Generalized Maxwell Equations and Their Solutions

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ABSTRACT. The generalized Maxwell equations are considered which include an additional gradient term. Such equations describe massless particles possessing spins one and zero. We find and investigate the matrix formulation of the first order of equations under consideration. All the linearly independent solutions of the equations for a free particle are obtained in terms of the projection matrix-dyads (density matrices).

1 Introduction

Now the Dirac-Kähler field in the framework of differential forms is of interest [1]. This is due to the possibility of using the Dirac-Kähler equation for describing fermions with spin 1/2 on the lattice [2]. Kähler [1] investigated an equation for inhomogeneous differential forms

$$(d-\delta+m)\,\Phi=0, \tag{1}$$

where m is the mass, d being the exterior derivative, $\delta=-\star^{-1}d\star$ turns $n-$forms into $(n-1)-$form; the $\star$ is the operator connecting a $n-$form with a $(4-n)-$form; $\star^2=1$, $d^2=\delta^2=0$; $(d-\delta)^2=-(d\delta+\delta d)=\partial_\mu^2$, $\partial_\mu=\partial/\partial x_\mu=(\partial/\partial x_m,\partial/i\partial t$, t is the time. The inhomogeneous differential form Φ is given by

$$\begin{gathered}\Phi=\varphi(x)+\varphi_\mu(x)dx^\mu+\frac{1}{2!}\varphi_{\mu\nu}(x)dx^\mu\wedge dx^\nu+\\+\frac{1}{3!}\varphi_{\mu\nu\rho}(x)dx^\mu\wedge dx^\nu\wedge dx^\rho+\frac{1}{4!}\varphi_{\mu\nu\rho\sigma}(x)dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma\end{gathered} \tag{2}$$

what is equivalent to introducing a scalar field $\varphi(x)$, vector field $\varphi_\mu(x)$, and antisymmetric tensor fields: $\varphi_{\mu\nu}(x)$, $\varphi_{\mu\nu\rho}(x)$, $\varphi_{\mu\nu\rho\sigma}(x)$. The antisymmetric tensors $\varphi_{\mu\nu\rho}(x)$, $\varphi_{\mu\nu\rho\sigma}(x)$ define a pseudovector and pseudoscalar fields, respectively:

$$\widetilde{\varphi}_\mu(x) = \frac{1}{3!}\varepsilon_{\mu\nu\rho\sigma}\varphi_{\nu\rho\sigma}(x), \qquad \widetilde{\varphi}(x) = \frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\varphi_{\mu\nu\rho\sigma}(x), \tag{3}$$

where $\varepsilon_{\mu\nu\alpha\beta}$ is an antisymmetric tensor Levy-Civita; $\varepsilon_{1234} = -i$. So, Eq.(1) includes two scalar and two vector fields. This means the consideration of fields with spins zero and one with the same mass m. Eq.(1) with the definitions (2), (3) can be represented as the following system of tensor fields [3]:

$$\begin{gathered}\partial_\nu\psi_{[\mu\nu]}(x) - \partial_\mu\psi(x) + m^2\psi_\mu(x) = 0, \qquad \partial_\nu\widetilde{\psi}_{[\mu\nu]}(x) - \partial_\mu\widetilde{\psi}(x) + m^2\widetilde{\psi}_\mu(x) = 0,\\ \partial_\mu\psi_\mu(x) = \psi(x), \qquad \partial_\mu\widetilde{\psi}_\mu(x) = \widetilde{\psi}(x),\\ \psi_{[\mu\nu]}(x) = \partial_\mu\psi_\nu(x) - \partial_\nu\psi_\mu(x) - \varepsilon_{\mu\nu\alpha\beta}\partial_\alpha\widetilde{\psi}_\beta(x),\end{gathered} \tag{4}$$

where $\widetilde{\psi}_{[\mu\nu]} = (1/2)\varepsilon_{\mu\nu\alpha\beta}\psi_{\alpha\beta}$ is the dual tensor. There is the doubling of the spin states of fields described because Eqs.(4) contain two four-vectors $\psi_\mu(x)$, $\widetilde{\psi}_\mu(x)$ and two scalars $\psi(x)$, $\widetilde{\psi}(x)$). Equations (4) can be represented as the 16 -dimensional first order Dirac equation [3]. That is why there is a connection between description of fermions with spin 1/2 and bosonic fields $\psi(x)$, $\psi_\mu(x)$, $\psi_{\mu\nu}(x)$, $\widetilde{\psi}(x)$, $\widetilde{\psi}_\mu(x)$. At the restrictions $\widetilde{\psi}_\mu = 0$, $\widetilde{\psi} = 0$ we arrive at the Proca equations [4]. Stueckelberg's equation [5], describing fields with spin one and zero, corresponds to the case $\widetilde{\psi}_\mu = 0$ in (4).

