strong coherent effect, as happens with certain earthquakes).

Let us then consider the N oscillators of the ensemble split into two different states, with N_1 of them having the same value ϕ_1 of the phase, while the remaining $N_2 = N - N_1$ have all the same phase ϕ_2. Each one of the N_1 oscillators is thus perturbed by a small wave

$$\kappa(N_1 - 1)\ \sin\phi_1 + \kappa N_2 \sin\phi_2 \cong \kappa N_1 \sin\phi_1 + \kappa N_2 \sin\phi_2 \equiv f(t)$$

arising from the other $N - 1$ resonators. The same is true for any among the N_2 oscillators with phase ϕ_2, since we have

$$\kappa N_1 \sin\phi_1 + \kappa(N_2 - 1) \sin\phi_2 \cong \kappa N_1 \sin\phi_1 + \kappa N_2 \sin\phi_2 \equiv f(t)$$

The evolution in time of phases ϕ_1 and ϕ_2 is then given by

$$\dot{\phi}_1 = -1 + \cos\phi_1 \cdot (\kappa N_1 \sin\phi_1 + \kappa N_2 \sin\phi_2)$$

$$\dot{\phi}_2 = -1 + \cos\phi_2 \cdot (\kappa N_1 \sin\phi_1 + \kappa N_2 \sin\phi_2)$$

The aim of the present paper is the study of this dynamical system on the torus $\phi_1 \in [0, 2\pi), \phi_2 \in [0, 2\pi)$, namely the existence of closed orbits, their behaviour and stability.

An obvious periodic solution is obtained when both phases coincide: $\phi_1 = \phi_2 \equiv \phi$. In this case ϕ is given by

$$\dot{\phi} = -1 + \cos\phi \cdot \kappa N \sin\phi = -1 + \frac{1}{2}\kappa N \sin 2\phi$$

The integration is easily performed and leads to

$$\phi(t) = tg^{-1}\left[\frac{\cos(-\varepsilon + (t-c)\cos\varepsilon)}{\sin((t-c)\cos\varepsilon)}\right]$$

where we have defined

$$\varepsilon \equiv \sin^{-1}\left(\frac{\kappa N}{2}\right) \qquad (\kappa N \ll 1)$$

$$c \equiv t_0 + \frac{1}{\cos\varepsilon}\tan^{-1}\left[\frac{\cos\varepsilon}{\sin\varepsilon - \tan(\phi(t=t_0))}\right]$$

This is a periodical orbit, with a period that increases with N and reduces to the unperturbed value 2π when $\kappa \to 0$:

$$\tau \equiv \frac{2\pi}{\cos\varepsilon} = \frac{2\pi}{\sqrt{1 - \left(\frac{\kappa N}{2}\right)^2}} \cong 2\pi + \pi\left(\frac{\kappa N}{2}\right)$$

In order to discuss the existence of periodic solutions other than in phase coincidence, let us define

$$X_1 \equiv \phi_1 + \phi_2 \qquad X_2 = \phi_1 - \phi_2$$

where $\frac{1}{2}X_1$ is the phase of a RP "equidistant" of phases ϕ_1 and ϕ_2, that is, the phase of the center of the interval $(\phi_1(t), \phi_2(t))$, while X_2 is the "length" of that same phase interval. In these new variables our dynamical system takes the form

$$\dot{X}_1 = -2 + A\sin X_1 \cdot (1 + \cos X_2) + B \sin X_2 \cdot (1 + \cos X_1)$$

$$\dot{X}_2 = B\sin X_1 \cdot (-1+\cos X_2) + A\sin X_2 \cdot (-1+\cos X_1)$$

with both X_1 and X_2 $2\pi-$ periodical on the torus $X_1 \in [0,2\pi), X_2 \in [0,2\pi)$ and the small constants A, B given by

$$A \equiv \frac{\kappa}{2}(N_1+N_2) = \frac{\kappa}{2}N \qquad B \equiv \frac{\kappa}{2}(N_1-N_2) \geq 0$$

