Is Bell's Locality Condition Necessary for The Derivation of Bell's Inequality ?

M. Golshani, A. Fahmi

Institute for Studies in Theoretical Physics and Mathematics (IPM),
P.O.Box 19395-5531, Tehran, Iran
and Department of Physics Sharif University of Technology, P.O.Box
11365-9061, Tehran, Iran
email: golshani@ihcs.ac.ir, fahmi@theory.ipm.ac.ir

ABSTRACT. In this paper, we have derived, in a very simple way, the original Bell's and CH's inequalities without using Bell's original locality condition.
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1 Introduction

Bell's inequality has been derived in different ways [1-10], using the following assumptions: locality, determinism, hidden variables and counterfactual definiteness. From this inequality, Bell concluded the so-called Bell's theorem: " In certain experiments all realistic local hidden variable theories are incompatible with quantum mechanics ". Clauser and Horne tried to drive an inequality without the assumption of determinism [11] and showed that this inequality isn't consistent with quantum mechanics. In some cases, people have tried to introduce the locality hypothesis as a natural condition (e. g. the Bell's inequality for the case of a single particle) and have shown the violation of Bell's inequality [12-15]. Ben-Aryeh used entangled singlet spin-state with spatial spherical wave functions and showed that Bell's inequalities are violated only for subensembles which are not pure states, and he concluded that locality is not violated [16]. G. Lochak [17] analyzed the proof given by J. S. Bell and showed that Bell's reasoning involves not only his locality assumption (with which he agrees) but also a statistical hypothesis which is exactly the equivalent to the one introduced by von Neumann. This hypothesis consists in the admitting that such a theory must restore the classical probabilistic pattern simultaneously in the statistics of all measurement results. This leads immediately to a contradiction with the calculation of mean values in quantum mechanics. E. Santos (and others) [18-19], considers experimental restrictions, and argues that from actual experiments, with low efficiency detectors, one cannot conclude the violation of Bell's inequality. In deriving Bell's inequality most authors assume the locality condition. Some authors use a weaker locality condition, For example, Helman [20] constructed a model with an almost ideal correlation and an almost determinism, and showed that this should suffice to derive an approximate Bell-type inequality. In this model, outcome-independence and factorizability of joint probability are not assumed, but rather an approximate from of factorizability is derived. Elby and Jones [21] used K.S. theorem, and derived a simple algebraic contradiction between various locality assumptions and the predictions of quantum mechanics. In this paper, we consider non-locality in a special form and drive Bell and CH inequalities. Consequently, our proof rules out a broader class of hidden variable theories.

2 Locality Condition and Bell's Inequality

A typical Bell-type experiment consists of two particles which originate from a source and propagate in opposite directions towards their corresponding measurement apparatuses and detectors. Each detector detects a dichotomic variable which takes the values ± 1. We represent the parameters which characterize the measurement apparatuses (1) and (2) by $\hat{a}$ and $\hat{b}$ respectively ($\hat{a}$, e.g., is the direction of magnetic field in a Stern-Gerlach experiment). The measurement results depend on the controllable variables $\hat{a}$ and $\hat{b}$ and a set of uncontrollable variables, the so-called hidden variables, which we collectively represent by λ.
In Bell's original approach [2], the result A of the spin measurement on particle 1 was taken to depend on $\hat{a}$, $\hat{b}$ and λ, i.e. we have $A(\hat{a}, \hat{b}, \lambda)$. Similarly, $B(\hat{a}, \hat{b}, \lambda)$ represents the result of a spin measurement on the particle 2. Bell, then, applied Einstein's locality in the following form:

$$\begin{aligned} A(\hat{a}, \hat{b}, \lambda) &= A(\hat{a}, \lambda) \\ B(\hat{a}, \hat{b}, \lambda) &= B(\hat{b}, \lambda) \end{aligned} \tag{1}$$

In our approach, we replace Bell's locality condition by the following condition:

$$\begin{aligned} A(\hat{a}, \hat{b}, \lambda) &= f_A(\hat{a}, \lambda) g_A(\hat{b}, \lambda) \\ B(\hat{a}, \hat{b}, \lambda) &= f_B(\hat{a}, \lambda) g_B(\hat{b}, \lambda) \end{aligned} \tag{2}$$

Here, Shimony's parameter independence is clearly violated. We are, as in Bell's case [2], interested in the correlation of the results of joint spin measurements performed on particles 1 and 2. Thus, we define the correlation function $C(\hat{a},\hat{b})$ in the form:

$$C(\hat{a},\hat{b}) = \int A(\hat{a},\hat{b},\lambda)B(\hat{a},\hat{b},\lambda)\rho(\lambda)\ d\lambda \tag{3}$$

where the probability distribution function for the uncontrollable hidden variables λ is represented by $\rho(\lambda)$, with

$$\int \rho(\lambda)\ d\lambda = 1, \quad \rho(\lambda) \geq 0 \tag{4}$$
$$A(\hat{a},\hat{b},\lambda) = \pm 1, \quad B(\hat{a},\hat{b},\lambda) = \pm 1$$

