Fields of charged particles in the causal theory

PEDRO SANCHO

GPV de Valladolid
Centro Zonal de Castilla y León
Orión 1, 47014 Valladolid

ABSTRACT. We present an analysis of the electromagnetic fields associated to charged particles in the framework of the causal interpretation.

RÉSUMÉ. Nous examinons le champ électromagnétique d'une charge dans le cadre de l'interprétation causale de la mécanique quantique.

1 Introduction

We analyze in this article the quantum electromagnetic fields associated in the causal interpretation to a quantum charged particle. When the polar decomposition of the wave function is introduced into the Schrödinger equation of the complete system (quantum electromagnetic fields interacting with quantum charged particles) we obtain the equations of motion for the particle and field coordinates.

The study is carried out in the Coulomb gauge. This particular choice of the gauge, usual in nonrelativistic treatments of radiation, breaks down the manifest covariance of the theory but simplifies the analysis.

The plan of the paper is as follows. In Sect. 2 we present the Hamilton-Jacobi equation for the classical field-charge interacting system. Section 3 deals with the quantization of the system and the causal equations of motion for the charge and fields. In the Conclusions we discuss the main results obtained in the paper.

2 Hamilton-Jacobi equation for the electromagnetic field

We present in this Section the classical theory of the electromagnetic field in interaction with charges within the framework of the Hamilton-Jacobi theory, in order to compare it later with the causal formulation of the system. We consider a classical electromagnetic field interacting with classical charges. As it is well-known, the motion equations of the system can be derived from its Lagrange function [1-3]:

$$L_{cl} = \int d^3 \vec{x} \; \mathfrak{L}_{cl} \tag{1}$$

with :

$$\mathfrak{L}_{cl} = \frac{1}{2}(\vec{E}_{cl}^{2} - \vec{B}_{cl}^{2}) - j_{\mu}^{cl} A_{cl}^{\mu} \tag{2}$$

the Lagrangian density.

In the above equations we have introduced the subscripts and superscripts "cl" (classical) in order to avoid any confussion with the functions introduced for the quantized system. $\vec{E}_{cl}$ and $\vec{B}_{cl}$ represent the electric and magnetic fields, given as a function of the classical potentials by the equations:

$$\vec{E}_{cl} = -\frac{\partial \vec{A}_{cl}}{\partial t} \; ; \; \vec{B}_{cl} = \Delta \wedge \vec{A}_{cl} \tag{3}$$

These equations have been written in the Coulomb gauge

$$A_o^{cl} = 0 \; ; \; \Delta . \vec{A}_{cl} = 0 \tag{4}$$

We use the Coulomb gauge because it is the usual election in nonrelativistic problems, as those we shall consider in the next section.

We have used units with c=1 and the usual relativistic notation $\mu = 0$ 5 for the temporal and $\mu = 1,2,3$ 6 for the spatial components of any four-vector. The scalar product of two four-vectors is $X_{\mu} Y^{\mu} = X_o Y_o - \vec{X}.\vec{Y}$ 7. j_{μ}^{cl} 8 represents the density charge current, given by $(\rho, \rho\vec{v})$ 9, with ρ 10 the charge density and $\vec{v}$ the velocity of the charges. The scalar product in (2) is $j_{\mu}^{cl} A_{cl}^{\mu} = - \vec{j}_{cl} \cdot \vec{A}_{cl}$ 14 because of the gauge condition.

The Hamiltonian formulation of the problem is introduced via the momenta conjugate to the potential variables. Their definition is [1-3]:

$$\pi_{\mu}^{cl} = \frac{\delta \, \mathfrak{L}_{cl}}{\delta(\partial A_{\mu}^{cl} / \partial t)} \tag{5}$$

where δ 16 represents the variational derivative [1].
In the electromagnetic case we have

$$\pi_o^{cl} = 0 \tag{6}$$

and

$$\vec{\pi}_{cl} = \frac{\partial \vec{A}_{cl}}{\partial t} \tag{7}$$

The Hamiltonian of the system is $H_{cl} = \int d^3\vec{x}\, \mathfrak{C}_{cl}$ 19, with the Hamiltonian density given by

$$\mathfrak{C}_{cl} = \vec{\pi}_{cl} \cdot \frac{\partial \vec{A}_{cl}}{\partial t} - \boldsymbol{L}_{cl} = \frac{1}{2}(\vec{E}_{cl}^{2} + \vec{B}_{cl}^{2}) + j_{\mu}^{cl} A_{cl}^{\mu} \tag{8}$$