(In what follows we assume that $N_1 > N_2$). From the o.d.e. for the orbits of the dynamical system,

$$\frac{dX_2}{dX_1} = \frac{B\sin X_1(-1+\cos X_2) + A\sin X_2(-1+\cos X_1)}{-2 + A\sin X_1(1+\cos X_2) + B\sin X_2(1+\cos X_1)},$$

we obtain three isoclines with null slope :

$$\frac{dX_2}{dX_1} = 0 \Leftrightarrow \begin{cases} X_1 = 0 \\ X_2 = 0 \\ \Gamma : f(X_2) = -\frac{B}{A} f(X_1), \quad \text{with} \quad f(\rho) \equiv \frac{\sin\rho}{-1+\cos\rho} \end{cases}$$

(The isocline $X_2 = 0$ corresponds to the above mentioned periodic orbit). As for the isocline Γ, if we write this curve under the explicit form $\Gamma : X_2 = g(X_1)$, we easily obtain some of its properties, namely

$$g(X_1 = 0) = 2\pi \qquad g(X_1 = 2\pi) = 0 \qquad -\infty < cte < \frac{dg}{dX_1} < 0$$

$$\left.\frac{dg}{dX_1}\right|_{X_1=0} = \left.\frac{dg}{dX_1}\right|_{X_1=2\pi} = -\frac{A}{B}$$

(NB : $\frac{d^2X_2}{dX_1^2}$ is null only in the intersection of Γ with $X_2 = X_1$). The curve also shows some symmetry, for if any point X_1, X_2 belongs to it, then the same happens with $2\pi - X_1, 2\pi - X_2 (X_1 = X_2 = \pi$ belongs to Γ). Besides, Γ clearly divides the torus X_1, X_2 in two regions where the trajectories have different signal for the derivatives $\frac{dX_2}{dX_1}$.

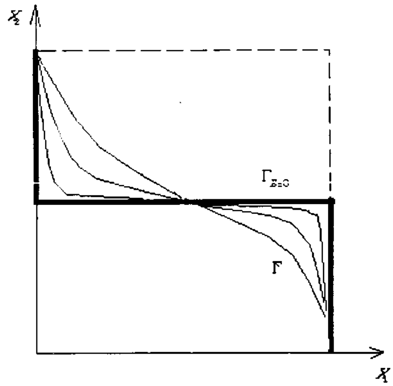

Figure 1.

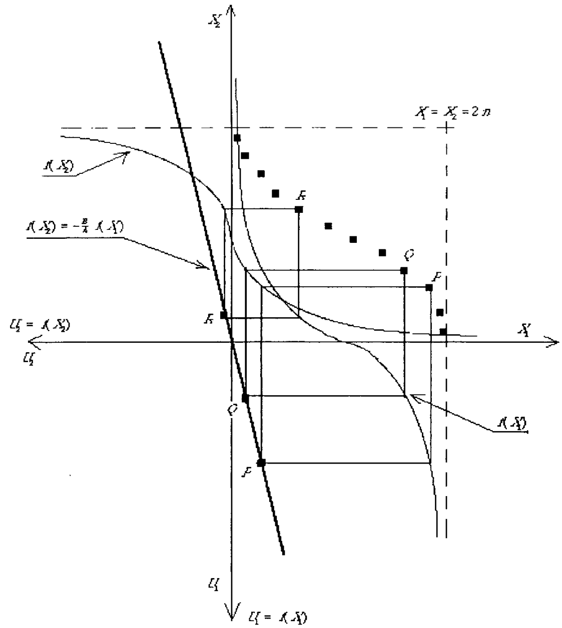

Figure 2.

Figure 1 shows isocline Γ (for different values of the parameter B) which can also be directly obtained by making use of a straightforward geometrical method. This is done in figure 2, where we represent four diagrams in the four quadrants : One with function $U_1 = f(X_1)$, another with $U_2 = f(X_2)$, another one with the straight line $U_2 = -\frac{B}{A}U_1$, and finally this same diagram in coordinates X_1, X_2, that is, the curve Γ itself.

