Mechanics of the Wave-Particle Dualism*

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ABSTRACT. A variational principle is proposed for the orbits of a lightlike point charge in external electromagnetic fields. The principle provides equations of motion which are Lorentz invariant and do not depend on the choice of any particular orbit parameter. The equations of motion are solved in general in the case that external fields are absent. Geometrical and physical properties of the solutions sensitively depend on the spacetime structure of the momentum vector associated by the variational principle to an orbit. Only timelike momenta allow to assign a restmass to lightlike trajectories. In that case however, lightlike trajectories completely may account for the wave-particle duality of matter and display a spatial spread in accord with the momentum-position uncertainty principle.

RÉSUMÉ. *On propose un principe variationnel pour les trajectoires d'une charge ponctuelle ou point isotrope se mouvant dans un champ électro-magnétique extérieur toujours à la vitesse de la lumière. Le principe impose que le vecteur tangent à chaque point d'une trajectoire soit sur le cône de lumière dans l'espace Minkowski. L'existence explicite de cette condition permet l'invariance des trajectoires par rapport au paramètre du mouvement. Les solutions générales des équations du mouvement sont construites pour le cas d'absence de champ extérieur. On trouve que le comportement d'une solution dépend fortement du genre du vecteur impulsion-énergie adjoint à chaque trajectoire par le principe* *variationnel. Si la quadri-impulsion est un vecteur de temps, on peut attribuer une masse à cette trajectoire. La projection de la trajectoire (du point isotrope) dans l'espace Euclidien est alors une hélice qui offre un modèle mécanique pour le dualisme onde-particule de la matière de même que la relation d'incertitude entre l'impulsion et la position dans l'espace.*

1. Introduction

Jan Weyssenhoff [1] investigated a variational principle for timelike and lightlike free motions. In the case of lightlike motions the Lagrangian he proposes however is **not** parameter invariant! Parameter invariance in his case only holds if the tangent- or velocity vector is restricted to be lightlike. Because of this defect, he is forced to approach lightlike motions as a limiting case of more conventional timelike motions. Yet this limit is undefined, as he admits in his comment in [1] below equation (4.56). This has the consequence that his spin bivector remains undetermined in [1] and recourse has to be made on sections 11 and 12 in [2].

In section 2 of this article we propose a variational principle for the motion of a lightlike point charge in external electromagnetic fields. Our variational integral involves second derivatives just as in [1]. However, as distinguished from [1], our principle is supplement by a subsidiary condition which restricts the velocity to the light cone. This constraint ensures the independence of the equations of motion from the choice of parameter. Equations of motion then are displayed in a manifestly parameter invariant form and are regrouped into a momentum law and an angular momentum law.

For a free lightlike point, the general solution of the equations of motion is constructed in section 3. Three different cases are distinguished according to whether the momentum vector is lightlike, spacelike or timelike.

Only those lightlike orbits with a timelike momentum allow a definition of a restmass (with respect to this momentum) and are capable to account for the wave-particle dualism of known matter. This is the topic of section 4.

Finally, in section 5, we demonstrate that a lightlike point with a timelike momentum may move in such a way, that its helicoidal orbit around its momentum dependent guiding center has a spatial spread in accord with the momentum-position uncertainty principle. In that case the fundamental length, introduced at the beginning of section 2, coincides with the Compton length of the restmass.

2. Dynamics of a lightlike point charge in an external electromagnetic field

Let λz be the position vector of the point charge in $\mathbb{R}^{1,3}$. The scalar λ **is a fundamental length** and the vector $z = (z_0 + \vec{z})\gamma_0$ is dimensionless. Now, if $\alpha \in \mathbb{R}$ is an arbitrary parameter, a curve $z = z(\alpha)$ is called **lightlike** if the tangent vector $\dot{z} = \frac{dz}{d\alpha}$ satisfies the **condition**

$$\dot{z}^2 = 0. \tag{2.1}$$

Writing $\dot{z} = w = (w_0 + \vec{w})\gamma_0$, $w^2 = 0$ implies $w = \pm(|\vec{w}| + \vec{w})\gamma_0$. So

$$\frac{d^2 z}{d\alpha^2} = \ddot{z} = \pm\left(\frac{\vec{w}}{|\vec{w}|} \cdot \dot{\vec{w}} + \dot{\vec{w}}\right)\gamma_0,$$

and hence

$$\ddot{z}^2 = -\left(\frac{\vec{w}}{|\vec{w}|} \times \dot{\vec{w}}\right)^2 \leq 0. \tag{2.2}$$

Thus, except for straight lightlike lines satisfying $\vec{w} \times \dot{\vec{w}} = \vec{0}$, the vector $\ddot{z}$ always is spacelike, i.e.,

$$\ddot{z}^2 < 0. \tag{2.3}$$

A smooth parameter change in equation (2.2) leads to the conclusion that equation (2.1) and relation (2.3) characterize all **curved** orbits of a lightlike point in a parameter independent, Lorentz invariant manner.

Equations of motion for a lightlike point in external fields have been derived in [3] employing the **Vessiot parameter** defined by

$$\ddot{z}^2 = -1. \tag{2.4}$$

Here we will obtain parameter independent equations of motion by variation of the integral

$$I = \int_{\alpha_1}^{\alpha_2} d\alpha L(z, \dot{z}, \ddot{z}) \tag{2.5}$$

between boundaries α_1 and α_2 subject to the constraint (2.1).

