Annales de la Fondation Louis de Broglie, Volume 26, n° spécial, 2001

Can one generalize the concept of energy-momentum tensor?

P. Teyssandier

Département d'Astronomie Fondamentale, CNRS/UMR 8630,
Observatoire de Paris, 61 Avenue de l'Observatoire, F-75014 Paris

ABSTRACT. The answer to the question is yes! Considering here the very simple case of a real Klein-Gordon field in Minkowski space-time, we find a class of rank 4 superenergy tensors generalizing the usual energy-momentum tensor. Then, we construct explicitely an infinite tower of rank $2(n+1)$ tensors that we call *weak n-superenergy tensors* and we determine the corresponding quantum operators.

1 Introduction

Historically, the need to extend the notion of energy-momentum of a field appeared for the first time in general relativity. Indeed, the impossibility of defining in a covariant way the local energy density of a gravitational field remains a crucial difficulty of the Einstein theory. Forty years ago, an attempt to overcome this difficulty led Bel [1] to introduce the four indices tensor $T_B^{\alpha\beta\gamma\delta}$ defined by

$$2T_B^{\alpha\beta\gamma\delta} = R^{\alpha\mu\gamma\nu}R^{\beta\ \delta}_{\ \mu\ \nu} + *R^{\alpha\mu\gamma\nu}*R^{\beta\ \delta}_{\ \mu\ \nu} + R*^{\alpha\mu\gamma\nu}R*^{\beta\ \delta}_{\ \mu\ \nu} + *R*^{\alpha\mu\gamma\nu}*R*^{\beta\ \delta}_{\ \mu\ \nu}, \quad (1)$$

where $R_{\mu\nu\rho\sigma}$ is the curvature tensor of the metric $g_{\mu\nu}$ and $*$ denotes the duality operator acting on the left or on the right pair of indices according to its position.[1]

The Bel tensor has many attractive properties. Its components may be expressed without using duality since they reduce to [2]

$$T_B^{\alpha\beta\gamma\delta} \quad = \quad \frac{1}{2}[R^{\alpha\mu\gamma\nu}R^{\beta\ \delta}_{\ \mu\ \nu} + R^{\beta\mu\gamma\nu}R^{\alpha\ \delta}_{\ \mu\ \nu} - \frac{1}{2}g^{\alpha\beta}R^{\gamma\lambda\mu\nu}R^{\delta}_{\ \lambda\mu\nu}$$

[1]By definition, $*R^{\alpha\mu\gamma\nu} = \frac{1}{2}\eta^{\alpha\mu\rho\sigma}R_{\rho\sigma}{}^{\gamma\nu}$ and $R*^{\alpha\mu\gamma\nu} = \frac{1}{2}\eta^{\gamma\nu\rho\sigma}R^{\alpha\mu}{}_{\rho\sigma}$, where $\eta^{\alpha\mu\rho\sigma}$ is the canonical volume element 4-form. Note the analogy of the Bel tensor with the electromagnetic energy-momentum tensor.

$$- \frac{1}{2} g^{\gamma\delta} R^{\alpha\lambda\mu\nu} R^{\beta}_{\ \lambda\mu\nu} + \frac{1}{8} g^{\alpha\beta} g^{\gamma\delta} R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}] \,. \qquad (2)$$

In any spacetime [2], $T_B^{\alpha\beta\gamma\delta} = T_B^{(\alpha\beta)(\gamma\delta)} = T_B^{(\gamma\delta)(\alpha\beta)}$. When $R_{\mu\nu} = 0$, $T_B^{\alpha\beta\gamma\delta}$ is totally symmetric and satisfies the conservation equation $\nabla_\alpha T_B^{\alpha\beta\gamma\delta} = 0$. For any timelike unit vector u, the scalar density

$$\epsilon_B(u) = T_B^{\alpha\beta\gamma\delta} u_\alpha u_\beta u_\gamma u_\delta \qquad (3)$$

is positive definite [1] and the vector density

$$p_B^\alpha(u) = T_B^{\alpha\beta\gamma\delta} u_\beta u_\gamma u_\delta \qquad (4)$$

is timelike or null [3].

