De Broglie – Bohm Quantum Theory and Perihelion Precession

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ABSTRACT. By studying the central force problem in the framework of de-Broglie–Bohm theory, we observe that the perihelion precession of planets may be understood as the effect of the quantum potential. In fact, we show that there is a specific solution to de-Broglie–Bohm equations, in which the radial dependence and the sign of the quantum potential is exactly what is needed for the perihelion precession. The free parameters can be chosen such that the correct value of precession is obtained.

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1 Introduction

de-Broglie–Bohm mechanics[1] is a *single-event* theory underlying quantum mechanics, which is a statistical theory. The relation between the two theories is just like the relation between kinetic theory and thermodynamics. Note that one needs to add some statistical postulates to the dynamics of single events in the kinetic theory to arrive at thermodynamics. In general one has the following formula:

SINGLE-EVENT THEORY + STATISTICAL POSTULATE(S) =
STATISTICAL THEORY

In de-Broglie–Bohm mechanics, any particle is always accompanied by an *objectively real* field, $\psi(\vec{x},t)$, satisfying an appropriate field equation (e.g. the Schrödinger equation for a non-relativistic particle). This field exerts a force –the quantum force– on the particle, which is given by:

$$\vec{F}_Q = -\vec{\nabla} Q \qquad ; \qquad Q = -\frac{\hbar^2}{2m}\frac{\nabla^2|\psi|}{|\psi|} \tag{1}$$

where Q is the quantum potential.

The extension to many-particle systems is straightforward. One must assume that ψ is a function of the position of all of the particles, i.e. $\psi(\vec{x}_1, \cdots, \vec{x}_n; t)$.[1] The quantum force exerted on the i-th particle is given by:

$$\vec{F}_{Qi} = -\vec{\nabla}_i Q \qquad ; \qquad Q = \sum_{i=1}^{n} -\frac{\hbar^2}{2m_i}\frac{\nabla_i^2|\psi|}{|\psi|} \tag{2}$$

As it was stated previously, there must be some statistical postulate(s) to arrive at the statistical aspect of de-Broglie–Bohm mechanics, i.e. quantum mechanics. This statistical postulate is:

$$\rho(\vec{x}_1, \cdots, \vec{x}_n; t) = |\psi(\vec{x}_1, \cdots, \vec{x}_n; t)|^2 \tag{3}$$

where ρ is the ensemble density of the system under consideration. There has been a lot of work to prove this postulate.[2]

An essential feature of the quantum potential is that it is not a predefined function of space and time. Its dependence on the position and time, depends upon the form of ψ. It varies from a specific solution to another. It is this feature of the quantum potential, that enables it to cancel almost any classical potential (i.e. the state in which $V + Q =$constant), leading to stationary states, i.e. the cases in which the particle is at rest.

In this way, de-Broglie and Bohm were able to explain the stability of atoms. In the ground state of atoms, the quantum force just cancels the Coulomb force. Thus the electron is at rest and does not emit electromagnetic radiation. So the atom is stable. In general, for any stable state of atoms, electron is either at rest or moving uniformly.

The above example considers the motion of a two-body system under the central Coulomb force. Now an important question arises. What about the gravitational interactions? That is, what is the stable solutions of a two-body system under their central gravity force? Certainly, the situation differs from the case of atoms, because the gravitational

[1] There is an important objection to the theory, here. How can a field in the configuration space, be an objectively real field exerting a force on the particles?

radiation (if exists) is very small and can be ignored. So there are approximately stable states, in which the particles have acceleration. In particular we are interested in the semi-classical domain, i.e. where the quantum force is very small in comparison to the classical force.[3] In this case the motion is Keplerian motion perturbed by small quantum effects.

In the present paper, we shall try to answer the above question. It will be shown that there exists stable solutions in which the quantum potential is just what is needed to have the correct (both in sign and in value) perihelion precession. This shows that this particular phenomenon which supports the general relativity *may* be explained as a quantum effect.

2 Gravitational force in de Broglie–Bohm mechanics

Our aim in this section, is to find a solution for de-Broglie–Bohm equations of motion with the central gravitational force, such that the motion only differs slightly from the classical one. Consider a two-body system, with the classical attraction:

$$V(\vec{x}_1, \vec{x}_2) = -\frac{Gm_1m_2}{|\vec{x}_1 - \vec{x}_2|} \equiv -\frac{k}{r} \tag{4}$$

To solve this problem as a de-Broglie–Bohm one, we must either solve the Schrödinger equation or solve the de-Broglie–Bohm equations of motion[1, 3]:

$$m_i\frac{d^2\vec{r}_i}{dt^2} = -\vec{\nabla}_i(V+Q); \qquad i=1,2 \tag{5}$$

and the continuity equation:

$$\frac{\partial\rho}{\partial t} + \sum_{i=1}^{2}\vec{\nabla}_i\cdot\left(\rho\frac{d\vec{r}_i}{dt}\right) = 0 \tag{6}$$

Equations (5) and (6) are identical to the Schrödinger equation.[3]

Since the potential (4) is a function of the relative distance r, the Schrödinger equation admits solutions in which:

$$|\psi| = |\psi|(r) \Longrightarrow Q = Q(r) \tag{7}$$

For such a solution, the central character of the problem survives. This is not the general case. $|\psi|$ can depend on θ and ϕ and so Q. Therefore the quantum potential can breaks down the classical symmetry. This is a general feature of quantum potential that can destroy the classical symmetries.