For a lightlike point charge minimally coupled to the external vector potential $a(z) \in \mathbb{R}^{1,3}$ we propose the gauge invariant Lagrangian

$$L(z, \dot{z}, \ddot{z}) = 2(-\ddot{z}^2)^{1/4} + \dot{z} \cdot a(z). \tag{2.6}$$

The vector potential $a(z)$ is dimensionless and may be related to the electromagnetic potential vector $A(\lambda z)$ at the spacetime point λz by

$$a(z) = \frac{q_0}{m_0 c^2} A(\lambda z), \tag{2.7}$$

where the constant q_0 is some charge unit, the constant m_0 defines a mass scale, c is the speed of light in vacuo and λ is the fundamental constant which defines the scale of length as indicated at the beginning of this section. The bivector of the electromagnetic fieldstrength $F(\lambda z) = \vec{E}(\lambda z) + i\vec{B}(\lambda z)$ at the point λz then is the exterior derivative of $a(z)$ with respect to z, viz.,

$$\partial \wedge a(z) = \mathcal{F}(z) = \frac{q_0 \lambda}{m_0 c^2} F(\lambda z). \tag{2.8}$$

We represent a **free, i.e., an unrestricted variation** of $z = z(\alpha)$ in the form

$$z \longrightarrow z + \varepsilon y,\ \varepsilon \in \mathbb{R},\ \dot{\varepsilon} = 0,\ y(\alpha_{1,2}) = 0,\ \dot{y}(\alpha_{1,2}) = 0, \tag{2.9}$$

implying

$$L \to L(z + \varepsilon y, \dot{z} + \varepsilon \dot{y}, \ddot{z} + \varepsilon \ddot{y}) = L + \varepsilon(y \cdot \partial_z + \dot{y} \cdot \partial_{\dot{z}} + \ddot{y} \cdot \partial_{\ddot{z}}) L(z, \dot{z}, \ddot{z}) + \mathcal{O}(\varepsilon^2)$$

and hence

$$I \to I + \varepsilon I_1 + \mathcal{O}(\varepsilon^2), \tag{2.10}$$

where

$$\begin{aligned} I_1 &= \int_{\alpha_1}^{\alpha_2} d\alpha\ (y \cdot \partial_z L + \dot{y} \cdot \partial_{\dot{z}} L + \ddot{y} \cdot \partial_{\ddot{z}} L) \\ &= \int_{\alpha_1}^{\alpha_2} d\alpha\ [y \cdot \partial_z L + \dot{y} \cdot (\partial_{\dot{z}} L - (\partial_{\ddot{z}} L)^{\cdot})]. \end{aligned} \tag{2.11}$$

The definition of the auxiliary vector k according to

$$\dot{k} = \partial_z L \tag{2.12}$$

permits a further integration by parts in (2.11) such that finally the first variation of I may be written in the form

$$I_1 = \int_{\alpha_1}^{\alpha_2} d\alpha \; \dot{y} \cdot [-k + \partial_{\dot{z}} L - (\partial_{\ddot{z}} L)^{\cdot}]. \tag{2.13}$$

The constraint (2.1) restricts the vector y in (2.9) according to

$$\dot{z}^2 = 0 \rightarrow (\dot{z} + \varepsilon\dot{y})^2 = \dot{z}^2 + 2\varepsilon\dot{y} \cdot \dot{z} + \varepsilon^2\dot{y}^2 = 0 = 2\varepsilon\dot{y} \cdot \dot{z} + \mathcal{O}(\varepsilon^2),$$

or

$$\dot{y} \cdot \dot{z} = 0. \tag{2.14}$$

This condition may be solved for $\dot{y}$ by putting

$$\dot{y} = \eta\dot{z} + Yi, \;\; \eta \in \mathbb{R}, \;\; Y \in \Lambda^3 \tag{2.15}$$

and fixing the trivector Y with the help of (2.14),

$$\dot{z} \cdot \dot{y} = \dot{z} \cdot (Yi) = (\dot{z} \wedge Y)i = 0,$$

in terms of an arbitrary bivector B, viz.,

$$Y = \dot{z} \wedge (Bi) = (\dot{z} \cdot B)i, \;\; B \in \Lambda^2. \tag{2.16}$$

So, equations (2.15) and (2.16) provide the general solution of (2.14)

$$\dot{y} = \eta\dot{z} + B \cdot \dot{z}, \;\; \eta \in \mathbb{R}, \;\; B \in \Lambda^2 \tag{2.17}$$

depending on seven arbitrary, scalar functions $\eta = \eta(\alpha)$ and $B = B(\alpha) \in \Lambda^2$ which have to vanish at the boundaries of the integration interval. Insertion of (2.17) into (2.13) casts the first variation I_1 in such a form

$$I_1 = \int_{\alpha_1}^{\alpha_2} d\alpha \; \eta\dot{z} \cdot [-k + \partial_{\dot{z}} L - (\partial_{\ddot{z}} L)^{\cdot}] + \int_{\alpha_1}^{\alpha_2} d\alpha \; B \cdot [\dot{z} \wedge (-k + \partial_{\dot{z}} L - (\partial_{\ddot{z}} L)^{\cdot})], \tag{2.18}$$

that the principal conclusion of variational calculus may be applied with respect to η and B: The extremals of (2.5) subjected to the subsidiary condition (2.1) necessarily have to fulfill equation (2.12), the scalar equation

$$\dot{z} \cdot [-k + \partial_{\dot{z}} L - (\partial_{\ddot{z}} L)^{\cdot}] = 0, \tag{2.19}$$

and the bivector equation

$$\dot{z} \wedge [-k + \partial_{\dot{z}} L - (\partial_{\ddot{z}} L)^{\cdot}] = 0. \tag{2.20}$$