The above mentioned properties of $\epsilon_B(u)$ and of $p_B^\alpha(u)$ allow to regard $T_B^{\alpha\beta\gamma\delta}$ as having the key properties of an energy-momentum tensor. This fundamental feature explains why $T_B^{\alpha\beta\gamma\delta}$ is named a *gravitational superenergy tensor*. Then $\epsilon_B(u)$ and $p_B^\alpha(u)$ may be regarded as defining respectively a *gravitational superenergy density* and a *gravitational supermomentum density vector relative to (an observer moving with) the unit 4-velocity u*.

We do not try here to elucidate the physical meaning of the Bel tensor, which remains very obscure in spite of a lot of discussions (see, e.g.,[4] ,[5] and Refs. therein). Our purpose is to present a general method enabling to construct a class of superenergy tensors for scalar and electromagnetic fields [3]. For the sake of simplicity, we focus our attention on a real scalar field ϕ satisfying the Klein-Gordon equation

$$(\Box + m^2)\phi = 0 \qquad (5)$$

in Minkowski space-time.

In our reasoning, the requirement that a superenergy tensor must be divergence-free plays a basic role. So our method is quite different from the purely algebraic construction of superenergy tensors recently developed by Senovilla [6, 7] for arbitrary fields. Nevertheless, the two methods lead to equivalent results for the scalar field.

[2] We put $A^{(\alpha\beta)} = 1/2(A^{\alpha\beta} + A^{\beta\alpha})$. More generally $A^{(\alpha_1\alpha_2\cdots\alpha_r)}$ denotes the totally symmetric part of a tensor $A^{\alpha_1\alpha_2\cdots\alpha_r}$. $R_{\mu\nu}$ is the Ricci tensor: $R_{\mu\nu} = R^{\lambda}_{\ \mu\lambda\nu}$.

[3] For the electromagnetic field, a rank 4 tensor formally analogous to the Bel tensor was given for the first time in [2].

It must be noted also that the tensor

$$C^{\alpha\beta\gamma\delta} = \frac{1}{2}\,(\phi\,\phi^{,\alpha\beta\gamma\delta} - \phi^{,\alpha}\phi^{,\beta\gamma\delta})\,, \tag{6}$$

has been previously considered by Komar [8] as a kind of Bel tensor for the Klein-Gordon field. Indeed, (6) satisfies the conservation equation $C^{\alpha\beta\gamma\delta}{}_{,\alpha} = 0$ when Eq. (5) holds. However, it is not obvious that the Komar tensor is an acceptable superenergy tensor since the positivity of $C^{\alpha\beta\gamma\delta}u_\alpha u_\beta u_\gamma u_\delta$ is not ensured. We avoid this flaw since we find a class of rank 4, divergence-free tensors with positive definite energy densities.

The plan is as follows. In Sect. 2, we give the general definition of what we call a *n-superenergy tensor* for the scalar field ϕ (the rank of such a tensor is $2(n+1)$). In Sect. 3, we show that the rank 4 tensors fulfilling our definition constitute a two-parameter family. Moreover, we show that this family reduces to a unique tensor W (up to a positive constant factor) when the complete symmetry on the four indices is required. This unicity implies that W can henceforth be regarded as defining the 1-superenergy tensor *par excellence*. In Sect. 4, we construct explicitly an infinite set of rank $2(n+1)$ tensors $U_{(n,n)}$ that we call *weak n-superenergy tensors* because they have almost all the good properties of the superenergy-momentum tensors defined in Sect. 2. We show that $U_{(1,1)}$ and W yield the same total 1-superenergy-momentum when the field ϕ and its first derivative with respect to time are functions of rapid decrease at spatial infinity. In Sect. 5, we give the superhamiltonian and the supermomentum operators corresponding to W and to each $U_{(n,n)}$ within the framework of canonical quantization. In Sect. 6, we give some concluding remarks.