From Eqs.(4) at $m = 0$ we arrive at the two-potential formulation of massless fields with two gradient terms

$$\begin{gathered}\partial_\nu\psi_{[\mu\nu]}(x) - \partial_\mu\psi(x) = 0, \qquad \partial_\nu\widetilde{\psi}_{[\mu\nu]}(x) - \partial_\mu\widetilde{\psi}(x) = 0,\\ \partial_\mu\psi_\mu(x) = \psi(x), \qquad \partial_\mu\widetilde{\psi}_\mu(x) = \widetilde{\psi}(x),\\ \psi_{[\mu\nu]}(x) = \partial_\mu\psi_\nu(x) - \partial_\nu\psi_\mu(x) - \varepsilon_{\mu\nu\alpha\beta}\partial_\alpha\widetilde{\psi}_\beta(x).\end{gathered} \tag{5}$$

Eqs.5 represent the generalized Maxwell equations which were studied in [6-11]. In [3] we found and investigated the matrix formulation of the first order of equations (5) and solutions for a free particle in the form of the projection matrix-dyads. The matrices of an equation obey the Dirac algebra. In this work we study "minimal" generalization of Maxwell's equations by setting $\widetilde{\psi}_\beta(x) = \widetilde{\psi}(x) = 0$ in (5). In this case there is no doubling of spin states of fields: there is one state with spin zero and two states with helicity ± 1.

2 Matrix form of massless bosonic fields

Let us consider the following generalized Maxwell equations (see also [12-14])

$$\begin{gathered}
\partial_\nu \psi_{[\mu\nu]} + \partial_\mu \psi_0 = 0, \\
\partial_\nu \psi_\mu - \partial_\mu \psi_\nu + \kappa \psi_{[\mu\nu]} = 0, \\
\partial_\mu \psi_\mu + \kappa \psi_0 = 0.
\end{gathered} \tag{6}$$

Eqs.(6) follow from (5) at the replacement $\widetilde{\psi}_\beta(x) = \widetilde{\psi}(x) = 0$, $\psi_{\mu\nu} \to \kappa\psi_{\mu\nu}$, $\psi(x) \to -\kappa\psi_0(x)$. Fields ψ_μ, ψ_0 are massless vector and scalar fields, respectively, and κ is a parameter which we introduced for convenience. So, equations (6) describe massless particles possessing spins one and zero without doubling of spin states. The classical Maxwell equations are obtained by setting $\psi_0 = 0$.

It is easy to get the massive fields by adding the term $m\psi_\mu$ in the first equation (6) (at $\kappa = m$). In this case we arrive at the massive (with mass m) Stueckelberg fields [5,15]. In [15] the matrix form of equations for massive fields and solutions in the form of the projection matrix-dyads were found.

Now we consider the matrix formulation of the first order of the field equations (6) for massless fields which is convenient for constructing the density matrix and for some electrodynamics calculations. Let us introduce the matrix $\varepsilon^{A,B}$ [16] with dimension $n \times n$; its elements consist of zeroes and only one element is unity where row A and column B cross. So the matrix elements and multiplication of these matrices are

$$\left(\varepsilon^{A,B}\right)_{CD} = \delta_{AC}\delta_{BD}, \qquad \varepsilon^{A,B}\varepsilon^{C,D} = \delta_{BC}\varepsilon^{A,D}, \tag{7}$$

where indexes $A, B, C, D = 1, 2, ...n$. After introducing the 11-dimensional function

$$\Psi(x) = \{\psi_A(x)\} = \begin{pmatrix} \psi_0 \\ \psi_\mu \\ \psi_{[\mu\nu]} \end{pmatrix} \qquad (A = 0, \mu, [\mu\nu]), \tag{8}$$

where μ, $\nu = 1, 2, 3, 4$, and using the elements of the entire algebra (7), Eq.(6) can be written in the form of one equation

$$\partial_\nu \left(\varepsilon^{\mu,[\mu\nu]} + \varepsilon^{[\mu\nu],\mu} + \varepsilon^{\nu,0} + \varepsilon^{0,\nu}\right)_{AB} \psi_B(x) + \\ +\kappa \left(\varepsilon^{0,0} + \frac{1}{2}\varepsilon^{[\mu\nu],[\mu\nu]}\right)_{AB} \psi_B(x) = 0. \tag{9}$$