Forming an inner product (contraction) of (2.20) with the vector $\dot{z}$, one notes that (2.19) is a consequence of (2.20) and hence may be discarded. The equations of motion therefore may be summarized in the system

$$\dot{z}^2 = 0,\ \dot{z} \wedge [k - \partial_{\dot{z}} L + (\partial_{\ddot{z}} L)^{\cdot}] = 0,\ \dot{k} = \partial_z L. \tag{2.21}$$

There is now the question, which functional dependence of the Lagrangian $L(z, \dot{z}, \ddot{z}) = L(z, w, b)$ on the vectors z, w and b generates parameter-independent equations (2.21)? A functional equation for $L(z, w, b)$ may be derived by studying a parameter change $\alpha \leftrightarrow \beta$ in the constraint (2.1) and the variational integral (2.5). Let $\beta = \beta(\alpha)$ and $z(\alpha) = z(\alpha(\beta)) = \overline{z}(\beta)$, $\dot{z} = \dot{\beta}\overline{z}'$, $0 < \dot{\beta} = \frac{d\beta}{d\alpha} < \infty$, $\ddot{z} = \ddot{\beta}\overline{z}' + \dot{\beta}^2\overline{z}''$, where dots denote derivatives with respect to α and primes derivatives with respect to β. Then $\dot{z}^2 = 0$ and hence $\dot{z} \cdot \ddot{z} = 0$ imply $\overline{z}'^2 = 0$ and $\overline{z}' \cdot \overline{z}'' = 0$. The integral (2.5) is transformed into $I = \int_{\beta_1}^{\beta_2} \frac{d\beta}{\dot{\beta}} L(\overline{z}, \dot{\beta}\overline{z}', \ddot{\beta}\overline{z}' + \dot{\beta}^2\overline{z}'')$, which leads to equation (2.21) with primes instead of dots if and only if the Lagrangian $L(z, w, b)$ satisfies the functional equation

$$L(z, \dot{\beta}w, \ddot{\beta}w + \dot{\beta}^2 b) = \dot{\beta}[L(z, w, b) + (w \cdot \partial_z + b \cdot \partial_w)\gamma(z, w)] \tag{2.22}$$

for all $z, w, b \in \mathbb{R}^{1,3}$ with $w^2 = 0$ and $b \cdot w = 0$. The scalar function $\gamma(z, w)$ generates a total differential in the integrand of I when the mapping from α to β is performed.

The Lagrangian (2.6) obviously fulfills (2.22) with $\gamma = 0$, i.e.,

$$L(z, \dot{\beta}w, \ddot{\beta}w + \dot{\beta}^2 b) = \dot{\beta}L(z, w, b),\ 0 < \dot{\beta} < \infty, \tag{2.23}$$

because $w^2 = 0$ and $w \cdot b = 0$ imply $(\ddot{\beta}w + \dot{\beta}^2 b)^2 = \dot{\beta}^4 b^2$.

The parameter independence of (2.21) in case of the Lagrangian (2.6) can be made manifest by defining for $\dot{z}^2 = 0$ the parameter-independent bivector [3]

$$\Sigma = \dot{z} \wedge \partial_{\ddot{z}} L = \frac{\ddot{z} \wedge \dot{z}}{(-\ddot{z}^2)^{3/4}}, \tag{2.24}$$

and the momentum vector

$$p = k - \partial_{\dot{z}} L = k - a(z), \tag{2.25}$$

which also does not depend on the choice of the parameter. With (2.24) and (2.25) the equations of motion (2.21) for the Lagrangian (2.6) become

$$\dot{z}^2 = 0, \ \Sigma(-\ddot{z}^2)^{3/4} = \ddot{z} \wedge \dot{z}, \ \dot{\Sigma} = p \wedge \dot{z}, \ \dot{p} = \mathcal{F} \cdot \dot{z}, \ \mathcal{F} = \partial \wedge a, \tag{2.26}$$

where $\mathcal{F}$ is related to the bivector of the electromagnetic fieldstrengthes by equation (2.8). As in section 8 of ref. [3] one may define a parameter-independent total angular momentum bivector

$$J = z \wedge p + \Sigma, \tag{2.27}$$

for which the **the momentum law**

$$\dot{p} = \mathcal{F} \cdot \dot{z} \tag{2.28}$$

in (2.26) implies **the angular momentum law**

$$\dot{J} = z \wedge (\mathcal{F} \cdot \dot{z}). \tag{2.29}$$

This section ends with a deduction of equation (2.19) in a form which for $\dot{z}^2 = 0$ manifestly does not depend on the choice of parameter, viz.,

$$p \cdot \dot{z} = (-\ddot{z}^2)^{1/4}. \tag{2.30}$$

With (2.24) one concludes from $\Sigma \cdot \dot{z} = 0$ by taking a derivative with respect to an arbitrary parameter α and making use of $\dot{\Sigma} = p \wedge \dot{z}$ in (2.26),

$$\dot{\Sigma} \cdot \dot{z} + \Sigma \cdot \ddot{z} = 0 = \Sigma \cdot \ddot{z} + (p \wedge \dot{z}) \cdot \dot{z}.$$

Insertion of the definition (2.24) for Σ on the right hand side and evaluation of the inner products with the help of $\dot{z}^2 = 0$ and $\dot{z} \cdot \ddot{z} = 0$ then proves (2.30).