2 Superenergy tensors for a scalar field

We use coordinates $x^\alpha = (x^0, \mathbf{x})$ such that the metric components $g_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$ and we denote by $\mathcal{G}$ the corresponding inertial frame of reference. Units are chosen so that $c = 1$ and $\hbar = 1$. We put $F_{,\alpha} = \partial_\alpha F$.

The energetic content of the field ϕ is described by the well-known symmetric, divergence-free energy-momentum tensor

$$T^{\alpha\beta}[\phi] = \phi^{,\alpha}\phi^{,\beta} - \frac{1}{2}\,g^{\alpha\beta}(\phi^{,\lambda}\phi_{,\lambda} - m^2\phi^2)\,. \tag{7}$$

We shall call *n-superenergy tensor* any tensor $T^{\alpha_1\beta_1\ldots\alpha_{n+1}\beta_{n+1}}$ of even rank $2(n+1) \geq 4$ possessing the following properties :

P1. Denoting by μ_A a block of indices $\{\mu_i, 1 \leq i \leq A\}$, $T^{\alpha_1\beta_1\ldots\alpha_{n+1}\beta_{n+1}}$ may be written in the form

$$T^{\alpha_1\beta_1\ldots\alpha_{n+1}\beta_{n+1}} = \sum_{A=0}^{A=n+1} C^{\alpha_1\beta_1\ldots\alpha_{n+1}\beta_{n+1}}{}_{\mu_A\nu_A}\phi^{,\mu_A}\phi^{,\nu_A}\,, \tag{8}$$

where the coefficients $C^{\alpha_1\beta_1\ldots\alpha_{n+1}\beta_{n+1}}{}_{\mu_A\nu_A}$ are tensorial quantities involving only the components of the metric (for $A = 0$, we put $\phi^{,\mu_A} = \phi$).

P2. $T^{\alpha_1\beta_1\ldots\alpha_{n+1}\beta_{n+1}}$ is symmetric in each pair (α_i, β_i) of indices; it is also symmetric in the interchange of two blocks (α_i, β_i) and (α_j, β_j).

P3. $T^{\alpha_1\beta_1\ldots\alpha_{n+1}\beta_{n+1}}$ satisfies a conservation equation :

$$(\,\Box + m^2)\phi = 0 \;\Rightarrow\; T^{\alpha_1\beta_1\ldots\alpha_i\beta_i\ldots\alpha_{n+1}\beta_{n+1}}{}_{,\alpha_i} = 0\,. \tag{9}$$

P4. The *n-superenergy density relative to u* defined by

$$\epsilon_{(n)}(T, u) = T^{\alpha_1\beta_1\ldots\alpha_{n+1}\beta_{n+1}}u_{\alpha_1}u_{\beta_1}\ldots u_{\alpha_{n+1}}u_{\beta_{n+1}}\,, \tag{10}$$

is positive definite ($\epsilon_{(n)}(T, u) > 0$ if $T^{\alpha_1\beta_1\ldots\alpha_{n+1}\beta_{n+1}} \neq 0$).

Of course, the arbitrariness on the superenergy tensors will be reduced if **P2** is replaced by the more restrictive requirement

P'2. $T^{\alpha_1\beta_1\ldots\alpha_{n+1}\beta_{n+1}}$ is totally symmetric.

3 Class of rank 4 superenergy tensors

Given a rank 4 tensor T, denote by $\overline{T}$ the totally symmetric part of T and define the tensor $\widetilde{T}$ by

$$\widetilde{T}^{\alpha\beta\gamma\delta} = \frac{1}{2}(T^{\alpha\gamma\beta\delta} + T^{\alpha\delta\beta\gamma})\,. \tag{11}$$

We have the following lemma. (The proof is immediate.)

Lemma 1 *Let $\mathcal{E}_{(1)}$ be the class of rank 4 tensors possessing properties* **P1**, **P2** *and* **P3**. *For any $T \in \mathcal{E}_{(1)}$, the following propositions hold :*

1. $\overline{T} \in \mathcal{E}_{(1)}$ *and* $\widetilde{T} \in \mathcal{E}_{(1)}$.
2. $\overline{T} = \frac{1}{3}T + \frac{2}{3}\widetilde{T}$.
3. $T = \widetilde{T} \Leftrightarrow T = \overline{T}$.
4. $T^{\alpha\beta\gamma\delta}u_\beta u_\gamma u_\delta = \widetilde{T}^{\alpha\beta\gamma\delta}u_\beta u_\gamma u_\delta = T^{(\alpha\beta\gamma\delta)}u_\beta u_\gamma u_\delta$.

