Parametric resonance as a possible cause of spontaneous transition from metastable states

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ABSTRACT. The recent results in dynamics of the first-order phase transitions in non-dissipative nonlinear systems are reviewed. Some parallels with quantum-mechanical transitions are discussed.

I am especially pleased to make a contribution to the issue of the Annales dedicated to the jubilee of my friend and teacher Georges Lochak. Many years he supports with real interest my bustling activity in nonlinear wave equations giving it a sense and physical content. I have been still remembering with pleasure that (now distant) time when we had succeeded in applying some methods of the soliton theory to the nonlinear spinor equation proposed by G. Lochak for the magnetic monopole [1].

But today I would like to discuss the results of the recent investigations in the first-order phase transitions pursued in collaboration with A. Shagalov from the Institute of Metal Physics [2]. Also, I would like to share some new considerations which came into my mind in this connection. For the sake of clarity I will try to explain the essence managing with a few simple formulas.

Let us consider a closed physical system having the potential energy of the form depicted in Fig. 1. The variable ϕ describes the state of the system and is usually refered to as an order parameter. It may be, for example, density in the system gas-liquid or liquid-solid, displacement of atoms in a nonlinear cristal, distortion in some shape-memory alloys and others. The transition of the system from a metastable state ϕ_1 to the true vacuum state ϕ_2 , known as a first-order phase transition, proceeds through the formation of local inhomogeneities of a sufficiently large magnitude (critical nuclei), their growth and the subsequent expansion of bubbles of a new phase inside the old one [3]. If the dissipation in the system is significant, the waves cannot propagate, and the energy transfer occurs via the diffusion mechanism. Such systems are discribed by the equations of the Ginzburg-Landau-Khalatnikov type [3, 4]. In this case the system being initially at rest in a metastable state remains in this state as long as a thermal fluctuation of anough magnitude happens to overcome the potential barrier and initiate the phase transition. Since the probability of a fluctuation is exponentially small in its energy, the system can remain in the metastable state for a fairly long time.

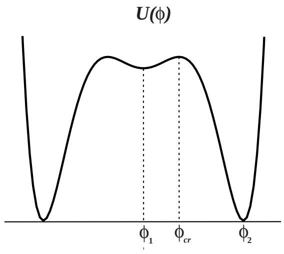

Figure 1: The shape of the potential $U(\phi)$.

However, the situation is drastically changed when the dissipation is negligibly small. The evolution of the order parameter field in now governed by the wave instead of the diffusion equation. The most important feature of nonlinear wave equations is the resonance interaction of wave modes. This can lead to instabilities which are often manifest themself as a fast increasing a field magnitude in small spatial domains. The further development of this process is known as a wave collapse. Thus, a new possibility appears to overcome the potential barrier in a non-thermal way, in those domains where the field exceeds the critical level due to resonance and collapse-like effects. Obviously, this mechanism, being pure dynamical, differs also from the quantum tunneling which is usually invoked when considering the cosmological phase transitions [5].

To illustrate how the resonance mechanism can initiate the first-order phase transition let us consider a physical system described by the non-linear Klein-Gordon equation,

$$\phi_{tt} - \Delta\phi + U'(\phi) = 0, \tag{1}$$

with the potential

$$U(\phi) = U(0) + \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4 + \frac{\epsilon}{6}\phi^6. \tag{2}$$

Nonlinearities of this kind occur frequently in the condense matter physics [6,7], quantum field theory [8, 9] and modern cosmology [5]. It is easy to check that for $0 < \epsilon < 3/16$ the model has the metastable false vacuum at $\phi = \phi_1 = 0$ and the true vacuum at $\phi = \pm\phi_2$ (Fig. 1).

Let us consider spatially uniform oscillations $\phi(t)$ in the potential well near the metastable vacuum. For simplicity we assume ϵ to be sufficiently small so that the depth of the well near ϕ_1 is $\sim 1/4$, its width is ~ 1, while the true vacuum value $\phi_2 \sim \epsilon^{-1/2}$. Hence, considering the oscillations in this well we can restrict ourselves to the ϕ^4 theory, neglecting the term ϕ^6. However, as we will see later, this term becomes important when considering the bubble dynamics once the phase transition has locally occurred. Taking the above into account, from (1), (2) we find

$$\phi(t) = \phi_{\max} sn\left(\frac{t - t_0}{\sqrt{1+\varkappa^2}}, \varkappa\right), \tag{3}$$

where

$$\varkappa^2 = \phi_{\max}^2/(2 - \phi_{\max}^2) \tag{4}$$

is the modulus squared of the elliptic sine, $0 < \varkappa^2 < 1$, $0 < \phi_{\max}^2 < 1$.

