A classical interpretation of Bell's inequality

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ABSTRACT. The author has been developing a classical particle model theory which satisfies almost all the requirements of quantum mechanics (QM) including known unsolved logical problems except the superposition principle. To make such a classical approach acceptable, it is necessary to examine how the model theory responds to Bell's inequality. In this paper a lack of required symmetry in positive detection pairs of correlated particles and ill-defined nature of the inequality are identified following the logic of classical physics.

RÉSUMÉ. L'auteur a développé un modèle de particule classique, qui satisfait presque toutes les exigences de la mécanique quantique, y compris des problèmes logiques non résolus, à l'exception du principe de superposition. Pour qu'une telle approche classique soit acceptable, il faut examiner comment cette théorie répond à l'inégalité de Bell. Dans cet article, en suivant la logique de la physique classique, on met en évidence un défaut de symétrie dans la détection de paires de particules corrélées ainsi que la nature mal définie de l'inégalité.

1. Motivation for this work

Based on an earlier work [1], the author has been developing a classical particle model theory, which is consistent with almost all required particle properties in QM, including known unsolved logical problems, but with exception of the superposition principle. Interestingly enough, Penrose has independently questioned the applicability of the principle in dealing with spacetime geometries in Hawking's quantum black hole model [2]. Therefore, the famous superposition principle can be a suspect as more fully investigated later in another paper. Although such an exclusion of the principle may suggest imposition of serious detriment and hindrance to QM theoretical formulation, the new model theory remains consistent with the currently established results of QM theories and supports all the basic facts without any need for alteration except the concept of mixed states. The model particle satisfies Lorentz transformations in its internal microscopic formulation, and yet macroscopically exhibits Galilean invariance. A comprehensive modeling of three generations of particles and a cosmological model both can logically be deduced from the simple model stipulations, and gravitational effects can easily be modeled. These discussions are left to other papers.

Bell's inequality has been recognized as the cornerstone of QM triumph over classical physics. Because of this, it is necessary to determine how the model theory reacts to the inequality as investigated in this paper. Lack of required symmetry and ill-defined nature of Bell's inequality are identified through the classical reasoning developed in the particle model theory. Without recognition of the simple but critically important results presented in this paper, phycisits in favor of QM would likely remain skeptical of the merit of the proposed particle model theory.

2. Preliminaries

In case of spin-1/2 particle pair, QM predict with certainty that, if measurement of the x-component $\sigma_x(1)$ of the spin for the particle-1 yields +1, the measurement of $\sigma_x(2)$ yields -1 for the partner particle-2. However, the actual chance of measurement yielding a positive detection result decreases in proportion to the cosine $\cos \boldsymbol{\sigma}(i) \cdot \mathbf{x}$ of the angle between the true spin orientation unit vector $\boldsymbol{\sigma}(i)$ of the i-th particle, i=1,2, and the detector orientation unit vector $\mathbf{x}$. This fact is incorporated in Bell's prediction formula for the case of quantum mechanics, and in the comparison between the predictions using a hidden variable and quantum mechanics.

In Bell's presentation the role of the hidden variable λ in $A(\mathbf{a}, \lambda)$ is not clearly stated. In this paper the author makes a classical assumption that each undisturbed classical particle pair satisfies a particular set of initial conditions defined by the initial spin (or polarization) orientation values $\boldsymbol{\sigma}(i)$, i=1,2, which remain unchanged until they reach detectors. Note that this assumption is not in agreement with general QM concepts and violates Bell's local hidden variable concept. However, it nicely eliminates the problem of communication between the pair at a speed faster than the speed of light, and requires no introduction of the λ. So, the introduction of initial conditions in classical sense immediately solves this logical problem long recognized in QM.

To investigate the symmetry of the spin correlation on three positive detection pairs, the incorporation of the cosine detection factor simply complicates the issue. It is the best to specify the angular range of true spin orientation as within $\pm\pi/2$ of the detector orientation for yielding a positive detection, and then analyze the angular ranges over which positve detections are declared. In this way, the computation of mathematical expectations with respect to random spin orientation can be substituted by interval statements. The true spin orientation $\boldsymbol{\sigma}(1)$ of the first particle can be assumed uniformly distributed over $[-\pi, \pi]$, and similarly for the second particle with $\boldsymbol{\sigma}(2) = \boldsymbol{\sigma}(1) + \pi$. However, the assumption of uniformity is not required so long as the assumed density is continuous.