We are now in a position to determine the class $\mathcal{E}_{(1)}$. The most general tensor $T^{\alpha\beta\gamma\delta}$ which fulfills conditions **P1** and **P2** may be written as [4]

$$\begin{aligned} T^{\alpha\beta\gamma\delta} &= a\phi^{,\alpha\beta}\phi^{,\gamma\delta} + b\phi^{,\gamma(\alpha}\phi^{,\beta)\delta} \\ &+ c_1(g^{\alpha\beta}\phi^{,\gamma\lambda}\phi^{\delta}{}_{\lambda} + g^{\gamma\delta}\phi^{,\alpha\lambda}\phi^{\beta}{}_{\lambda}) \\ &+ c_2(g^{\gamma(\alpha}\phi^{,\beta)\lambda}\phi^{,\delta}{}_{\lambda} + g^{\delta(\alpha}\phi^{,\beta)\lambda}\phi^{,\gamma}{}_{\lambda}) \\ &+ d_1 m^2(g^{\alpha\beta}\phi^{,\gamma}\phi^{,\delta} + g^{\gamma\delta}\phi^{,\alpha}\phi^{,\beta}) \\ &+ d_2 m^2(g^{\gamma(\alpha}\phi^{,\beta)}\phi^{,\delta} + g^{\delta(\alpha}\phi^{,\beta)}\phi^{,\gamma}) \\ &+ K_1 g^{\alpha\beta}g^{\gamma\delta} + K_2 g^{\gamma(\alpha}g^{\beta)\delta}\,, \end{aligned}$$

with

$$K_s = p_s\phi^{,\rho\sigma}\phi_{,\rho\sigma} + q_s m^2\phi^{,\lambda}\phi_{,\lambda} + r_s m^4\phi^2\,,$$

$a, b, c_1, c_2, d_1, d_2, p_s, q_s$ and r_s being dimensionless constants ($s = 1, 2$). A straightforward calculation shows that $T^{\alpha\beta\gamma\delta}{}_{,\alpha} = 0$ holds if and only if the coefficients c_s, d_s, p_s, q_s and r_s are given by

$$c_1 = \frac{1}{2}(a-b)\,,\ c_2 = -a\,,\ d_1 = \frac{1}{2}(b-a)\,,\ d_2 = a\,,$$

and

$$p_s = -\frac{1}{2}c_s\,,\ q_s = \frac{1}{2}(c_s - d_s)\,,\ r_s = \frac{1}{2}d_s\,,$$

the parameters a and b being chosen arbitrarily.

Thus we obtain the first theorem of this paper [9].

Theorem 1 *Any tensor $T \in \mathcal{E}_{(1)}$ may be written as*

$$T^{\alpha\beta\gamma\delta} = a\,T_1^{\alpha\beta\gamma\delta} + b\,T_2^{\alpha\beta\gamma\delta}\,, \tag{12}$$

where

$$\begin{aligned} T_1^{\alpha\beta\gamma\delta} &= \phi^{,\alpha\beta}\phi^{,\gamma\delta} + \frac{1}{2}(g^{\alpha\beta}\tau^{\gamma\delta} + g^{\gamma\delta}\tau^{\alpha\beta}) \\ &- g^{\alpha(\gamma}\tau^{\delta)\beta} - g^{\beta(\gamma}\tau^{\delta)\alpha}\,, \end{aligned} \tag{13}$$

[4] We suppose that Eq. (5) is taken into account. Thus we exclude terms like $g^{\alpha\beta}\phi^{,\gamma\delta}\,\Box\phi$ since such a term reduces to $-m^2\phi g^{\alpha\beta}\phi^{,\gamma\delta}$, a form which is not compatible with **P1** .