3. Free motions of a lightlike point

A lightlike point charge moves freely if $\mathcal{F} = \partial \wedge a = 0$. In that case, according to equations (2.28) and (2.29), the momentum vector p and the angular momentum bivector J are constants of the motion,

$$\dot{p} = 0, \;\; \dot{J} = 0. \tag{3.1}$$

From (2.27)

$$z \wedge p = J - \Sigma \tag{3.2}$$

we find $(z \wedge p) \cdot p = zp^2 - p(p \cdot z) = J \cdot p - \Sigma \cdot p$, or,

$$zp^2 = J \cdot p + p(p \cdot z) - \Sigma \cdot p. \tag{3.3}$$

Because of $\dot{p} = 0$, equation (2.30) becomes

$$p \cdot \dot{z} = (-\ddot{z}^2)^{1/4} = (p \cdot z)^{\cdot}. \tag{3.4}$$

This suggests the choice of the Vessiot parameter [3]

$$(-\ddot{z}^2)^{1/4} = 1, \tag{3.5}$$

whence

$$p \cdot \ddot{z} = 0, \tag{3.6}$$

and equation (3.4), up to an unimportant constant, may be integrated to

$$p \cdot z = \alpha. \tag{3.7}$$

The choice (3.5) simplifies the definition of Σ, equation (2.24),

$$\Sigma = \ddot{z} \wedge \dot{z}, \tag{3.8}$$

such that according to (3.4) and (3.6)

$$\Sigma \cdot p = \ddot{z}, \tag{3.9}$$

and equation (3.3) may be written

$$p^2 z = J \cdot p + \alpha p - b, \tag{3.10}$$

where

$$b = \Sigma \cdot p = \ddot{z}. \tag{3.11}$$

Taking two derivatives of equation (3.10) with respect to α, one finds for b the differential equation

$$\ddot{b} + p^2 b = 0, \tag{3.12}$$

displaying the linear dependence of the fourth derivative of z from the second one. Recalling the definition of curvature classes in [3], we therefore conclude that the orbits of a freely moving lightlike point either are straight lines or at most doubly curved. The solutions of equation (3.12) are restricted by equations (3.5) and (3.6). In addition, $\dot{z}^2 = 0$ together with (3.10) and (3.2) require further constraints which in the sequel are considered separately according to the space-time properties of the momentum vector p. Three cases are to be distinguished:

If the momentum p of a lightlike orbit z is a timelike vector, i.e., $p^2 > 0$, we speak of a massive free particle (antiparticle) of momentum p and restmass $\sqrt{p^2} > 0$. Following G. Feinberg [4], we call such an object a **tardyon**.

If lightlike orbits z are such that the momentum vectors $p \neq 0$ are lightlike, i.e., $p^2 = 0$, we call these massless objects **luxons** [4] (like e.g. a neutrino).

If finally a lightlike orbit z has a spacelike momentum p, i.e., $p^2 < 0$, we say that this orbit z describes a **tachyon** [4] of momentum p, total angular momentum J and eventually additional specifications needed for a unique characterization.

At any rate, the necessary specifications will be found during the process of solving the above equations of motion for the three classes of momenta. We now start with the luxon.

3.1 Luxons: $p^2 = 0$

For $p^2 = 0$, equation (3.12) falls back on equations (3.10) and (3.11), viz.,

$$\ddot{z} = J \cdot p + \alpha p = b, \tag{3.13}$$

and fulfills condition (3.6). So,

$$\dot{z} = q + \alpha J \cdot p + \frac{\alpha^2}{2} p, \ \dot{q} = 0, \tag{3.14}$$

and

$$z = z(0) + \alpha q + \frac{\alpha^2}{2} J \cdot p + \frac{\alpha^3}{6} p, \ q = \dot{z}(0). \tag{3.15}$$

Thus, a lightlike point z with the properties of a luxon moves on a cubic! The restriction (3.5) or $\dddot{z}^2 = -1$ implies the condition

$$(J \cdot p)^2 = -1, \tag{3.16}$$

and $\dot{z}^2 = 0$ is satisfied if

$$p^2 = 0 = q^2, \ p \cdot q = 1, \ p \cdot J \cdot q = 0. \tag{3.17}$$

With (3.13) and (3.14), the bivector $\Sigma = \ddot{z} \wedge \dot{z}$ becomes $\Sigma = (J \cdot p) \wedge q + \alpha p \wedge q + \frac{\alpha^2}{2} p \wedge (J \cdot p)$ and $z \wedge p = z(0) \wedge p + \alpha q \wedge p + \frac{\alpha^2}{2} (J \cdot p) \wedge p$. Hence, we obtain

$$z \wedge p + \Sigma = z(0) \wedge p + (J \cdot p) \wedge q = J, \tag{3.18}$$

if (3.2) is to be warranted. Inner multiplication of (3.18) with q and making use of $p \cdot q = 1$ leads to $z(0) = \mu p + J \cdot q$ where $\mu = q \cdot z(0)$, whence J has to be restricted in such a way, that $J = (J \cdot q) \wedge p + (J \cdot p) \wedge q$ holds. With the help of $p \cdot q = 1$, this condition may be rearranged into the multivector equation

$$pJq + qJp = 0, \tag{3.19}$$

and (3.15) may be written

$$z = \mu p \sqrt{2} + J \cdot q + \alpha q + \frac{\alpha^2}{2} J \cdot p + \frac{\alpha^3}{6} p, \ \mu \in \mathbb{R}. \tag{3.20}$$