As in the classical case, the motion can be divided to the center of mass motion (which is either at rest or moving uniformly, because the total external force is zero) and the relative motion. Now let us to investigate the effect of a small ϕ-dependence of the quantum potential. This is interested to us because we search for a solution that differs from the classical one, slightly. Assume that the quantum potential is a function of spherical coordinates, r, θ and ϕ. Then the equations of motion are:

$$\mu\frac{d^2r}{dt^2} - \mu r\left(\frac{d\theta}{dt}\right)^2 - \mu r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2 + \frac{dV}{dr} + \frac{\partial Q}{\partial r} = 0 \qquad (8)$$

$$\frac{d}{dt}\left(\mu r^2\frac{d\theta}{dt}\right) - \mu r^2\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2 + \frac{\partial Q}{\partial\theta} = 0 \qquad (9)$$

$$\frac{d}{dt}\left(\mu r^2\sin^2\theta\frac{d\phi}{dt}\right) + \frac{\partial Q}{\partial\phi} = 0 \qquad (10)$$

where μ is the reduced mass:

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \qquad (11)$$

If $Q \neq Q(\theta)$, the solution $\theta = \frac{\pi}{2}$ is a stable solution of equation (9). That is, the motion is restricted to a plane.

Now suppose that the quantum potential has a small ϕ-dependence:

$$Q = Q_1(r) + \epsilon\phi Q_2(r) \qquad (12)$$

where ϵ is a small number. Equation (10) can be integrated to:

$$\ell^2 = \ell_0^2 - 2\epsilon\mu\int d\phi r^2 Q_2(r) \qquad (13)$$

where $\ell = \mu r^2 d\phi/dt$ is the actual angular momentum, and ℓ_0 is the orbital angular momentum ignoring the ϕ-dependence of Q. Since we search for a nearly classical solution and since the classical solution is periodic, it can be seen that ℓ is a periodic function. So as an approximation we can use the average value of the orbital angular momentum in the equations of motion.

One can combine the equations of motion to obtain the orbit equation:

$$\frac{d^2u}{d\phi^2} + u \simeq -\frac{\mu}{\overline{\ell}^2}\frac{d}{du}\left(V(\frac{1}{u}) + Q(\frac{1}{u})\right) \tag{14}$$

where $u = 1/r$, and $\overline{\ell}$ is the average value of the angular momentum. This is an approximate equation since we have ignored the ϕ–dependence of Q.

The continuity equation leads to:

$$\frac{d}{dr}\left(\frac{|\psi|^2}{r}\right) \simeq 0 \tag{15}$$

Now consider for $|\psi|$ a solution in the form:

$$|\psi| = r^n e^{-\alpha r} \tag{16}$$

where α and n are constants. This wavefunction satisfies the appropriate boundary conditions. The solution (16) is a valid solution of equation (15) provided $|\psi|$ peaks about:

$$r \simeq \frac{2n-1}{2\alpha} \tag{17}$$

The quantum potential arising from (16) is:

$$Q(r) = -\frac{\hbar^2}{2\mu}\frac{\nabla^2|\psi|}{|\psi|} = -\frac{\hbar^2}{2\mu}\left(\frac{n(n-1)}{r^2} - \frac{2n\alpha}{r} + \alpha^2\right) \tag{18}$$

So the quantum potential consists of a constant term which can be ignored, a term just like the Newtonian gravity, and a term proportional to $1/r^2$. Using this quantum potential, the orbit equation (14) reads as:

$$\frac{d^2u}{d\phi^2} + \left(1 - \frac{\hbar^2}{\overline{\ell}^2}n(n-1)\right)u \simeq \frac{\mu}{\overline{\ell}^2}\left(k - \frac{\hbar^2\alpha}{\mu}n\right) \tag{19}$$

which has the solution:

$$u = \frac{1}{r} = \frac{1}{r_0}\left(1 + e\cos(\omega\phi)\right) \tag{20}$$

where:

$$r_0 = \frac{\overline{\ell}^2 - \hbar^2 n(n-1)}{k\mu - \hbar^2\alpha n} \tag{21}$$

$$\omega = \sqrt{1 - \frac{\hbar^2}{\overline{\ell}^2} n(n-1)} \tag{22}$$

where e is the eccentricity. Since the eccentricity is small (for an ellipse it is between zero and one), equations (21) and (17) could be consistent if one demands:

$$\alpha = \frac{k\mu}{2\overline{\ell}^2} \frac{2n-1}{1+\hbar^2 n/2\ell^2} \tag{23}$$

The orbit (20) is an ellipse with an advanced perihelion precession given by:

$$\delta\phi = \frac{\pi\hbar^2}{\overline{\ell}^2} n(n-1) \qquad \text{per revolution} \tag{24}$$

Note that:

$$r_0 \simeq (r)_{classical} \equiv \frac{\overline{\ell}^2}{k\mu} \tag{25}$$

$$\alpha \simeq \frac{n}{(r)_{classical}} \tag{26}$$

So the wavefunction given by (16) represents a packet localized at $(r)_{classical}$.