Introducing 11-dimensional matrices

$$\alpha_\nu = \varepsilon^{\mu,[\mu\nu]} + \varepsilon^{[\mu\nu],\mu} + \varepsilon^{\nu,0} + \varepsilon^{0,\nu}, \\ \overline{P} = \varepsilon^{\mu,\mu}, \qquad P = \varepsilon^{0,0} + \frac{1}{2}\varepsilon^{[\mu\nu],[\mu\nu]}, \tag{10}$$

Eq.(9) takes the form of the relativistic wave equation of the first order:

$$\left(\alpha_\mu \partial_\mu + \kappa P\right) \Psi(x) = 0. \tag{11}$$

So, matrix equation (11) gives a unified description of a scalar and vector massless fields.

Matrices $\overline{P}$, P are the projective matrices (see [16,17]) which obey the relations:

$$P^2 = P, \qquad \overline{P}^2 = \overline{P}, \qquad P + \overline{P} = I_{11}, \\ \alpha_\mu \overline{P} + \overline{P}\alpha_\mu = \alpha_\mu, \qquad \alpha_\mu P + P\alpha_\mu = \alpha_\mu, \tag{12}$$

where I_{11} is the unit matrix in 11−dimensional space. The Stueckelberg equation for massive fields in the matrix form is given by [15].

$$\left(\alpha_\mu \partial_\mu + m\right) \Psi(x) = 0. \tag{13}$$

It should be noted that the matrices α_μ can be represented as

$$\alpha_\mu = \beta_\mu^{(1)} + \beta_\mu^{(0)}, \\ \beta_\nu^{(1)} = \varepsilon^{\mu,[\mu\nu]} + \varepsilon^{[\mu\nu],\mu}, \\ \beta_\nu^{(0)} = \varepsilon^{\nu,0} + \varepsilon^{0,\nu}, \tag{14}$$

where the 10−dimensional $\beta_\mu^{(1)}$ and 5−dimensional $\beta_\mu^{(0)}$ matrices obey the Petiau-Duffin-Kemmer [18-20] algebra:

$$\beta_\mu \beta_\nu \beta_\alpha + \beta_\alpha \beta_\nu \beta_\mu = \delta_{\mu\nu} \beta_\alpha + \delta_{\alpha\nu} \beta_\mu, \tag{15}$$

so that the equations for massive spin-1 and spin-0 particles are (see [16])

$$\left(\beta_\mu^{(1)}\partial_\mu + m\right)\Psi^{(1)}(x) = 0, \qquad \Psi^{(1)}(x) = \begin{pmatrix} \psi_\mu \\ \psi_{[\mu\nu]} \end{pmatrix}, \tag{16}$$

$$\left(\beta_\mu^{(0)}\partial_\mu + m\right)\Psi^{(0)}(x) = 0, \qquad \Psi^{(0)}(x) = \begin{pmatrix} \psi_0 \\ \psi_\mu \end{pmatrix}. \tag{17}$$

The 10−dimensional Petiau-Duffin-Kemmer equation (16) is equivalent to the Proca equations [4] for spin-1 particles and the 5−dimensional Eq. (17) is equivalent to the Klein-Gordon-Fock equation for scalar particles. The 11−dimensional Eq.(11) describes massless fields with two spins 0, 1 (multi-spin 0,1). It is not difficult to verify (using Eqs.(7)) that the 11−dimensional matrices α_μ (10) satisfy the algebra (see also [21]):

$$\begin{gathered}\alpha_\mu\alpha_\nu\alpha_\alpha + \alpha_\alpha\alpha_\nu\alpha_\mu + \alpha_\mu\alpha_\alpha\alpha_\nu + \alpha_\nu\alpha_\alpha\alpha_\mu + \alpha_\nu\alpha_\mu\alpha_\alpha + \alpha_\alpha\alpha_\mu\alpha_\nu = \\ = 2\left(\delta_{\mu\nu}\alpha_\alpha + \delta_{\alpha\nu}\alpha_\mu + \delta_{\mu\alpha}\alpha_\nu\right).\end{gathered} \tag{18}$$

This algebra is more complicated than the Petiau-Duffin-Kemmer algebra (15). Different representations of the Petiau-Duffin-Kemmer algebra (15) were considered in [22-27].