The task now is, to decouple equations (3.16), (3.17) and (3.19) in terms of more appropriate new variables. To that end we put

$$p\sqrt{2} = v + s, \ q\sqrt{2} = v - s, \tag{3.21}$$

where

$$v^2 = 1 = -s^2, \ v \cdot s = 0. \tag{3.22}$$

This linear mapping from the vectors p, q to the vectors v, s solves the conditions $p^2 = 0 = q^2$, $p \cdot q = 1$ and transforms (3.19) into

$$vJv = sJs. \tag{3.23}$$

Making use of (3.22) one notes that (3.23) as well may be written in the form $vJs = sJv$, whose scalar part leads to $v \cdot J \cdot s = 0$ and (3.21) implies $p \cdot J \cdot q = 0$. So, equations (3.21)-(3.23) comprise (3.17) and (3.19).

The general solution of (3.23) and (3.16) is obtained in a most succinct form when the two vectors v and s orthonormally are supplemented by a spacelike third vector t,

$$v \cdot t = 0 = s \cdot t, \ t^2 = -1 \tag{3.24}$$

to a triad which spans a $\mathbb{R}^{1,2}$. In terms of this triad we obtain the result

$$J = \frac{\sqrt{2}}{|\eta + \zeta|}(\eta v - \zeta s)t, \tag{3.25}$$

where

$$\eta = \eta_0 + i\eta_4, \ \eta_{0,4} \in \mathbb{R}, \ \zeta = \zeta_0 + i\zeta_4, \ \zeta_{0,4} \in \mathbb{R}, \tag{3.26}$$

and

$$\begin{aligned} |\eta + \zeta| &= \sqrt{(\eta + \zeta)(\eta + \zeta)^*} \geq 0, \\ (\eta + \zeta)^* &= \gamma_0(\eta + \zeta)\gamma_0 = \eta_0 + \zeta_0 - i(\eta_4 + \zeta_4). \end{aligned} \tag{3.27}$$

The general solution (3.25) implies

$$J \cdot p|\eta + \zeta| = -(\eta_0 + \zeta_0)t + (\eta_4 + \zeta_4)Ti, \tag{3.28}$$

$$J \cdot q|\eta + \zeta| = (\zeta_0 - \eta_0)t + (\zeta_4 - \eta_4)Ti, \tag{3.29}$$

where the trivector T is defined by

$$T = vst, \tag{3.30}$$

and

$$J^2 = \frac{2(\eta - \zeta)}{(\eta + \zeta)^*}. \tag{3.31}$$

Inserting (3.29), (3.28) and (3.21) in (3.20), the general orbit $z = z(\alpha)$, $\alpha \in \mathbb{R}$, of a luxon becomes

$$\begin{gathered} z = \mu(v+s) + \frac{\eta_0 + \zeta_0}{|\eta + \zeta|} t + \frac{\zeta_4 - \eta_4}{|\eta + \zeta|} Ti + \\ + \frac{\alpha}{\sqrt{2}}(v - s) + \frac{\alpha^2}{2}[-\frac{\eta_0 + \zeta_0}{|\eta + \zeta|} t + \frac{\eta_4 + \zeta_4}{|\eta + \zeta|} Ti] + \frac{\alpha^3}{6\sqrt{2}}(v + s). \end{gathered} \tag{3.32}$$

3.2 Tachyons: $p^2 < 0$ and Tardyons: $p^2 > 0$

For $p^2 \neq 0$ the trajectory $z = z(\alpha)$ of the lightlike point z is determined by (3.10) and (3.11) provided that $b = b(\alpha)$ is known. Equation (3.12) only is a necessary condition to determine b since it follows from (3.10) or (3.3) by inner multiplication of (3.2) with the vector p. In order to exploit the angular momentum law completely we now study the additional implications of

$$J \wedge p = \Sigma \wedge p = \ddot{z} \wedge \dot{z} \wedge p \tag{3.33}$$

concerning b and J. With the help of a derivative of (3.10) with respect to α and making use of (3.11), the vector z may be eliminated from (3.33) in favour of the vector, viz.,

$$p^2(J \wedge p) = -b \wedge \dot{b} \wedge p = -p \wedge b \wedge \dot{b}. \tag{3.34}$$

Equation (3.11) implies $b \cdot p = 0$ and hence $\dot{b} \cdot p = 0$, so that $p \cdot (b \wedge \dot{b}) = 0$, which means, that (3.34) can be written in the form

$$b \wedge \dot{b} = -p(J \wedge p). \tag{3.35}$$

The Vessiot choice (3.5) leads to $b^2 = -1$, and consequently, $b \cdot \dot{b} = 0$, whence (3.35) becomes $b\dot{b} = -p(J \wedge p)$ or after Clifford multiplication with b from the left,

$$\dot{b} = b\Omega, \tag{3.36}$$

where

$$\Omega = p(J \wedge p) = (J \wedge p)p = p \cdot (J \wedge p). \tag{3.37}$$

Since the bivector Ω is constant, the general solution of (3.36) is

$$b = b(0)e^{\alpha\Omega} = e^{-\frac{\alpha}{2}\Omega}b(0)e^{\frac{\alpha}{2}\Omega}. \tag{3.38}$$