Consider now a small perturbation on this background,

$$\phi(t, \mathbf{r}) = \phi(t) + \chi(t, \mathbf{r}) \qquad (|\chi| \ll |\phi|). \tag{5}$$

In the linear approximation χ satisfies the equation

$$\chi_{tt} - \triangle\chi + (1 - 3\phi^2(t))\chi = 0. \tag{6}$$

Performing the Fourier transform,

$$\chi(t, \mathbf{r}) = \int A(t, \mathbf{k}) e^{i\mathbf{kr}} d\mathbf{k}, \tag{7}$$

and introducing

$$\tau = (t - t_0)(1 + \varkappa^2)^{-1/2}, \tag{8}$$

we arrive at the Lamé equation

$$a_{\tau\tau} + [(1+k^2)(1+\varkappa^2) - 6\varkappa^2 sn^2(\tau,\varkappa)]a = 0 \tag{9}$$

(here a means the real or imaginary part of A).

This equation belongs to the class of the Hill equations describing the physical systems in which the parametric resonance can occur. According to the Floquet theory [10], the growing resonant solutions have the generic form $\varphi(\tau)e^{\mu\tau}$, where $\varphi(\tau)$ is a periodic function and $\mu > 0$ is the characteristic exponent, which depend on coefficients of an equation. For the Lamé equation with arbitrary coefficients a more detailed treatment of the solutions is a rather complicated problem involving the theory of the theta functions. Fortunately, in the case of the equation (9) the resonant solutions can be found in a simple form, in terms of integrals of algebraic functions, using the so-called Lindemann-Stieltjes procedure [10]. Referring to the paper [2] for details I present here the result for the characteristic exponent only. It turned out that there are two domains on the plane $(\varkappa^2, k^2)$ where μ is positive, i.e., the parametric resonance takes place. Namely,

$$\mu(\varkappa^2, k^2) = \frac{\sqrt{(k^2/3)(1+\varkappa^2)}}{K(\varkappa)}\left[\sqrt{\frac{z_2}{z_1}}\Pi\left(z_1^{-1},\varkappa\right) - \sqrt{\frac{z_1}{z_2}}\Pi\left(z_2^{-1},\varkappa\right)\right] \tag{10}$$

in the domain

$$0 < k^2 < \frac{3\varkappa^2}{1+\varkappa^2}, \tag{11}$$

and

$$\mu(\varkappa^2, k^2) = \frac{\sqrt{(k^2/3)(1+\varkappa^2)}}{K(\varkappa)}\left[\sqrt{\frac{z_1}{z_2}}\Pi\left(z_2^{-1},\varkappa\right) - \sqrt{\frac{z_2}{z_1}}\Pi\left(z_1^{-1},\varkappa\right)\right] \tag{12}$$

in the domain

$$\frac{3}{1+\varkappa^2} < k^2 < 1 + \frac{2\sqrt{1-\varkappa^2+\varkappa^4}}{1+\varkappa^2}. \tag{13}$$

Here $K(\varkappa)$ and $\Pi\left(z^{-1},\varkappa\right)$ are the complete elliptic integrals of the first and third kinds respectively, $z_{1,2}(\varkappa^2,k^2)$ are determined as

$$z_{1,2}=\frac{1}{6\varkappa^2}[(1+\varkappa^2)(3-k^2)\pm\sqrt{3(3-k^2)(1+k^2)(1+\varkappa^2)^2-36\varkappa^2}]. \tag{14}$$

The surface $\mu(\varkappa^2,k^2)$ over the stability-instability chart is shown in Fig.2. Well seen are the domains (11), (13): at their boundaries $\mu=0$. Within the domain (11) the values of μ are tangibly greater than the ones within the domain (13). The shells are in contact at the point $\varkappa^2=1,\ \ k^2=3/2$.