and

$$T_2^{\alpha\beta\gamma\delta} = \widetilde{T}_1^{\alpha\beta\gamma\delta} = \phi^{,\alpha(\gamma}\phi^{,\delta)\beta} - \frac{1}{2}(g^{\alpha\beta}\tau^{\gamma\delta} + g^{\gamma\delta}\tau^{\alpha\beta})\,, \qquad (14)$$

$\tau^{\alpha\beta}$ *being defined by*

$$\tau^{\alpha\beta} = \phi^{,\alpha\lambda}\phi^{,\beta}{}_{\lambda} - m^2\phi^{,\alpha}\phi^{,\beta} - \frac{1}{4}g^{\alpha\beta}(\phi^{,\rho\sigma}\phi_{,\rho\sigma} - 2m^2\phi^{,\lambda}\phi_{,\lambda} + m^4\phi^2)\,. \qquad (15)$$

The coefficients a and b are arbitrary constants.

Lemma 1 enables us to complete the above theorem by the following one.

Theorem 2 *Any tensor $T \in \mathcal{E}_{(1)}$ which is totally symmetric may be written as*

$$T^{\alpha\beta\gamma\delta} = k\,W^{\alpha\beta\gamma\delta}\,, \qquad (16)$$

where k is a constant and $W^{\alpha\beta\gamma\delta} = T_1^{(\alpha\beta\gamma\delta)}$, i.e.

$$W^{\alpha\beta\gamma\delta} = \frac{1}{3}(\phi^{,\alpha\beta}\phi^{,\gamma\delta} + 2\phi^{,\alpha(\gamma}\phi^{,\delta)\beta}) - \frac{1}{6}(g^{\alpha\beta}\tau^{\gamma\delta} + g^{\gamma\delta}\tau^{\alpha\beta} + 2g^{\alpha(\gamma}\tau^{\delta)\beta} + 2g^{\beta(\gamma}\tau^{\delta)\alpha})\,. \qquad (17)$$

Let us put now

$$s^{\alpha}(u) = W^{\alpha\beta\gamma\delta}u_{\beta}u_{\gamma}u_{\delta}\,, \qquad (18)$$

and

$$w(u) = s^{\alpha}(u)u_{\alpha}\,. \qquad (19)$$

Some algebra leads to the third theorem.

Theorem 3 *For any timelike unit vector u, $w(u)$ is positive and $s^{\alpha}(u)$ is timelike or null :*

$$w(u) \geq 0\,, \qquad (20)$$

$$s^{\alpha}(u)s_{\alpha}(u) \geq 0\,. \qquad (21)$$

When $m \neq 0$,the equality $w(u) = 0$ is possible iff $\phi = 0$. When $m = 0$, $w(u) = 0$ iff $\phi_{,\alpha\beta} = 0$.

Proof. In any Cartesian coordinate system x^α, we have

$$\begin{aligned} W^{0000} &= \frac{1}{4}(\phi_{,00})^2 + \frac{1}{2}\sum_i(\phi_{,0i})^2 + \frac{1}{4}\sum_{i,j}(\phi_{,ij})^2 \\ &\quad + \frac{1}{2}m^2\left[(\phi_{,0})^2 + \sum_i(\phi_{,i})^2 + \frac{1}{2}m^2\phi^2\right] \end{aligned} \tag{22}$$

and

$$W^{i000} = -\frac{1}{2}\left[\phi_{,00}\phi_{,0i} + \sum_j \phi_{,0j}\phi_{,ij} + m^2\phi_{,0}\phi_{,i}\right]. \tag{23}$$

The timelike unit vector u being given, choose for this proof coordinates x^α such that $\partial_0 = u$. Then we have

$$s^\alpha(u) = W^{\alpha 000}. \tag{24}$$

With this choice, $w(u) = s^0(u) = W^{0000}$. Taking into account (22), we immediately obtain (20).