The vector $b(0)$ has to satisfy

$$b(0)\Omega + \Omega b(0) = 0 = 2[b(0) \wedge \Omega] \tag{3.39}$$

in order that $b(0)e^{\alpha\Omega}$ has no trivector part. Equation (3.6) restricts the solution (3.38) according to $b \cdot p = 0$, which because of $p\Omega = \Omega p$ (see (3.37)) is guaranteed if

$$b(0) \cdot p = 0. \tag{3.40}$$

Similarly, $b^2 = -1$ is fulfilled, when

$$(b(0))^2 = -1 \tag{3.41}$$

holds. In the same manner as (2.17) has been derived, one may conclude from (3.40) that

$$b(0) = K \cdot p, \tag{3.42}$$

where K is a constant bivector. The most general constant bivector we may construct from J and Ω is $K = \rho_1 e^{i\varphi_1} J + \rho_2 e^{i\varphi_2}\Omega$, $\rho_{1,2} > 0$, $\varphi_{1,2} \in \mathbb{R}$. This leads to

$$b(0) = K \cdot p = \rho J \cdot p + i\sigma(J \wedge p), \;\; \rho, \sigma \in \mathbb{R}, \tag{3.43}$$

which must satisfy condition (3.39). The result is

$$b(0) = \rho J \cdot p, \;\; \rho \in \mathbb{R}\backslash\{0\}, \tag{3.44}$$

and the condition $(J \cdot p)(J \wedge p) = (J \wedge p)(J \cdot p)$, i.e., $J^2 p = pJ^2$ implying that J^2 must be a scalar,

$$J^2 = \gamma_0 J^2 \gamma_0 = (J^2)^* \in \mathbb{R}. \tag{3.45}$$

This holds for $J \neq 0$ in $\mathcal{C}\ell_{1,3}$ if and only if J is an outer product of two vectors. There is a further restriction imposed on J by equation (3.12). From (3.38) one notes $\Omega^2 = -p^2 = p^2(J \wedge p)^2$ and hence

$$(J \wedge p)^2 = -1. \tag{3.46}$$

Condition (3.41) in conjunction with (3.44) leads to

$$(J \cdot p)^2 < 0 \tag{3.47}$$

and

$$\rho\sqrt{|(J \cdot p)^2|} = \pm 1. \tag{3.48}$$

Defining the **spin bivector** S according to

$$S\sqrt{|p^2|} = \Omega = p(J \wedge p) = p(\Sigma \wedge p), \tag{3.49}$$

equation (3.46) implies

$$S^2 = -\frac{p^2}{|p^2|} = -sign(p^2) \tag{3.50}$$

and the expression (3.10) for every trajectory of tachyons and tardyons finally may be summarized in the form

$$p^2 z = J \cdot p + \alpha p \mp \frac{(J \cdot p)}{\sqrt{|(J \cdot p)^2|}} e^{\alpha S\sqrt{|p^2|}}, \ \alpha \in \mathbb{R}. \tag{3.51}$$

For a tardyon, $p^2 > 0$, the spin bivector is spacelike, i.e. $S^2 = -1$ as is seen from (3.50). So, only for tardyons the orbits (3.51) are helices, elliptically wound along the straight line $\frac{1}{p^2}(J \cdot p + \alpha p)$, $-\infty < \alpha < \infty$ with the timelike tangent vector $\frac{1}{p}$. This (elliptic) rotation around the timelike guiding center conventionally is called the "Zitterbewegung". In the following sections it will be shown that this rotation spatially is confined to a cylinder enclosing the guiding straight line and having a small cross section of the order of a Compton length.

As distinguished from the tardyon, there is no such spatial confinement transverse to the guiding straight line for the tachyon, since the spin bivector is timelike and hence periodicity in space can not emerge. The following sections may therefore lead to the conclusion, that **quantum phenomena like the wave-particle dualism and spatial uncertainty only occur for tardyons, i.e., for massive particles and antiparticles!**

4. Wave-Particle Duality of Tardyons

For a tardyon, $p^2 > 0$, still, the bivector of total angular momentum J has to be restricted in such a way that conditions (3.45)–(3.47) hold, before formula (3.51) may be applied. A simple tool for solving these conditions is to transform the timelike momentum $p = (p_0 + \vec{p})\gamma_0$, with energy component p_0 and space part $\vec{p}$, to its restframe in which p is proportional to γ_0. Inversely, every timelike vector may be obtained by applying a suitable Lorentz transformation on a real multiple of the vector γ_0. Let us call such a representation of a vector p a polar form. This form extends the more familiar construction of spherical polar coordinates for a vector $\vec{r} \in \mathbb{R}^3$ to a subset of timelike vectors in the Lorentz space $\mathbb{R}^{1,3}$. The subset of all timelike vectors p with $0 < p^2 < \infty$ however consists of two disconnected manifolds distinguished by $p_0 = p \cdot \gamma_0 > 0$ and $p_0 = p \cdot \gamma_0 < 0$. The two parts cannot be mapped onto each other by means of a **proper** Lorentz transformation (determinant $= +1$). Consequently one needs two different polar forms (charts) in order to cover both of the disconnected parts, viz.,

$$p = \varepsilon |p| \nu, \;\; \varepsilon = \pm 1, \;\; |p| = \sqrt{p^2} > 0, \tag{4.1}$$

$$\nu = (\nu_0 + \vec{\nu})\gamma_0, \;\; \nu_0 = \sqrt{1 + \vec{\nu}^2} \geq 1, \;\; \vec{\nu} \in \mathbb{R}^3, \tag{4.2}$$