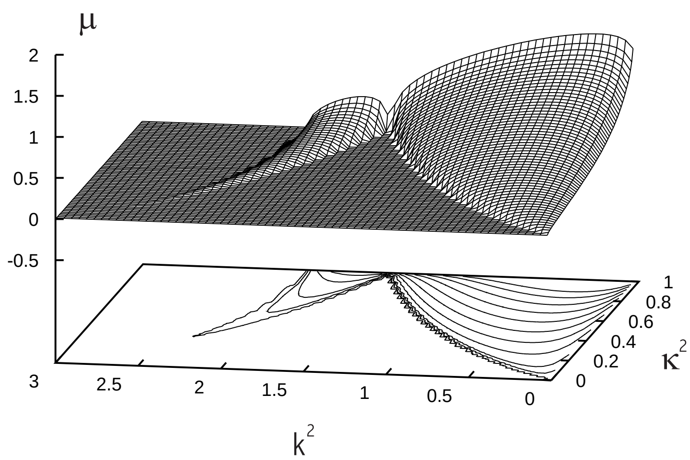

Figure 2: The characteristic exponent surface $\mu(\varkappa^2,k^2)$ over the stability-instability chart.

In the small-amplitude limit, when $\varkappa^2\ll 1$, from (10)-(14) we obtain

$$\mu(\varkappa^2,k^2)\approx\frac{1}{2}\sqrt{k^2(3\varkappa^2-k^2)} \qquad (0<k^2<3\varkappa^2\ll 1), \tag{15}$$

$\mu_{\max}\approx\frac{3}{4}\varkappa^2\quad(k^2\approx\frac{3}{2}\varkappa^2)$ in the domain (11), and, introducing the deviation $\delta = k^2 - 3 + 3\varkappa^2$,

$$\mu(\varkappa^2, k^2) \approx \frac{1}{8}\sqrt{(\delta - 3\varkappa^4)(15\varkappa^4 - 4\delta)} \qquad (3\varkappa^4 < \delta < \frac{15}{4}\varkappa^4 \ll 1), \tag{16}$$

$\mu_{\max} \approx \frac{3}{32}\varkappa^4$ $(\delta \approx \frac{27}{8}\varkappa^4)$ in the domain (13). Notice, that (15) coincides with the well-known dispersion relation for the Langmuir waves. This is not a surprise since in the small-amplitude long-wavelength limit the nonlinear Klein-Gordon equation reduces to the nonlinear Schrödinger equation (see, e.g., [11]).

To demonstrate the efficiency of the resonant mechanism and investigate the dynamics of the phase transition itself we have performed the direct numerical integration of the model (1), (2) in $2 + 1$ dimensions [2]. In so doing we have included the term ϕ^6 in the potential since it is responsible for the existence of the true vacuum and, hence, must be taken into account when considering the formation of the bubbles of the new phase. In agreement with (5) we have taken a spatially uniform background at a level $\phi_{\max}$ ($\phi_{\max} \ll \phi_{cr}$) and superimposed on it a small perturbation with an amplitude much less than $\phi_{\max}$. Thus, the total initial amplitude has been much less than the critical one. The characteristic wavelength of the perturbation has been chosen in such a way as to work into the domain (11) where μ is sufficiently large. For calculation purposes we have confined the perturbation in a two-dimensional box and imposed periodic boundary conditions.

As a result we have observed the significant amplification (up to the level close to ϕ_{cr}) of the field oscillations in the box where the initial perturbation had been placed. The estimation has shown that the growth rate of the oscillations is in agrement with the value of μ calculated from (10). This suggests that the amplification is of resonance origin. Thus, the localized oscillating inhomogeneity appears on the spatially uniform oscillating background. Simultaneously with the growth of the inhomogeneity we have observed a decrease of the background oscillations. This means that the energy of the spatially uniform oscillations concentrates progressively on the inhomogeneity resulting in the exponential growth of the field via the resonance mechanism.

In Fig. 3 are shown the consecutive stages of evolution of the inhomogeneity in terms of time dependence of the field at its center. The parametric amplification of the oscillations is well seen. After the amplitude of the oscillations exceeds a critical value the phase transition begins. On a short interval we have observed the explosive growth of the field within the nucleus up to the true vacuum value ϕ_2. Simultaneously with this, the nucleus shows evidence of significant contraction. For sufficiently small ϵ the field dynamics at this stage is mainly determined by the term ϕ^4 in the potential. In this case the nucleus growth rate can be judged from the integral estimate $\int \phi^2(t, \mathbf{r}) d\mathbf{r} \gtrsim const \times (t_c - t)^{-2}$ derived in [12, 13] for the collapse in the ϕ^4 theory.