Finally, we can introduce the Hamilton-Jacobi equation for the system

$$\frac{\partial S_{cl}}{\partial t} + H_{cl} = 0 \tag{9}$$

where, by resemblance with the particle theory, the functional S_{cl} (it depends on the potentials A_{μ}^{cl} 22) is defined by [4]:

$$\frac{\delta S_{cl}}{\delta A_{\mu}^{cl}} = \pi_{\mu}^{cl} \tag{10}$$

Then, remembering Eqs. (6) and (7), we have

$$\frac{\delta S_{cl}}{\delta A_o^{cl}} = 0 \;\; ; \;\; \frac{\delta S_{cl}}{\delta \vec{A}_{cl}} = \frac{\partial \vec{A}_{cl}}{\partial t} \tag{11}$$

From Eq. (9) we can derive the equations of motion for the potentials. We present this derivation because it is similar to that we shall use to obtain the Hamilton-Jacobi-type equation valid for the causal formulation of the quantized problem.

Equation (9) in the case of the electromagnetic field interacting with charges becomes explicitly

$$\frac{\partial S_{cl}}{\partial t} + \int \left(\frac{1}{2} \left(\left(\frac{\delta S_{cl}}{\delta \vec{A}_{cl}} \right)^2 + (\Delta x\, \vec{A}_{cl})^2 \right) + j_{\mu}^{cl} A_{cl}^{\mu} \right) d^3 \vec{x} = 0 \qquad (12)$$

We apply the functional derivative $\delta / \delta A_{\mu}^{cl}$ 26 to this equation. We decompose this derivative into two parts. The first one is

$$\frac{\delta}{\delta A_{\mu}^{cl}} \frac{\partial S_{cl}}{\partial t} + \frac{\delta}{\delta A_{\mu}^{cl}} \frac{1}{2} \int d^3 \vec{x} \left(\frac{\delta S_{cl}}{\delta \vec{A}_{cl}} \right)^2 = \frac{\partial}{\partial t} \frac{\delta S_{cl}}{\delta A_{\mu}^{cl}} +$$

$$\int d^3 \vec{x} \left(\frac{\delta S_{cl}}{\delta \vec{A}_{cl}} \right) . \frac{\delta}{\delta A_{\mu}^{cl}} \left(\frac{\delta S_{cl}}{\delta \vec{A}_{cl}} \right) = \frac{d}{dt} \left(\frac{\delta S_{cl}}{\delta A_{\mu}^{cl}} \right) = \frac{\partial^2 A_{\mu}^{cl}}{\partial t^2} \qquad (13)$$

In the above derivation we have used the relation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \int d^3 \vec{x} \left(\frac{\partial \vec{A}_{cl}}{\partial t} \right) . \frac{\delta}{\delta \vec{A}_{cl}} \qquad (14)$$

and, as usual, we have identified $d \dot{A}_{\mu}^{cl} / dt$ 29 with $\partial \dot{A}_{\mu}^{cl} / \partial t$ 30, and $\dot{A}_{\mu}^{cl} = \delta S_{cl} / \delta A_{\mu}^{cl}$ 31 [4].

Note that in the case of this equation is purely formal, because both sides equate zero.

On the other hand, the A_{o}^{cl} second part of the decomposition is:

$$\frac{\delta}{\delta A_{\mu}^{cl}} \int d^3 \vec{x} \left(\frac{1}{2} (\Delta \wedge \vec{A}_{cl})^2 + j_{\mu}^{cl} A_{cl}^{\mu} \right) \qquad (15)$$

This expression can be evaluated using the well-known relation between the functional derivative of the Hamiltonian and the usual derivatives of the Hamiltonian density [1]:

$$\frac{\delta H_{cl}}{\delta A_{\mu}^{cl}}=\frac{\partial \mathfrak{C}_{cl}}{\partial A_{\mu}^{cl}}-\sum_{k=1}^{3}\frac{d}{dx_k}\left(\frac{\partial \mathfrak{C}_{cl}}{\partial(\partial A_{\mu}^{cl}/\partial x_k)}\right) \tag{16}$$

Using Eqs. (16) and (4) it is simple to see that (15) is for A_o^{cl} 34 identically zero, and for i=1,2,3

$$-j_i^{cl}-\Delta A_i^{cl} \tag{17}$$

with $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$ 36.