We now study the behaviour of the dynamical system in the neighbourhood of the periodical orbit $X_2 = 0$, that is, for $0 < X_2 \equiv \eta \ll 1$. if we develop in powers of η the o.d.e. for the orbits, we find

$$\frac{d\eta}{dX_1} = \frac{A}{2} \cdot \frac{(-1+\cos X_1)}{(-1+A\sin X_1)} \cdot \eta + \frac{B}{4} \cdot \frac{\sin X_1}{(1-A\sin X_1)^2} \cdot \eta^2 + 0(\eta^3)$$

Now the truncated linear equation

$$\frac{d\nu}{dX_1} = \frac{A}{2} \cdot \frac{(-1+\cos X_1)}{(-1+A\sin X_1)} \cdot \nu$$

in easily integrated, leading to the solution

$$\nu = Cte \cdot (1 - A\sin X_1)^{+\frac{1}{2}} \cdot \exp\left[+\frac{A}{\sqrt{1-A^2}} \cdot \tan^{-1}\left(\frac{-A+\tan\frac{X_1}{2}}{\sqrt{1-A^2}}\right)\right]$$

Then the method of variation of the constants provides the general solution for $\eta(X_1)$:

$$\eta(X_1) = \frac{(1-A\sin X_1)^{-\frac{1}{2}} \cdot \exp\left[\frac{A}{\sqrt{1-A^2}} \cdot \tan^{-1}\left(\frac{-A+\tan\frac{X_1}{2}}{\sqrt{1-A^2}}\right)\right]}{C - \frac{B}{4} \cdot I(X_1)} \times \text{Cte}$$

where C is an arbitrary constant and $I(X_1)$ is given by

$$I(X_1) \equiv \int_0^{X_1} \frac{\sin s}{(1-A\sin s)^{\frac{5}{2}}} \cdot \exp\left(-\frac{A}{\sqrt{1-A^2}}\tan^{-1}\left(\frac{-A+\tan\frac{s}{2}}{\sqrt{1-A^2}}\right)\right) ds$$

In order to know wether we have stability or not for the periodical solution $X_2 = 0$ we just have to follow an orbit with initial condition

$$X_2(X_1 = 0) \equiv \eta(X_1 = 0) = \delta \ll 1$$

and calculate $\eta(X_1 = 2\pi - 0) \equiv \delta'$. The considerations above concerning the shape and properties of Γ, then allow us to conclude stability if $\delta' < \delta$ and instability for $\delta' > \delta$. Now the particular solution satisfying $\eta(X_1 = 0) = \delta$ is easily found to be

$$\eta(X_1) = \frac{(1-A\sin X_1)^{-\frac{1}{2}} \cdot \exp\left[+\frac{A}{\sqrt{1-A^2}.} \cdot \tan^{-1}\left(\frac{-A+\tan\frac{X_1}{2}}{\sqrt{1-A^2}}\right)\right]}{\delta^{-1} \cdot \exp\left(+\frac{A}{\sqrt{1-A^2}}\tan^{-1}\left(\frac{+A}{\sqrt{1-A^2}}\right)\right) - \frac{B}{4} \cdot I(X_1)}$$

and we obtain

$$\begin{aligned}\delta' &\equiv \eta(X_1 = 2\pi - 0) \\ &= \left[\delta^{-1} - \frac{B}{4} \cdot I(X_1 = 2\pi) \cdot \exp\left(\frac{A}{\sqrt{1-A^2}}\tan^{-1}\left(\frac{A}{\sqrt{1-A^2}}\right)\right)\right]^{-1}\end{aligned}$$