3 Solutions of generalized Maxwell's equations

Let us now consider the solutions of the matrix equation (11) for massless fields. In the momentum space, Eq.(11) is given by

$$D\Psi_k = 0, \qquad D = i\widehat{k} + \kappa P, \tag{19}$$

where $\widehat{k} = \alpha_\mu k_\mu$, $k_\mu^2 = \mathbf{k}^2 - k_0^2 = 0$ and the matrix D obeys the minimal equation

$$D\left(D-\kappa\right)^2 = 0. \tag{20}$$

It should be noted that this matrix equation of generalized Maxwell's equations with multi-spin $0,1$ is simpler than the minimal equation for Maxwell's equations with pure spin 1 [16,28]. Using the general scheme [17] we find that the projection operator corresponding to eigenvalue 0 of the operator D is

$$\gamma = \left(\frac{D-\kappa}{\kappa}\right)^2, \tag{21}$$

so that $\gamma^2 = \gamma$.

Every column of the matrix γ can be considered as an eigenvector Ψ_k of equation (19) with eigenvalue 0. Eq.(19) for projection operators tells that matrix γ can be transformed into diagonal form, with the diagonal containing only ones and zeroes. So the γ acting on any function Ψ will retain components which are solutions of Eq.(19).

The generators of the Lorentz group in the 11−dimensional space being considered are given by

$$J_{\mu\nu} = \beta_\mu^{(1)}\beta_\nu^{(1)} - \beta_\nu^{(1)}\beta_\mu^{(1)}. \tag{22}$$

It should be noted that matrices (22) act in the 10−dimensional subspace $\left(\psi_\mu, \psi_{[\mu\nu]}\right)$ because the scalar ψ_0 is an invariant of the Lorentz transformations. So matrices (22) are also generators of the Lorentz group for the Petiau-Duffin-Kemmer fields of Eq.(16). Using properties (7), we get the commutation relations

$$[J_{\rho\sigma}, J_{\mu\nu}] = \delta_{\sigma\mu}J_{\rho\nu} + \delta_{\rho\nu}J_{\sigma\mu} - \delta_{\rho\mu}J_{\sigma\nu} - \delta_{\sigma\nu}J_{\rho\mu}, \tag{23}$$

$$[\alpha_\lambda, J_{\mu\nu}] = \delta_{\lambda\mu}\alpha_\nu - \delta_{\lambda\nu}\alpha_\mu. \tag{24}$$

Relationship (23) is a well known commutation relation for generators of the Lorentz group $SO(3,1)$. Equation (11) is form-invariant under the Lorentz transformations since relation (24) is valid. To guarantee the existence of a relativistically invariant bilinear form

$$\overline{\Psi}\Psi = \Psi^+\eta\Psi, \tag{25}$$

where Ψ^+ is the Hermitian-conjugate wave function, we should construct a Hermitianizing matrix η with the properties [16,17,24]:

$$\eta\alpha_i = -\alpha_i\eta, \qquad \eta\alpha_4 = \alpha_4\eta \qquad (i = 1,2,3). \tag{26}$$

Such a matrix exists and is given by

$$\eta = -\varepsilon^{0,0} + 2\beta_4^{(1)2} - I_{10}, \tag{27}$$

$$I_{10} = \varepsilon^{\mu,\mu} + \frac{1}{2}\varepsilon^{[\mu\nu],[\mu\nu]},$$

where the matrix $\eta^{(1)} = 2\beta_4^{(1)2} - I_{10}$ plays the role of a Hermitianizing matrix for the Petiau-Duffin-Kemmer equation (16) [16]. The operator of the squared spin (squared Pauli-Lubanski vector) is given by

$$\sigma^2 = \left(\frac{1}{2k_0}\varepsilon_{\mu\nu\alpha\beta}k_\nu J_{\alpha\beta}\right)^2 = \frac{1}{k_0^2}J_{\mu\sigma}J_{\sigma\nu}k_\mu k_\nu. \tag{28}$$

It may be verified that this operator obeys the minimal equation

$$\sigma^2\left(\sigma^2 - 2\right) = 0, \tag{29}$$

so that eigenvalues of the squared spin operator σ^2 are $s(s+1) = 0$ and $s(s+1) = 2$. This confirms that the considered fields describe the superposition of two spins $s = 0$ and $s = 1$. To separate these states we use the projection operators

$$S^2_{(0)} = 1 - \frac{\sigma^2}{2}, \qquad S^2_{(1)} = \frac{\sigma^2}{2} \tag{30}$$

having the properties $S^2_{(0)}S^2_{(1)} = 0$, $\left(S^2_{(0)}\right)^2 = S^2_{(0)}$, $\left(S^2_{(1)}\right)^2 = S^2_{(1)}$, $S^2_{(0)} + S^2_{(1)} = 1$, where $1 \equiv I_{11}$ is the unit matrix in $11-$ dimensional space. In accordance with the general properties of the projection operators, the matrices $S^2_{(0)}$, $S^2_{(1)}$ acting on the wave function extract pure states with spin 0 and 1, respectively. Now we introduce the operator of the spin projection on the direction of the momentum **k** (helicity) :

$$\sigma_k = -\frac{i}{2k_0}\epsilon_{abc}k_a J_{bc} = -\frac{i}{k_0}\epsilon_{abc}k_a\beta_b^{(1)}\beta_c^{(1)}. \tag{31}$$