Combining with Eq. (13) we have finally that for A_o^{cl} 37 we obtain an equation identically zero, and for i=1,2,3

$$\Box A_i^{cl}=j_i^{cl} \tag{18}$$

with $\Box=\partial^2/\partial t^2-\Delta$ 39.

In conclusion, the variational method applied to the Hamilton-Jacobi equation of the system gives an equation identically zero for A_o^{cl} 40, as it corresponds to the gauge choice $A_o^{cl}=0$ 41, and the usual Eq. (18) for the spatial components.

3 Quantization

We shall derive the complete set of equations ruling the motion of particles and fields in the causal interpretation for the interacting field-charge system. In order to simplify the analysis we shall assume that the particles are nonrelativistic. In this approximation the field is usually treated in the Coulomb gauge [5]. This choice simplifies the mathematical treatment, but breaks the manifest Lorentz covariance of the system. The quantization in a manifest covariant way introduces into the problem a number of technical difficulties [6], which would obscure its physical interpretation.

In the causal interpretation of quantum theory [4,7,8] the system is described by the Schrödinger equation of the complete system, given by

$$i\frac{\partial\phi}{\partial t}=\hat{H}\phi=\hat{H}_p\phi+\hat{H}_F\phi+\hat{H}_I\phi \tag{19}$$

where ϕ 43 is the wave function of the complete system and the Hamiltonian operator is the sum of the particle "p", field "F" and interaction "I" Hamiltonian operators. The field operator is given by the following equation:

$$\hat{H}_F = \frac{1}{2} \int \left(- \frac{\delta^2}{\delta \vec{A}^2} + (\Delta x \vec{A})^2 \right) d^3 \vec{x} \tag{20}$$

We use natural units, $\hbar = c = 1$ 45.

To quantize the field system we treat the field coordinate A_μ 46 and its momentum conjugate as time-independent Schrödinger operators. We work in a representation in which the Hermitian operator A_μ 47 representing the field coordinate is diagonal (in this representation we do not need to use the operator notation for these diagonal operators). Then, following the usual rule for any quantum field [4,6], the momentum conjugate of the variable A_k^{cl} 48 (k=1,2,3), π_k^{cl} 49, is replaced in H_{cl} by the operator $i\delta/\delta A_k$ 50. Note that in the Coulomb gauge $\pi_o^{cl} = 0$ 51 is not present in H_{cl} and we cannot introduce the operator $i\delta/\delta A_o$ 52 that should be present in a manifest covariant quantization.

On the other hand, the interaction Hamiltonian is

$$\hat{H}_I = \int A^\mu \hat{j}_\mu d^3 \vec{x} \tag{21}$$

where the operator $\hat{j}_\mu$ 54 represents the charge current operator associated to the particle.

We assume that the particle is nonrelativistic; the expression for the nonrelativistic particle Hamiltonian is

$$\hat{H}_p = - \frac{1}{2m} \Delta_{\vec{y}} + V(\vec{y}) \tag{22}$$

The variable $\vec{y}$ 56 refers to the particle subsystem and is an external potential acting on the particle.

Now, for the interaction Hamiltonian we must use its $V(\vec{y})$ nonrelativistic expression, which is given by the well-known equation:

$$\hat{H}_I = \frac{e^2}{2m}\vec{A}^2 + \frac{ie}{m}\vec{A}.\Delta_{\vec{y}} + e A_o = \frac{e^2}{2m}\vec{A}^2 + \frac{ie}{m}\vec{A}.\Delta_{\vec{y}} \tag{23}$$

We have neglected the term depending on A_o because we assume, as in Sect. 2, the Coulomb gauge.