Now put $\vec{s}(u) = s^i(u)\partial_i$ and $s^2(u) = s^\alpha(u)s_\alpha(u)$. If $\vec{s}(u) = 0$, $s^2(u) \geq 0$. If $\vec{s}(u) \neq 0$, choose ∂_1 so that $\partial_1 = \vec{s}(u)/\parallel \vec{s}(u) \parallel$. Then, $s^2(u) = [s^0(u)]^2 - [s^1(u)]^2$. From (24), (22) and (23), it results that

$$\begin{aligned} s^0(u) \pm s^1(u) &= \frac{1}{4}\sum_\lambda(\phi_{,0\lambda} \mp \phi_{,1\lambda})^2 + \frac{1}{4}\sum_\lambda\sum_{j\neq 1}(\phi_{,\lambda j})^2 \\ &\quad + \frac{1}{4}m^2\left[(\phi_{,0} \mp \phi_{,1})^2 + \sum_{j\neq 1}(\phi_{,j})^2 + \sum_\lambda(\phi_{,\lambda})^2 + m^2\phi^2\right], \end{aligned}$$

which implies that $[s^0(u)]^2 - [s^1(u)]^2 \geq 0$. Therefore $s^2(u) \geq 0$. Q. E. D.

We can now state the central theorem of this paper.

Theorem 4 *The tensor $W^{\alpha\beta\gamma\delta}$ given by (17) is the unique (up to an arbitrary constant positive factor) totally symmetric 1-superenergy tensor of ϕ.*

Moreover, it follows from Lemma 1 and Eq. (17) that for any $T^{\alpha\beta\gamma\delta}$ given by (12)

$$T^{\alpha\beta\gamma\delta}u_\beta u_\gamma u_\delta = (a+b)W^{\alpha\beta\gamma\delta}u_\beta u_\gamma u_\delta. \tag{25}$$

As a consequence, $w(u)$ and $s^\alpha(u)$ may be henceforth considered without loss of generality as defining respectively the 1-*superenergy density* and the 1-*supermomentum density* of ϕ relative to u.

Since $W^{\alpha\beta\gamma\delta}$ is divergence-free, the integrals

$$W_{(1)}[\phi] = \int W^{0000}(x^0, \mathbf{x}')d^3\mathbf{x}' \,, \qquad S^i_{(1)}[\phi] = \int W^{i000}(x^0, \mathbf{x}')d^3\mathbf{x}' \tag{26}$$

are constants of the motion : $W_{(1)}[\phi]$ and $S^i_{(1)}[\phi]$ will be respectively called the *total 1-superenergy* and the *total spatial 1-supermomentum* of the field ϕ in the inertial frame $\mathcal{G}$. Comparing (22) and (23) with (7), we find that

$$\int W^{\alpha 000}(x^0, \mathbf{x}')d^3\mathbf{x}' = \frac{1}{2}\sum_\lambda P^\alpha[\phi_{,\lambda}] + \frac{1}{2}m^2 P^\alpha[\phi] \,, \tag{27}$$

where $P^\alpha[F]$ is defined by

$$P^\alpha[F] = \int T^{\alpha 0}[F](x^0, \mathbf{x}')d^3\mathbf{x}' \tag{28}$$

for any scalar function $F(x^0, \mathbf{x})$.

4 Weak n-superenergy tensors

It would be of course possible to determine the n-superenergy tensors for $n > 1$ by the method of Sect. 3. However, this procedure becomes heavier and heavier as n is increasing. So we introduce the tensors $U_{(n,n)}$ defined by [10]

$$\begin{aligned} U^{\alpha\beta}{}_{\mu_1...\mu_n\nu_1...\nu_n} \quad &= \quad \phi^{(,\alpha}{}_{\mu_1...\mu_n}\phi^{,\beta)}{}_{\nu_1...\nu_n} - \frac{1}{2}g^{\alpha\beta}(\phi^{,\lambda}{}_{\mu_1...\mu_n}\phi_{,\lambda\nu_1...\nu_n} \\ &\quad - m^2\phi_{,\mu_1...\mu_n}\phi_{,\nu_1...\nu_n}) \,. \end{aligned} \tag{29}$$