implying

$$\nu^2 = \nu_0^2 - \vec{\nu}^2 = 1. \tag{4.3}$$

We call a tardyon a particle if $\varepsilon = +1$ **and an antiparticle if** $\varepsilon = -1$. The time component of the vector $\nu \in \mathbb{R}^{1,3}$ is positive as is seen from (4.2). So, there is (at least) one unimodular spinor L such that for all $\vec{\nu} \in \mathbb{R}^3$, the vector ν has the polar form

$$\nu = L \gamma_0 \tilde{L}, \tag{4.4}$$

and in particular one may choose

$$L = \frac{1 + \nu_0 + \vec{\nu}}{\sqrt{2(1 + \nu_0)}}. \tag{4.5}$$

Equation (4.4) finally yields the following two polar forms of the momentum p of a tardyon,

$$p = \varepsilon |p| L \gamma_0 \tilde{L}, \;\; \varepsilon = \pm 1. \tag{4.6}$$

In conditions (3.45)–(3.47) the vector $J \cdot p$ and the trivector $J \wedge p$ is needed. These quantities are obtained from the Clifford product Jp by a split into the grades one and three. Hence, we evaluate with the help of (4.6) and

$$L\tilde{L} = 1 = \tilde{L}L \tag{4.7}$$

the product $Jp = \varepsilon|p|JL\gamma_0\tilde{L} = \varepsilon|p|L(\tilde{L}JL\gamma_0)\tilde{L}$. From $(\tilde{L}JL)^2 = \tilde{L}J^2L = J^2$ one concludes, that J^2 is a scalar if and only if $(\tilde{L}JL)^2$ is a scalar. On the algebra $\mathcal{C}\ell_{1,3}$ of spacetime $\mathbb{R}^{1,3}$ **every bivector** $\tilde{L}JL$ may be decomposed according to

$$\tilde{L}JL = \vec{f} + i\vec{g}, \;\; \vec{f}, \vec{g} \in \mathbb{R}^3 \tag{4.8}$$

into $\vec{f}$ and $\vec{g}$, **which are vectors of grade one with respect to the even subalgebra** $\mathcal{C}\ell_3$ **of** $Cl_{1,3}$. This implies

$$Jp = \varepsilon|p|L[(\vec{f} + i\vec{g})\gamma_0]\tilde{L} = \varepsilon|p|L\vec{f}\gamma_0\tilde{L} + \varepsilon|p|Li\vec{g}\gamma_0\tilde{L} = J \cdot p + J \wedge p, \tag{4.9}$$

and hence

$$(J \cdot p)^2 = -\vec{f}^2 p^2 < 0, \tag{4.10}$$

$$(J \wedge p)^2 = -\vec{g}^2 p^2 = -1, \tag{4.11}$$

in order to fulfill (3.47) and (3.46). Condition (3.45) is satisfied by

$$\vec{f} \cdot \vec{g} = 0, \tag{4.12}$$

as is noted after squaring equation (4.8).

Let us now study the consequences of (4.9)–(4.12) in equations (3.49) and (3.51). From (4.9) and (4.6) one infers that the spin bivector (3.49) is the Lorentz boost

$$S = Li\vec{s}\tilde{L}, \tag{4.13}$$

where $\vec{s} = \vec{g}|p|$, as a consequence of (4.11), is a unit vector in $\mathbb{R}^3$,

$$\vec{s}^2 = 1. \tag{4.14}$$

According to (4.12), $\vec{f}$ must be perpendicular to $\vec{s}$. We write

$$\vec{f} = \mp\varepsilon|\vec{f}|\vec{n}\times\vec{s},\ |\vec{f}| = \sqrt{\vec{f}^{\,2}} > 0, \tag{4.15}$$

where

$$\vec{n}\cdot\vec{s} = 0, \tag{4.16}$$

and

$$\vec{n}^2 = 1. \tag{4.17}$$

With (4.9), (4.13)–(4.17), the most general trajectory of a tardyon (3.51) may be displayed in the final form

$$z = z(\beta) = z(0) + \frac{\beta}{|p|}\nu + \frac{1}{p^2}L(\vec{n}\times\vec{s})e^{i\varepsilon\vec{s}\beta|p|}L\gamma_0,\ -\infty < \beta < \infty, \tag{4.18}$$

where $|p|z(0) = \varepsilon|\vec{f}|L(\vec{n}\times\vec{s})L\gamma_0$ and use has been made of $\gamma_0\tilde{L} = L\gamma_0$.

Note, that in passing from (3.51) to (4.18), a parameter change has been performed from α to $\beta = \alpha\varepsilon$. This change normalizes the direction of motion of the guiding center on the straight line

$$x = x(\beta) = z(0) + \frac{\beta}{|p|}\nu, \tag{4.19}$$

as β runs from $-\infty$ to $+\infty$. As a consequence of this normalization the factor $\varepsilon = +1$ (particle) and $\varepsilon = -1$ (antiparticle) appears in the exponential of the periodic part of (4.18),

$$y = y(\beta) = \frac{1}{p^2}L(\vec{n}\times\vec{s})e^{i\varepsilon\vec{s}\beta|p|}L\gamma_0, \tag{4.20}$$