3. Bell's inequality

Let **a**,**b**,**c** denote three unit vectors identifying the detector orientations. When the first particle A of a pair with a hidden variable λ enters the detector pointed along **a**, and the second particle B with the same λ enters the second detector along **b**, the outcomes of

$$A(\mathbf{a}, \lambda) = \pm 1 \text{ or } 0, \quad \text{and} \quad B(\mathbf{b}, \lambda) = \pm 1 \text{ or } 0, \quad \text{and}$$

are obtained. Bell introduced the locality assumption that the outcome of $A(\mathbf{a}, \lambda)$ does not affect the outcome of $B(\mathbf{b}, \lambda)$, and vice versa. He chose to analyse a set of particle pairs, A and B, simultaneously tested for ± 1's on both detectors, in which the symmetry

$$A(\mathbf{a}, \lambda) = -B(\mathbf{a}, \lambda) \quad \text{and} \quad B(\mathbf{b}, \lambda) = -A(\mathbf{b}, \lambda) \tag{3.1}$$

hold on the detectors pointed along **a** and **b**, and introduced the expectation of joint positive detections of A and B as

$$E(\mathbf{a}, \mathbf{b}) = \int_{\Lambda} A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda) \rho(\lambda) d\lambda \tag{3.2}$$

where $\rho(\lambda)$ is the probability density of the λ. By simple manipulations using (3.1)

$$E(\mathbf{a},\mathbf{b}) - E(\mathbf{a},\mathbf{c}) = \int_\Lambda [A(\mathbf{a},\lambda)A(\mathbf{b},\lambda) - A(\mathbf{a},\lambda)A(\mathbf{c},\lambda)]\rho(\lambda)d\lambda$$
$$= -\int_\Lambda A(\mathbf{a},\lambda)A(\mathbf{b},\lambda)[1 - A(\mathbf{b},\lambda)A(\mathbf{c},\lambda)]\rho(\lambda)d\lambda \tag{3.3}$$

Since $A, B = \pm 1$ must hold for each simultaneously registered pair,

$$|E(\mathbf{a},\mathbf{b}) - E(\mathbf{a},\mathbf{c})| \le \int_\Lambda [1 - A(\mathbf{b},\lambda)A(\mathbf{c},\lambda)]\rho(\lambda)d\lambda \tag{3.4}$$

holds. Using (3.1), (3.2) and $\int \rho(\lambda)d\lambda = 1$, Bell derived the final inequality

$$|E(\mathbf{a},\mathbf{b}) - E(\mathbf{a},\mathbf{c})| \le 1 + E(\mathbf{b},\mathbf{c}) \tag{3.5}$$

4. Deduced symmetry

In order to verify (3.5) the third particle C in

$$E(\mathbf{b},\mathbf{c}) = \int_\Lambda B(\mathbf{b},\lambda)C(\mathbf{b},\lambda)\rho(\lambda)d\lambda$$

by definition must be introduced. Note that the expression of

$$\int_\Lambda A(\mathbf{b},\lambda)A(\mathbf{c},\lambda)\rho(\lambda)d\lambda = -\int_\Lambda B(\mathbf{b},\lambda)A(\mathbf{c},\lambda)\rho(\lambda)d\lambda$$

in (3.4) holds using (3.1), but the unmodified term $A(\mathbf{c},\lambda)$ still remains in it. If a perfect symmetry exists between all paired positive detections along $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, it is necessary to stipulate an universal symmetry between A, B, C particles satisfying

$$\begin{aligned} A(\mathbf{a},\lambda) &= -B(\mathbf{a},\lambda) = -C(\mathbf{a},\lambda) \\ B(\mathbf{b},\lambda) &= -A(\mathbf{b},\lambda) = -C(\mathbf{b},\lambda) \\ C(\mathbf{c},\lambda) &= -A(\mathbf{c},\lambda) = -B(\mathbf{c},\lambda) \end{aligned} \tag{4.1}$$