We develop in powers of the small constant A the expression of

$$\exp\left(\frac{A}{\sqrt{1-A^2}}\tan^{-1}\left(\frac{A}{\sqrt{1-A^2}}\right)\right) = 1 + 0(A^2)$$

and

$$I(X_1 = 2\pi) \equiv \int_0^{2\pi} \sin s \cdot \left(1 + \frac{5}{2} A \sin s\right) \times$$
$$\times \exp\left(-A \cdot \left(\frac{s}{2} - A\cos^2 \frac{s}{2}\right)\right) ds + 0(A^3) =$$
$$= \frac{5}{2}\pi A + 0(A^2)$$

and finally obtain δ' :

$$\delta' = \delta'(\delta) \equiv \eta(X_1 = 2\pi - 0) = \delta(1 + \frac{5}{8}\pi AB \cdot \delta) + 0(A^2) > \delta$$

In order to assure that the periodical orbit $X_2 = 0$ is totally unstable (that is, doubly unstable and not merely semi-unstable), we should also verify instability in the neighbourhood $2\pi - \xi < X_2 \leq 2\pi \quad (0 \leq \xi \ll 1)$. This can be done in a similar way, and the result is also instability. More precisley, a trajectory starting from $X_2(X_1 = 0) = 2\pi - \mu \quad (0 < \mu \ll 1)$ will return over $X_1 = 0$ with $X_2(X_1 = 2\pi) \equiv 2\pi - \mu'$, where μ' is given by

$$\mu' = \mu'(\mu) = \quad \mu(1 + \frac{1}{8}\pi AB \cdot \mu \cdot) + 0(A^2) > \mu.$$

It is now clear that (at least) one peridical orbit must then exist (other than $X_2 = constant = 0$) with a non-constant value for X_2. (The study of the limit case $N_1 = N_2$ will enable us to conclude the uniqueness of this periodical orbit).

Let us then consider $B = 0$. Since $N_1 = N_2 = N/2$ we now have

$$\dot{\phi}_1 = -1 + \frac{\kappa N}{2} \cos\phi_1 \cdot (\sin\phi_1 + \sin\phi_2)$$
$$\dot{\phi} = -1 + \frac{\kappa N}{2} \cos\phi_2 \cdot (\sin\phi_1 + \sin\phi_2)$$

that is, in coordinates X_1, X_2,

$$\dot{X}_1 = -2 + A \sin X_1 \cdot (1 + \cos X_2)$$
$$\dot{X}_2 = A \sin X_2 \cdot (-1 + \cos X_1).$$

The orbits in the torus are now given by

$$\frac{dX_2}{dX_1} = \frac{A \sin X_2 \cdot (-1 + \cos X_1)}{-2 + A \sin X_1 \cdot (1 + \cos X_2)}$$

and we find for the null slope isoclines

$$\frac{dX_2}{dX_1} = 0 \quad \Leftrightarrow \qquad X_1 = 0 \qquad X_2 = 0, \quad X_2 = \pi$$

This means that Γ now degenerates into the form $\Gamma_{B=0}$ (the broad line shown in figure 1) dividing the torus in two halves in which we have for the derivatives

$$X_2 \in (0, \pi), \quad \forall X_1 \Rightarrow \frac{dX_2}{dX_1} > 0 \qquad X_2 \in (\pi, 2\pi), \quad \forall X_1 \Rightarrow \frac{dX_2}{dX_1} < 0$$

The only possible periodical orbits are now

$$X_2 = 0 \qquad \dot{X}_1 = -2 + \kappa N \sin X_1$$

(which reduces to the above considered case of the phase coincidence), and

$$X_2 = \pi, \qquad X_1 : \dot{X}_1 = -2$$

that is, the opposition of the phases, with the whole ensemble vibrating as a non-perturbed oscillator. The stability behaviour of the RP in the neighbourhood of this ortbit is easily inspected and we find :

$$v : 0 < |v| \ll 1 \qquad X_2(X_1 = 0) \equiv \pi + v \Rightarrow X_2(X_1 = 2\pi) \equiv \pi + v'$$

and we find a doubly stable limit cycle :

$$|v'| = |v| \cdot \exp(-\pi A) < |v|$$

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