It is obvious that the preccesion value can be determined by choosing an appropriate value of n. In order to interpret the mercury prihelion preccesion in this way one should set:

$$n = 34 \times 10^{68} \tag{27}$$

so that:

$$\alpha = 5.9 \times 10^{68} \text{ m}^{-1} \tag{28}$$

This leads to the correct value 41 seconds of arc per century.

The conclusion is that, *if one considers that the wavefunction is a packet localized at the classical solution, the quantum potential produces an $1/r^3$ force which forces the orbit to have precession with the correct sign, which given by experiment and by general relativity*. This is not a strange result because as it has been shown currently[4], that there is a very close connection between gravity and de-Broglie–Bohm quantum theory.

3 Concluding remarks

As we saw in the previous section, the quantum potential for a two-body gravitational system produces an $1/r^3$ attractive force, which in its turn causes an advanced perihelion precession. The free parameter n can be chosen in such a way that the observed value of precession would result.

The ability of the quantum force to describe this motion can be understood physically. Consider that, in general, the solution to the Schrödinger equation be a wave-packet around the classical solution:

$$|\psi| = f(|\vec{r}-\vec{r}_{classical}|) = f(0)+f'(0)|\vec{r}-\vec{r}_{classical}|+\frac{f"(0)}{2}|\vec{r}-\vec{r}_{classical}|^2+\cdots \tag{29}$$

where $f(0)$ is positive and $f'(0) = 0$ and $f"(0)$ is negative (otherwise f would not be a packet). The quantum force is then given by:

$$\vec{F}_Q = -\frac{6\hbar^2}{m}\frac{\vec{r}-\vec{r}_{classical}}{|\vec{r}-\vec{r}_{classical}|^4} \tag{30}$$

If we assume that $\vec{r}$ differs from $\vec{r}_{classical}$ slightly and set $\vec{r} \simeq (1+\epsilon)\vec{r}_{classical}$ the result would be an attractive force proportional to the inverse of the cube of r. This means we have an advanced precession.

Note that this is a result of de-Broglie–Bohm theory. It can not be achieved from the standard quantum mechanics, at least in this way. This is because, if one insists to *define* a path for particles in the standard quantum mechanics, two ways are imaginable. One consistent way is to use the Ehrenfest theorem. But, it can be easily shown that the effective motion given by the Ehrenfest theorem, i.e. $\left\langle \hat{\vec{x}} \right\rangle$, and the trajectories of de-Broglie–Bohm theory differ dramatically. In fact $\left\langle \hat{\vec{x}} \right\rangle$ is a constant for a stationary state. Another way to define path in the standard quantum mechanics, is to use the concept of the effective equations of motion.[5] Let us define the effective position (usually called the classical position) as:

$$\tilde{\vec{x}}[\vec{J}] = \frac{\left\langle \hat{\vec{x}} e^{i\vec{J}\cdot\hat{\vec{x}}} \right\rangle}{\left\langle e^{i\vec{J}\cdot\hat{\vec{x}}} \right\rangle} \tag{31}$$

It can be easily shown that:

$$\tilde{\vec{x}}[\vec{0}] = \left\langle \hat{\vec{x}} \right\rangle \tag{32}$$

and that $\tilde{\vec{x}}$ satisfies the effective equations of motion:

$$\left(\frac{\delta\mathcal{A}}{\delta\vec{x}}\right)_{\vec{x}\to\tilde{\vec{x}}-i\hbar\frac{\delta}{\delta\vec{J}}}+\vec{J}=0 \tag{33}$$

where $\mathcal{A}$ is the classical action. Again, the motion given by $\tilde{\vec{x}}$, differs from the one predicted in de-Broglie–Bohm theory. In particular for a harmonic oscillator, the classical and the effective equation of motions are identical. But for this problem, the quantum potential of de-Broglie–Bohm theory, is not necessarily zero. The difference between these two motions, i.e. the effective and de-Broglie–Bohm motions, is considered more carefully elsewhere.[6]

We conclude this paper by observing that the de-Broglie–Bohm mechanics can provide a different picture for observations supporting general relativity. So it seems that there must be some fundamental theory leading to the both gravity and quantum. It is obvious that the present work *only shows the possibility of this* and it can be used as a motivation for the search for such a fundamental theory.

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