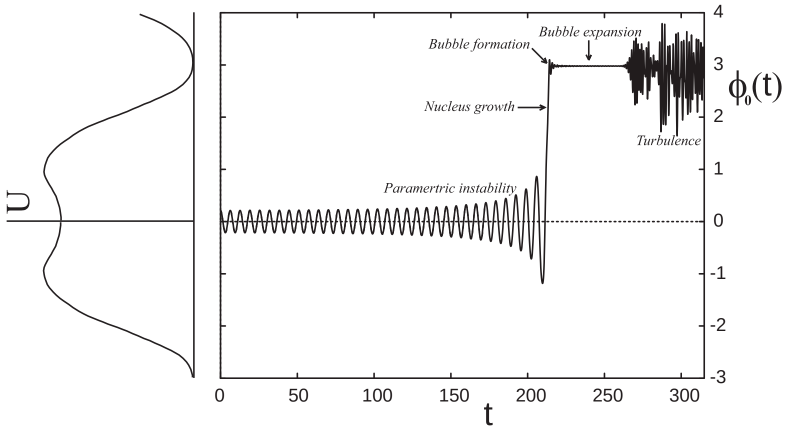

Figure 3: The dynamics of the phase transition in terms of the field $\phi_0\,(t)$at the center of the nucleus.

After several fast oscillations the field at the center of the nucleus has been established equal to the true vacuum value ϕ_2 due to the effect of the term ϕ^6, and the bubble of the new phase has formed. Immediately afterwards the bubble has began to expand fulling the box. When the bubble wall has reached the boundaries of the box the bubble has began to interact intensively with other bubbles which had been formed in neighbouring boxes owing to the periodic boundary conditions. This interaction leads to establishment of a highly inhomogeneous quasi-turbulent state. Nevertheless, it is seen that the mean level of the field coincides with the true vacuum value.

Thus we have got the serious evidence for our conjecture that the parametric resonance can be the cause of the first-order phase transitions in non-dissipative nonlinear systems.

In conclusion, I would like to draw some parallels between the transitions we have considered and transitions in quantum mechanics. Although these phenomena are commonly recognized as fundamentally different, they also exhibit a number of similarities.

Let us first consider a *point* particle placed, for example, in the potential shown in Fig. 1. The coordinate of the particle, say q, plays now the role of the variable ϕ. Obviously, the states of the particle q_1 and $\pm q_2$ are classically stable: small oscillations around them do not increase with time. This means that in the framework of classical mechanics the transitions $q_1 \rightleftarrows \pm q_2$ are forbidden. Such transitions are possible only through the quantum-mechanical tunnelling.

In contrast, when evolving *extended* classical objects the potential barrier separating the false and the true vacua can be overcome *locally,* in small spatial domains. As it was demonstrated above, this occurs dynamically, due to a re-distribution of the energy. The total energy is conserved in the process, i.e., the transitions $\phi_1 \to \pm\phi_2$ are classically allowed. Eventually the difference of the vacuum energies turns mainly into the kinetic energy of the expanding true-vacuum babbles. However, from the viewpoint of a *fixed localized* observer intersected by a moving thin bubble wall all happens as if there were a jump from the level $U(\phi_1)$ to the level $U(\phi_2)$, as with the quantum-mechanical transition of a particle.

Note that, as in the quantum case, the upward transitions $\pm\phi_2 \to \phi_1$ are also possible. It can be proved that in our model the oscillations around the true vacuum are also parametrically unstable with respect to the formation of spatial inhomogeneities. Hence a spontaneous formation of the false-vacuum bubbles is conceptually possible, however the probability of these events is likely to be small. A more effective mechanism of formation of such bubbles is the collisions of the true-vacuum bubbles [14, 15]. In these processes the kinetic energy of the colliding bubble walls turns back into the false vacuum energy. The equilibrium $\phi_1 \rightleftarrows \pm\phi_2$ gives rise to a stationary foam-like spatial structure. Being evolved, e.g., from the cosmological scalar fields it could manifest itself as the cellular structure of the Universe observed at the superscales.

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