The minimal matrix equation for the spin projection operator is

$$\sigma_k\left(\sigma_k - 1\right)\left(\sigma_k + 1\right) = 0 \tag{32}$$

and the corresponding projection operators are given by

$$\widehat{S}_{(\pm 1)} = \frac{1}{2}\sigma_k\left(\sigma_k \pm 1\right), \qquad \widehat{S}_{(0)} = 1 - \sigma_k^2. \tag{33}$$

Operators $\widehat{S}_{(\pm 1)}$ correspond to the spin projections $s_k = \pm 1$. It is easy to verify that the required commutation relations hold:

$$\left[S^2_{(0)}, \widehat{k}\right] = \left[S^2_{(1)}, \widehat{k}\right] = \left[\widehat{S}_{(\pm 1)}, \widehat{k}\right] = \left[\widehat{S}_{(0)}, \widehat{k}\right] = 0,$$

$$\left[S^2_{(0)}, \widehat{S}_{(\pm 1)}\right] = \left[S^2_{(1)}, \widehat{S}_{(\pm 1)}\right] = \left[S^2_{(0)}, \widehat{S}_{(0)}\right] = 0. \tag{34}$$

Thus the projection matrices extracting pure states with definite spin (0 and 1), and spin projections (helicity ± 1) take the form

$$\begin{aligned} \Pi_{(0)} &= \left(1 - \frac{\sigma^2}{2}\right)\left(\frac{D-\kappa}{\kappa}\right)^2, \\ \Pi_{(\pm 1)} &= \frac{1}{2}\sigma_k\left(\sigma_k \pm 1\right)\left(\frac{D-\kappa}{\kappa}\right)^2, \end{aligned} \tag{35}$$

where we took into account that $\left(\sigma^2/2\right)\sigma_k = \sigma_k$. Projection operators $\Pi_{(0)}$, $\Pi_{(\pm 1)}$ extract states with spin 0 and 1, respectively. The $\Pi_{(0)}$, $\Pi_{(\pm 1)}$ are the density matrices for pure spin spates. It is easy to consider impure states by summation of Eqs.(35) over spin projections and spins. Projection operators for pure states can be represented as matrices-dyads [17]:

$$\Pi_{(0)} = \Psi_{(0)} \cdot \overline{\Psi}_{(0)}, \qquad \Pi_{(\pm 1)} = \Psi_{(\pm)} \cdot \overline{\Psi}_{(\pm)}, \tag{36}$$

where the wave functions $\Psi_{(0)}$, $\Psi_{(\pm)}$ correspond to spin 0 and 1, respectively. Solutions of Eq.(13) for massive particles with spins 0 and 1 in the form of matrix-dyads were found in [15].

Expressions (35), (36) are convenient for calculating different electrodynamics processes involving polarized massless particles. It is possible to make evaluations of different physical quantities in a covariant manner without using the matrices of first-order equations in a definite representation.

4 Conclusion

Compared to the Maxwell equations which describe left and right polarized waves (helicity ± 1), Eqs.(6) admit also an additional longitudinal state corresponding to spin-zero of the field. This state gives the negative contribution to the Hamiltonian of fields under consideration and it is necessary to introduce an indefinite metric to quantize such a field (see [15]). To eliminate the additional state with spin-zero one may impose the constraint $\psi_0(x) = 0$ in equations (6), and we arrive at the classical Maxwell equations, where $\psi_{\mu\nu}(x)$ is the strength tensor; $E_m = i\psi_{m4}$, $H_m = (1/2)\varepsilon_{mnk}\psi_{nk}$ are electric and magnetic fields, respectively. It is possible also to treat the scalar field $\psi_0(x)$ as non-physical one in the general gauge $\psi_0(x) \neq 0$ (an orthodox point of view). In this way after some calculations one should eliminate the contribution of this non-physical scalar field in this general gauge. In extraordinary point of view, vector and scalar states of the system (6) can be treated on the same footing with introducing indefinite metric. This, however, requires the further development and physical interpretation of quantum field theory with indefinite metric.

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