If the parameter of the measurement apparatus (1) can take values $\hat{a}$ or $\hat{a}'$ and that of the measurement apparatus (2) can take values $\hat{b}$ or $\hat{b}'$, then, we have:

$$\begin{aligned} C(\hat{a},\hat{b}) - C(\hat{a},\hat{b}') &= \int \rho(\lambda)\ d\lambda \{A(\hat{a},\hat{b},\lambda)B(\hat{a},\hat{b},\lambda) - A(\hat{a},\hat{b},\lambda)B(\hat{a},\hat{b}',\lambda)\} \\ &= \int \rho(\lambda)\ d\lambda \{A(\hat{a},\hat{b},\lambda)B(\hat{a},\hat{b},\lambda)[\ 1 \pm A(\hat{a}',\hat{b}',\lambda)B(\hat{a}',\hat{b}',\lambda)] \\ &\quad - A(\hat{a},\hat{b}',\lambda)B(\hat{a},\hat{b}',\lambda)[\ 1 \pm A(\hat{a}',\hat{b},\lambda)B(\hat{a}',\hat{b},\lambda)]\ \} \end{aligned} \tag{5}$$

Taking the absolute values of both sides and using

$$\begin{aligned} |A(\hat{a},\hat{b},\lambda)| = |B(\hat{a},\hat{b},\lambda)| = |A(\hat{a},\hat{b}',\lambda)| = |B(\hat{a},\hat{b}',\lambda)| = 1 \\ |1 \pm A(\hat{a}',\hat{b},\lambda)B(\hat{a}',\hat{b},\lambda)| = 1 \pm A(\hat{a}',\hat{b},\lambda)B(\hat{a}',\hat{b},\lambda) \\ |1 \pm A(\hat{a}',\hat{b}',\lambda)B(\hat{a}',\hat{b}',\lambda)| = 1 \pm A(\hat{a}',\hat{b}',\lambda)B(\hat{a}',\hat{b}',\lambda) \end{aligned} \tag{6}$$

We have:

$$\begin{aligned} |C(\hat{a},\hat{b}) - C(\hat{a},\hat{b}')| &\leq \int \rho(\lambda)\ d\lambda \{\ [\ 1 \pm A(\hat{a}',\hat{b}',\lambda)B(\hat{a}',\hat{b}',\lambda)] \\ &\qquad + [\ 1 \pm A(\hat{a}',\hat{b},\lambda)B(\hat{a}',\hat{b},\lambda)]\ \} \\ &\leq 2 \pm [\ C(\hat{a}',\hat{b}') + C(\hat{a}',\hat{b})] \end{aligned} \tag{7}$$

or

$$|C(\hat{a},\hat{b}) - C(\hat{a},\hat{b}')| + |C(\hat{a}',\hat{b}') + C(\hat{a}',\hat{b})| \leq 2 \tag{8}$$

which is Bell's original inequality [2]. This shows that one can obtain Bell's inequality without appealing to Bell's locality condition (1). Furthermore, it shows that if there is any non-locality in nature it is not in the form of the eq. (2).

3 Locality Condition and CH's Inequality

Here, we first review the locality condition for the CH inequality. The schematic diagram for the experiment is the same as CH's experiment given in ref.[5]: A source produces two correlated particles at the origin O. Each particle goes through an analyzer and a detector. For the case of spin $\frac{1}{2}$ particles, the analyzers are simply Stern-Gerlach magnetic apparatuses and for the case of photons the analyzers are simply polarization filters. The detectors discover only the number of particles. We assume that the probability for the detection of a particle by detector D_1 depends only on the parameter of it's corresponding analyzer and a set of uncontrollable variables (the so-called hidden variables), which we represent collectively by λ. The probability of the detection of particle 1 is represented, by $P_1(\hat{a},\lambda)$ and that for the particle 2 by $P_2(\hat{b},\lambda)$. The joint probability function for detection of both particles is represented by $P_{12}(\hat{a},\hat{b},\lambda)$. The distribution function for the uncontrollable hidden variable λ is represented by $\rho(\lambda)$ which satisfies the following relations:

$$\int \rho(\lambda)\ d\lambda = 1, \quad \rho(\lambda) \geq 0$$

Averaging $P_1(\hat{a},\lambda), P_2(\hat{b},\lambda)$ and $P_{12}(\hat{a},\hat{b},\lambda)$ over λ, we get:

$$P_1(\hat{a}) = \int P_1(\hat{a},\lambda)\rho(\lambda)\ d\lambda$$

$$P_2(\hat{b}) = \int P_2(\hat{b},\lambda)\rho(\lambda)\ d\lambda$$

$$P_{12}(\hat{a},\hat{b}) = \int P_{12}(\hat{a},\hat{b},\lambda)\rho(\lambda)\ d\lambda \tag{9}$$