in correspondence to (3.1). The right hand side of (3.4) then becomes

$$\int_\Lambda [1 - B(\mathbf{b},\lambda)C(\mathbf{c},\lambda)]\rho(\lambda)d\lambda$$

and therefore (3.5) must be modified into

$$|E(\mathbf{a},\mathbf{b}) - E(\mathbf{a},\mathbf{c})| \leq 1 + |E(\mathbf{b},\mathbf{c})| \tag{4.2}$$

to account for the possibility of $E(\mathbf{b},\mathbf{c})$ taking on negative values. With the change of the last term, the (4.2) still produces the same numerical inconsistency $\sqrt{2} \leq 1$ in the prediction that the case with the hidden variable λ yields the same results as the case of quantum mechanics in the particular set of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ with $\mathbf{a} = (\mathbf{b}-\mathbf{c})/|\mathbf{b}-\mathbf{c}|$ where $\mathbf{b} \perp \mathbf{c}$.

Focus of inquiry

Certain logical and notational problems of Bell's presentations [3-6] can now be identified as the reminder :

1) The identity of the said hidden variable λ is unclear, while the λ may directly identify the true particle spin orientation values, σ, $\sigma+\pi$, of each particle pair. Neither the density function $\rho(\lambda)$ nor the set Λ of λ over which the expectation is taken is specified. The $\rho(\lambda)$ may choose a continuous density over $\Lambda = [-\pi, \pi]$ while $\lambda = \sigma$, or $\sigma + \pi$ can be assumed.

2) The notation $E(\mathbf{a},\mathbf{b})$ is confusing, because the expectation operator E should be defined for the random variable λ, while the identifier $(\mathbf{a},\mathbf{b})$ actually indicates a particular subensemble $\Lambda(\mathbf{a},\mathbf{b})$ of the Λ of positive detection pairs satisfying (3.1) for the fixed detector orientations $(\mathbf{a},\mathbf{b})$. The fact that (3.1) represents a conditional expectation over the $\Lambda(\mathbf{a},\mathbf{b})$ is not clearly indicated in the expression.

3) Note that Bell's inequality (3.5) has a general mathematical expression of

$$f(x,y) + f(x,z) \leq f(y,z) \tag{5.1}$$

where $x \in X$, $y \in Y$, and $z \in Z$, $(x,y) \in X \times Y$, $(x,z) \in X \times Z$, and $(y,z) \in Y \times Z$. Such an equation is very uncommon, and unquestionably is insufficiently defined. Unless the three domains of the variable pairs, (x,y), (x,z), and (y,z), of the f are proven exactly identical, (5.1) becomes ill-defined and meaningless. The additivity of the f on the left-hand side of (5.1) is not definable unless $X \times Y = X \times Z$ can be demonstrated. Similarly the inequality is undefinable unless $Y \times Z = X \times Y$ and/or $X \times Z$, or $Y \times Z = X \times Y \times Z$ can be shown.

4) The above argument of a mathematical function fails to apply to $E(\cdot,\cdot)$'s of (3.5), because $(\mathbf{a},\mathbf{b})$, $(\mathbf{a},\mathbf{c})$, and $(\mathbf{b},\mathbf{c})$ identify three distinct

subensembles $\Lambda(\mathbf{a}, \mathbf{b})$, $\Lambda(\mathbf{a}, \mathbf{c})$, and $\Lambda(\mathbf{b}, \mathbf{c})$ of the Λ with these given fixed detector orientations. To have the additivity or equality of two conditional expectations unambiguously defined over Λ_α and Λ_β, the two subensembles Λ_α and Λ_β must satisfy that the set difference $\Lambda_\alpha - \Lambda_\beta$ has zero probability. Otherwise neither the additivity nor equality can be established.

Given the above findings, it becomes necessary to investigate whether the conditional expectations taken over different subensembles can produce any meaningful mathematical relationship between them as indicated in (3.5) under the required symmetry specified in (4.1). If Bell's analysis was confined to particle pairs identified as A and B, no such complication should have arisen, because the detector orientations $\mathbf{a}$ and $\mathbf{b}$ could simply be chosen π apart to satisfy (3.1). However, the necessity of introducing the third particle C as needed in the Section 4 has complicated the problem.