It is easily seen that each $U_{(n,n)}$ possesses property **P1**, is symmetric in (α, β), is completely symmetric in $(\mu_1, ..., \mu_n)$ and in $(\nu_1, ..., \nu_n)$, satisfies the conservation law

$$U^{\alpha\beta}{}_{\mu_1...\mu_n\nu_1...\nu_n,\alpha} = 0 \tag{30}$$

and satisfies the following inequality for any timelike unit vector u :

$$U^{\alpha\beta}{}_{\mu_1...\mu_n\nu_1...\nu_n}u_\alpha u_\beta u^{\mu_1}...u^{\mu_n}u^{\nu_1}...u^{\nu_n} \geq 0 \,. \tag{31}$$

Therefore the tensor $U_{(n,n)}$ has almost all the properties of a n-superenergy tensor. So, we call $U_{(n,n)}$ the *weak n-superenergy tensor* .

Let us put

$$U_{(n)}[\phi] = \int U^{00}{}_{0_n 0_n}(x^0, \mathbf{x}')d^3\mathbf{x}' , \qquad R^i_{(n)}[\phi] = \int U^{i0}{}_{0_n 0_n}(x^0, \mathbf{x}')d^3\mathbf{x}' , \tag{32}$$

where 0_n denotes a block of n timelike indices. It follows from (30) that these quantities are constants of the motion, that we shall call respectively the *weak n-superenergy* and the *weak spatial n-supermomentum* of ϕ in the frame $\mathcal{G}$.

For $n = 1$, a straightforward calculation yields the relation

$$\widetilde{T}_1^{\alpha\beta\mu\nu} \quad = \quad U^{\alpha\beta\mu\nu} - \frac{1}{2}(U^{\alpha\beta\lambda}{}_\lambda - m^2 T^{\alpha\beta})g^{\mu\nu} . \tag{33}$$

This relation leads to the following theorem.

Theorem 5 *For any Klein-Gordon field, the following equalities hold:*

$$W_{(1)}[\phi] = U_{(1)}[\phi] + [Surf] \tag{34}$$

and

$$S^i_{(1)}[\phi] = R^i_{(1)}[\phi] + [Surf] , \tag{35}$$

where $[Surf]$ *denotes surface terms which cancel if* ϕ *and its derivative* $\phi_{,0}$ *are functions of sufficiently rapid decrease at spatial infinity.*

Proof. Using Lemma 1 and (17) yield $W^{\alpha000} = \widetilde{T}_1^{\alpha000}$. So we deduce from (33) that

$$W^{\alpha000} = U^{\alpha000} - \frac{1}{2}(U^{\alpha0\lambda}{}_\lambda - m^2 T^{\alpha0}) . \tag{36}$$

Each term $U^{\alpha0\lambda}{}_\lambda - m^2 T^{\alpha0}$ is a 3-divergence. Indeed, we find

$$U^{00\lambda}{}_\lambda - m^2 T^{00} = -\{\frac{1}{2}\phi_{,i}[\phi^{,ij} - \delta^{ij}(\triangle\phi - 2m^2\phi)]\}_{,j} , \tag{37}$$

$$U^{i0\lambda}{}_\lambda - m^2 T^{i0} = \{\phi_{,0}[\phi^{,ij} - \delta^{ij}(\triangle\phi - m^2\phi)]\}_{,j} , \tag{38}$$

where $\triangle\phi = \delta^{kl}\phi_{,kl}$. Thus the theorem is established. Q. E. D.

5 Quantum superhamiltonian and supermomentum operators

Within the canonical quantization procedure, ϕ becomes a Hermitian operator which can be expanded on the basis of the plane wave solutions :

$$\phi(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k} \frac{1}{\sqrt{2\omega_k}} \left[a(\mathbf{k})e^{-ikx} + a^\dagger(\mathbf{k})e^{ikx}\right]_{k^0=\omega_k} , \quad (39)$$

with

$$\omega_k = \sqrt{\mathbf{k}^2 + m^2} , \quad (40)$$

the operators $a(\mathbf{k})$ and their Hermitian conjugates $a^\dagger(\mathbf{k})$ satisfying the usual commutation relations. Substituting for ϕ from (39) into (26) yields the *1-superhamiltonian* and the *spatial 1-supermomentum operators*

$$\widehat{W}_{(1)} = \frac{1}{2} \int d^3\mathbf{k}\, \omega_k^3 [a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k})] , \quad (41)$$

$$\widehat{S}^i_{(1)} = \int d^3\mathbf{k}\, k^i \omega_k^2\, a^\dagger(\mathbf{k})a(\mathbf{k}) . \quad (42)$$