modeling what we call the "Zitterbewegung" of the lightlike point

$$z = x + y. \tag{4.21}$$

The lightlike point z is dual in the sense that its motion is composed of two parts: the motion along a straight line with a timelike tangent vector, usually attributed to a massive particle; and a space-like periodic motion circumscribing in $\mathbb{R}^3$ an ellipse, conventionally attributed to a wave. During one revolution period of (4.20)

$$\beta_0 = \frac{2\pi}{|p|}, \tag{4.22}$$

the guiding center is displaced along the straight line (4.19) by the vector

$$\lambda = \varepsilon\lambda\!\!\!^{-}[x(\beta+\beta_0) - x(\beta)] = \varepsilon\lambda\!\!\!^{-}\frac{\beta_0}{|p|}\nu = \varepsilon 2\pi\lambda\!\!\!^{-}\frac{\nu}{p^2}, \tag{4.23}$$

where **the constant $\lambda\!\!\!^{-}$ is the fundamental length introduced at the beginning of section 2**. In the sense of Louis de Broglie [5] we equate the (inner) product of this displacement vector with the momentum vector (4.1) with Planck's constant h, viz.,

$$\lambda p m_0 c = h = \frac{2\pi\lambda\!\!\!^{-}}{|p|}m_0 c, \tag{4.24}$$

thus fixing the fundamental length $\lambda\!\!\!^{-}$ on a value proportional to the Compton length of the mass m_0,

$$\lambda\!\!\!^{-} = \frac{\hbar}{m_0 c}|p|. \tag{4.25}$$

5. Uncertainty in Space

In order to relate the spatial spread of the orbit (4.21) due to the high-frequency "Zitterbewegung" (4.20) to a quantity Δ with the dimension of an angular momentum, we first have to establish the notions restmass m and proper time τ. Our conjecture is, that both of these two macro-quantities characterize properties of the guiding center motion (4.19).

The momentum vector (4.6) is dimensionless. We only need to rescale it with the factor m_0c (see below equation (2.7)) in order to obtain the physical momentum vector m_0cp. So, the restmass m is given by $(m_0cp)^2 = m^2c^2$, i.e.,

$$m = m_0|p| > 0 \tag{5.1}$$

according to (4.6). As is customary, we define the differential $d\tau$ of the proper time by

$$c d\tau = \lambdabar \sqrt{(dx)^2}, \tag{5.2}$$

whereupon (4.19) and (4.25) lead to

$$\tau = \frac{\beta \lambdabar}{c|p|} = \frac{\beta \hbar}{m_0 c^2}. \tag{5.3}$$

Insertion of (4.22) into (5.3) yields the timeperiod of Broglie's clock

$$\tau_0 = \frac{2\pi\hbar}{m_0|p|c^2} = \frac{h}{mc^2}. \tag{5.4}$$

The "Zitterbewegung" is defined by the vector $y \in \mathbb{R}^{1,3}$ which moves as β varies according to (4.20),

$$y\gamma_0 = y_0 + \vec{y} = L\vec{e}L, \ \vec{e}p^2 = (\vec{n} \times \vec{s})e^{i\varepsilon\vec{s}\varphi}, \ \varphi = \beta|p| = \frac{c\tau p^2}{\lambdabar}. \tag{5.5}$$

We claim that the spatial part $\vec{y}$ of the vector y is responsible for the microscopic spread of the orbit (4.21). A calculation of $\vec{y}$ is straightforward if

$$\vec{e}p^2 = (\vec{n} \times \vec{s})e^{i\varepsilon\vec{s}\varphi} = \vec{n} \times \vec{s}\cos\varphi + \varepsilon\vec{n}\sin\varphi \tag{5.6}$$

is decomposed into components parallel to $\vec{\nu}$ (defining L in (4.5)) and perpendicular to $\vec{\nu}$. The result is

$$\vec{y} = \vec{e} + \frac{\vec{\nu}(\vec{\nu} \cdot \vec{e})}{1 + \nu_0}, \ \nu_0 = \sqrt{1 + \vec{\nu}^2}. \tag{5.7}$$

As the parameter φ in (5.6) varies from $\varphi = 0$ to $\varphi = 2\pi$, the vector $\vec{y}$ in (5.7) surrounds an ellipse, whose area we consider as a measure for the spatial uncertainty, viz.,

$$\Delta = \frac{m\lambdabar^2}{\tau_0}|\frac{1}{2}\oint \vec{y} \times d\vec{y}| = \frac{\hbar p^4}{4\pi}|\oint \vec{y} \times d\vec{y}|. \tag{5.8}$$

Making use of (4.14), (4.16)–(4.17), we derive

$$\frac{\varepsilon p^4}{2\pi}\int_0^{2\pi} d\varphi \ \vec{y} \times \frac{d\vec{y}}{d\varphi} = \vec{s} + \frac{\vec{\nu} \times (\vec{s} \times \vec{\nu})}{1 + \nu_0} \tag{5.9}$$

and find for Δ the lower bound

$$\Delta \geq \frac{\hbar}{2}|\vec{s}| = \frac{\hbar}{2}. \tag{5.10}$$

From this result we draw the conclusion that **the properties of a free tardyon are compatible with the uncertainty principle for momentum and position in** $\mathbb{R}^3$. For $p^2 = 1$, the restmass (5.1) equals m_0 and the fundamental length (4.25) $\lambda\!\!\!^-$ **coincides** with the Compton length.

Now it is obvious, that the orbits of a freely moving lightlike point with a momentum vector which either is spacelike or lightlike, do not exhibit the conventional wave-particle dualism.

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