Clauser and Horne applied Einstein's locality condition in the following form:

$$P_{12}(\hat{a},\hat{b},\lambda) = P_1(\hat{a},\lambda)P_2(\hat{b},\lambda) \tag{10}$$

Now we replace CH's locality condition by the following conditions.

$$\begin{aligned} P_1(\hat{a},\hat{b},\lambda) &= f_1(\hat{a},\lambda)g_1(\hat{b},\lambda) \\ P_2(\hat{a},\hat{b},\lambda) &= f_2(\hat{a},\lambda)g_2(\hat{b},\lambda) \end{aligned} \tag{11}$$

then by considering the following inequality, which is always true,

$$x_a x_b y_a y_b - x_a x_{b'} y_a y_{b'} + x_{a'} x_{b'} y_{a'} y_{b'} + x_{a'} x_b y_{a'} y_b - x_{a'} - y_b \leq 0$$

$$0 \leq x_i, y_j \leq 1 \qquad i,j = a, a', b, b'$$

$$x_a = f_1(\hat{a},\lambda), \qquad x_b = g_1(\hat{b},\lambda), \qquad y_b = g_2(\hat{b},\lambda), \qquad y_a = f_2(\hat{a},\lambda)$$

We can drive CH's inequality. Here, we don't follow this path; rather, we use the approach of Arthur Fine in ref.[22], i.e. we begin from Bell's inequality and derive CH's inequality. representing the average values of measurements on particles (1) and (2) by $E^1(\hat{a},\hat{b},\lambda)$ and $E^2(\hat{a},\hat{b},\lambda)$, respectively, is:

$$\begin{aligned} E^1(\hat{a},\hat{b},\lambda)E^2(\hat{a},\hat{b},\lambda) = {} & P_{12}(\hat{a},\hat{b},\lambda) - P_{12}(\hat{a},-\hat{b},\lambda) \\ & -P_{12}(-\hat{a},\hat{b},\lambda) + P_{12}(-\hat{a},-\hat{b},\lambda) \end{aligned} \tag{12}$$

Of course, the assumptions of the previous section for A's and B's can be adopted for E^i, i.e. they are in product form, and all results of Sec. (2) hold for E_1 and E_2 .But we have neither assumed parameter or outcome independence, nor have made the assumption of there about the form of P_{12}. Now, we know that:

$$P_{12}(\hat{a},-\hat{b}) = P_1(\hat{a}) - P_{12}(\hat{a},\hat{b})$$

$$P_{12}(-\hat{a},\hat{b}) = P_2(\hat{b}) - P_{12}(\hat{a},\hat{b}) \tag{13}$$

$$P_{12}(-\hat{a},-\hat{b}) = 1 - P_{12}(-\hat{a},\hat{b}) - P_{12}(\hat{a},-\hat{b}) - P_{12}(\hat{a},\hat{b})$$

From equations (3), (12) and (13), we get:

$$C(\hat{a},\hat{b}) = 4P_{12}(\hat{a},\hat{b}) - 2P_1(\hat{a}) - 2P_2(\hat{b}) + 1 \tag{14}$$

and from the equation (14) and similar equations for $C(\hat{a},\hat{b}')$, $C(\hat{a}',\hat{b}')$ and $C(\hat{a}',\hat{b})$,and using Bell's inequality (8), we get:

$$\begin{aligned}-2 \leq [\ 4P_{12}(\hat{a},\hat{b}) - 2P_1(\hat{a}) - 2P_2(\hat{b}) + 1 - 4P_{12}(\hat{a},\hat{b}') + 2P_1(\hat{a}) + 2P_2(\hat{b}') - 1] \\ +[\ 4P_{12}(\hat{a}',\hat{b}) - 2P_1(\hat{a}') - 2P_2(\hat{b}) + 1 \\ +4P_{12}(\hat{a}',\hat{b}') - 2P_1(\hat{a}') - 2P_2(\hat{b}') + 1]\ \leq 2 \qquad (15)\end{aligned}$$

or

$$-1 \leq P_{12}(\hat{a},\hat{b}) - P_{12}(\hat{a},\hat{b}') + P_{12}(\hat{a}',\hat{b}') + P_{12}(\hat{a}',\hat{b}) - P_1(\hat{a}') - P_2(\hat{b}) \leq 0$$

which is the usual CH inequality [5].

4 Conclusion

In this paper,we have replaced Bell's locality condition by a more general condition to obtain the Bell's and CH's inequalities. The least conclusion we can draw is that if there were any non-locality in nature it would not be of the restricted form of eq.(2), and that for all hidden variable theories in which non-locality is assumed in the aforementioned form, one can obtain Bell's inequality. In other words, one can obtain CH's inequality without using the factorizability condition. Thus, we can conclude that the violation of Bell's inequalities is not necessarily the violation of Bell's locality condition, and that if there is any non-locality in nature, it is not in the form of the relation (2). Furthermore, all hidden variable theories of this sort are inconsistent with quantum mechanics. In a paper in progress, we shall study some more general cases of non-locality

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