More generally, we deduce from (32) the following operators

$$\widehat{U}_{(n)} = \frac{1}{2} \int d^3\mathbf{k}\, \omega_k^{2n+1} \left[a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k})\right] , \quad (43)$$

$$\widehat{R}^i_{(n)} = \int d^3\mathbf{k}\, k^i \omega_k^{2n}\, a^\dagger(\mathbf{k})a(\mathbf{k}) . \quad (44)$$

Putting $n = 0$ in these equations, we recover the usual Hamiltonian and momentum operators $\widehat{H}$ and $\widehat{P}^i$. When $n = 1$, a comparison of Eqs. (43) and (44) with Eqs. (41) and (42) shows that $\widehat{W}_{(1)} = \widehat{U}_{(1)}$ and $\widehat{S}^i_{(1)} = \widehat{R}^i_{(1)}$, equations which can also be immediately deduced from (34) and (35).

6 Concluding remarks

We have established the existence of rank 4 superenergy tensors for the Klein-Gordon field. These tensors form a two-parameter family. This last feature is not embarrassing, however, because the unicity (up to an arbitrary factor) is obtained by requiring the total symmetry in the four indices. Thus it is possible to speak about "the" 1-superenergy tensor of the field.

We have built divergence-free tensors $U_{(n,n)}$ of rank $2(n+1)$ which have almost all the good properties of the superenergy tensors. We have shown that $U_{(1,1)}$ and W yield the same total 1-superenergy and the same total spatial 1-supermomentum. This theorem leads to conjecture that for $n > 1$ $U_{(n,n)}$ can replace advantageously the n-superenergy tensors to evaluate the total n-superenergy and the total spatial n-supermomentum of the field.

Acknowledgments

We are deeply grateful to Ll. Bel and to J. M. M. Senovilla for many stimulating discussions and helpful suggestions.

References

[1] Ll. Bel, C. R. Acad. Sci. Paris **248**, 1297 (1959); *Thèse* (Paris, 1960) .

[2] M. Chevreton, Nuovo Cimento **34**, 901 (1964).

[3] M. Á. G. Bonilla and J. M. M. Senovilla, Phys. Rev. Lett. **78**, 783 (1997). See also G. Bergqvist, J. Math. Phys. **39**, 2141 (1998).

[4] J. Garecki, Int. J. Theor. Phys. **35**, 2195 (1996); J. Math. Phys. **40**, 4035 (1999).

[5] B. Mashhoon, J. C. McClune and H. Quevedo, Class. Quantum Grav. **16**, 1137 (1999).

[6] J. M. M. Senovilla, in *Gravitation and Relativity in General, Proceedings of the Spanish Relativity Meeting in Honour of the 65th Birthday of Lluis Bel*, eds. J. Martín, E. Ruiz, F. Atrio and A. Molina (World Scientific, Singapore, 1999), p. 175; http://xxx.laln.gov/ps/gr-qc/9901019.

[7] J. M. M. Senovilla, Preprint http://xxx.lanl.gov/ps/gr-qc/9905057 and 9906087.

[8] A. Komar, Phys. Rev. **164**, 1595 (1967).

[9] The tensor $T_2^{\alpha\beta\gamma\delta}$ in Theorem 1 is obtained also in [7]. When $m = 0$, $T_2^{\alpha\beta\gamma\delta}$ reduces to a tensor previously found by Bel (unpublished result). See also [6].

[10] More general divergence-free tensors are given in P. Teyssandier, Preprint http:// xxx.lanl.gov/ps/gr-qc/9905080.

(Manuscrit reçu le 10 octobre 2000)