6. Constraints of particle pairs and conditional subensembles

It is necessary to investigate what kind of dependency exists for the specification of spin angular ranges satisfying (4.1). This task can be initiated with the identifications of subensembles from which chosen positive detection pairs can be acquired.

Consider the proposed analysis of angular intervals in which each positive detection must results over the range of $[-\pi/2, \pi/2)$ about the detector orientation. Let σ and $\sigma + \pi$ denote the true spin orientation values of a particle pair over $[-\pi, \pi)$ through hidden specification of the λ (although this role is unclear, unless we assume $\lambda = \sigma$ for one and $\sigma + \pi$ for the other). The detector orientations chosen for the following analysis are identified by three unit vectors, $\mathbf{b} = (1, 0)^T$, $\mathbf{c} = (0, 1)^T$, and $\mathbf{a} = (\mathbf{b} - \mathbf{c})/|\mathbf{b} - \mathbf{c}| = (1, -1)^T/\sqrt{2}$, as a special case of importance. Any rotational changes of the orientations should result in equivalent cases. Let $a^\alpha b^\beta c^\gamma \wedge a^\xi b^\eta c^\zeta$ with superscripts $\pm$ denote the spin qualifications α, β, γ on the first particle and ξ, η, ζ on the second particle when the given spin values are analyzed with respect to the detection ranges of the three detectors oriented along the $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. They represent hypothetical detection results if each particle can be subjected to three successive idealized detector tests of 100% positive detections over the respectively specified detection ranges without ever changing the original spin orientation, while actual observational constraints of the cosine factor are disregarded.

Only the following six possible angular ranges of joint detections defined as disjoint subensembles can exist by exhaustion:

$S_1:$	$-\pi \le \sigma < -3\pi/4$	&	$0 \le \sigma+\pi < \pi/4$	$a^-b^-c^- \wedge a^+b^+c^+$
$S_2:$	$-3\pi/4 \le \sigma < -\pi/2$	&	$\pi/4 \le \sigma+\pi < \pi/2$	$a^+b^-c^- \wedge a^-b^+c^+$
$S_3:$	$-\pi/2 \le \sigma < 0$	&	$\pi/2 \le \sigma+\pi < \pi$	$a^+b^+c^- \wedge a^-b^-c^+$
$S_4:$	$0 \le \sigma < \pi/4$	&	$-\pi \le \sigma+\pi < -3\pi/4$	$a^+b^+c^+ \wedge a^-b^-c^-$
$S_5:$	$\pi/4 \le \sigma < \pi/2$	&	$-3\pi/4 \le \sigma+\pi < -\pi/2$	$a^-b^+c^+ \wedge a^+b^-c^-$
$S_6:$	$\pi/2 \le \sigma < \pi$	&	$-\pi/2 \le \sigma+\pi < 0$	$a^-b^-c^+ \wedge a^+b^+c^-$

where the last three subensembles are in mirrored images of the first three subensembles.

The obtainable paired positive detections on specified pair of detectors can now be derived and enumerated:

$S_1:$	$a^-b^- \wedge a^+b^+$	$a^-c^- \wedge a^+c^+$	$b^-c^- \wedge b^+c^+$
$S_2:$	$[a^+b^- \wedge a^-b^+]$	$[a^+c^- \wedge a^-c^+]$	$b^-c^- \wedge b^+c^+$
$S_3:$	$a^+b^+ \wedge a^-b^-$	$[a^+c^- \wedge a^-c^+]$	$[b^+c^- \wedge b^-c^+$
$S_4:$	$a^+b^+ \wedge a^-b^-$	$a^+c^+ \wedge a^-c^-$	$b^+c^+ \wedge b^-c^-$
$S_5:$	$[a^-b^+ \wedge a^+b^-]$	$[a^-c^+ \wedge a^+c^-]$	$b^-c^+ \wedge b^+c^-$
$S_6:$	$a^-b^- \wedge a^+b^+$	$[a^-c^+ \wedge a^+c^-]$	$[b^-c^+ \wedge b^+c^-]$

where the required paired positive detections satisfying (4.1) can be found only in those four subensembles with the paired detection results indicated inside the square brackets. The A, B pairs satisfying the paired detections of (4.1) are only possible in Subensembles S_2 and S_5, the A, C paired positive detections are only possible in Subensembles S_2, S_3, S_5 and S_6, and the B, C paired detections are only possible in Subensembles S_3 and S_6. From this finding, the subensembles over which the three conditional expectations are taken can be identified as:

$$E(\mathbf{a},\mathbf{b}) \quad \text{on Subensemble} \quad \Lambda(\mathbf{a},\mathbf{b}) = \{S_2, S_5\}$$
$$E(\mathbf{a},\mathbf{c}) \quad \text{on Subensemble} \quad \Lambda(\mathbf{a},\mathbf{c}) = \{S_2, S_3, S_5, S_6\}$$
$$E(\mathbf{b},\mathbf{c}) \quad \text{on Subensemble} \quad \Lambda(\mathbf{b},\mathbf{c}) = \{S_3, S_6\}$$

respectively. The three domains of definition for the $E(.,.)$ are therefore not identical. Clearly $\Lambda(\mathbf{a},\mathbf{c})-\Lambda(\mathbf{a},\mathbf{b}) = \{S_2, S_3\}$ and $\Lambda(\mathbf{a},\mathbf{c})-\Lambda(\mathbf{b},\mathbf{c}) = \{S_2, S_5\}$ where neither one of them possesses zero probability for any continuous density $\rho(\lambda)$.

The above example shows that the required symmetry in positive detection pairs of (4.1) cannot be satisfied for the chosen detector orientations of **a**, **b**, **c** of the particular special interest over the entire Λ, and that (3.5) and (4.2) become ill-defined.

7. A simple combinatorial problem

It is possible to generalize the above findings independent of the chosen detector orientations and of the exact σ, $\sigma+\pi$ spin pair requirement. In other words, a similar but more general argument should hold for the photon pair polarization analysis equally well.

Write down the binary numbers 000 through 111 in correspondence to 0 for "-" and 1 for "+" as the equivalent expressions of the superscripts of particle detection results for a given particle as shown in Section 6. Obviously these binary numbers consisting of three bits are unrelated to any particular choice of spin angular ranges, which were used in the previous analysis. Also, by treating only these binary numbers in three bits combinations, the analysis is neither affected by the σ, $\sigma+\pi$ requirement, nor confined to either one of the particle pair. Take two bits out of the three bits of each binary number. Three two-bits combinations can be acquired from each binary number. For example, from the binary number 010 the three two-bits combinations of 01, 10, and 00 are obtained. It is easy to verify that none of the three two-bits combinations of every binary number can satisfy 01 or 10 combinations in all three throughout the binary numbers 000 to 111. This "01 or 10" requirement reflects the constraints of (4.1). The negative finding shows the inevitable general constraint of choosing two-bits combinations out of the three bits constituting the binary numbers. The required symmetry of (4.1) therefore can never be universally satisfied as demonstrated in this analysis.

8. Summary and conclusions

A classical assumption of initial conditions was first introduced specifying that undisturbed particles of a correlated pair should maintain the original spin (or polarization) orientation values until they reached detectors. The derivation of Bell's inequality required the symmetry specified in (4.1). An examination revealed that the sign of the last term of the inequality appeared to be reversed, and the modified form of (4.2) was obtained. To understand Bell's inequality the author first identified logical and notational problems, and proposed to examine the relationship between subensembles over which the conditional expectations were defined. A particular case of interest with $\mathbf{a} = (\mathbf{b}-\mathbf{c})/|\mathbf{b}-\mathbf{c}|$ where $\mathbf{b} \perp \mathbf{c}$

had shown that these subensembles were not identical, and that the set differences between the identified subensembles were not empty. For a continuous density $\rho(\lambda)$, the set difference should have positive probability. As the result, Bell's inequality of the conditional expectations became ill-defined and meaningless for the critically important case of the analyzed example. The inability of getting the symmetry of (4.1) was however mathematical and fundamental as demonstrated in the simple combinatorial problem. The above conclusion suggests that no superiority exists for QM theories in analysis of particle physics, and that a variant of hidden variable concept, e.g., the concept of hidden initial values, is now considered viable, leading to potential founding of new classical particle model theories consistent with the established findings of quantum mechanics.

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(Manuscrit reçu le 21 octobre 1999)