This expression is the usual one in the study of the behaviour of a particle in a classical external field. As usual [5], we suppose that this equation is also valid when the field is quantized.

In the case of a particle with spin we must add to the above expression a spin-dependent term. However, in order to simplify the analysis we shall only consider spinless particles.

As usual in the causal theory we introduce the polar decomposition of the wave function $\phi = Re^{iS}$ 58 [4,8]. Note that R and S, as well as the wave function, are simultaneously functionals and functions (they depend simultaneously on $\vec{y}$ 59 and A_μ 60). Introducing the polar form we obtain the following equations:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\Delta_{\vec{y}} S)^2 + V(\vec{y}) + \int \frac{1}{2}\left(\left(\frac{\delta S}{\delta \vec{A}}\right)^2 + (\Delta x \vec{A})^2\right) d^3\vec{x} + \frac{e^2}{2m}\vec{A}^2 - \frac{e}{m}\vec{A}.\Delta_{\vec{y}} S + Q = 0 \tag{24}$$

and

$$\frac{\partial R^2}{\partial t} + \frac{1}{m}\Delta_{\vec{y}}.(R^2(\Delta_{\vec{y}} S - e\vec{A})) + \int d^3\vec{x} \frac{\delta}{\delta \vec{A}}.\left(R^2 \frac{\delta S}{\delta \vec{A}}\right) = 0 \tag{25}$$

where

$$Q = -\frac{1}{2m}\frac{\Delta_{\vec{y}} R}{R} - \frac{1}{2R}\int d^3\vec{x} \frac{\delta^2 R}{\delta \vec{A}^2} \tag{26}$$

Equation (25) is a conservation law, which justifies the assumption that $R^2(\vec{y}, A_\mu, t)$ 64 represents at time t the probability for the particle and field to lie, respectively, in an element of volume of the space around $\vec{y}$ 65 and a volume of the wave functions space around the configuration $A_\mu(\vec{x})$ 66 for all $\vec{x}$ 67.

Now, we can derive the motion equations for the particle and the field coordinate. We assume that at each instant the particle has a well-defined position and the field has a well-defined value for all $\vec{x}$ 68, as in the classical particle and field theories. In particular, this statement must also be assumed in the case of the fields associated to charges. The time evolution may be obtained from the solution of the coupled guidance equations [4]:

$$m\frac{d\vec{y}}{dt} = \Delta_{\vec{y}} S - e\vec{A} \tag{27}$$

and

$$\frac{\partial A_\mu}{\partial t} = \frac{\delta S}{\delta A_\mu} \tag{28}$$

As the phase of the wave function is in general a nonfactorizable function of all the variables the two last equations are coupled ones.

Next, we derive the equations in the Newtonian form. We begin with the particle equation. We apply the operator $\Delta_{\vec{y}}$ 71 to Eq. (24). Using standard techniques and Eq. (27) we obtain:

$$m\frac{d^2\vec{y}}{dt^2} = -\Delta_{\vec{y}}(Q+V) + \vec{F} \tag{29}$$

with $\vec{F} = e(\vec{E} + \vec{v}x\vec{B})$ 73 the Lorentz force and $\vec{v}$ 74 the velocity of the particle in the causal theory.

Now we want to derive the equation of motion of the field coordinate. It is simple to obtain using Eq. (27) the following expression, which we shall use in the derivation

$$\begin{aligned} &\frac{e^2}{2m}\vec{A}^2 - \frac{e}{m}\vec{A}.\Delta_{\vec{y}} S = \\ &-\frac{e^2}{m}\vec{A}^2 - e\vec{A}.\vec{v} = -\frac{e^2}{m}\vec{A}^2 + J^c_\mu A^\mu \end{aligned} \tag{30}$$

where $J^c_\mu = (e, e\vec{v}_c)$ 76 ($J^c_\mu A^\mu = -\vec{j}^c.\vec{A}$ 77 in the Coulomb gauge and with $\vec{v}_c$ 78 the velocity of the causal trajectory) is the causal charge current, that is, the charge current associated to the motion of the particle in the causal theory. The introduction of the charge current in the causal interpretation is possible because this theory assumes the existence of trajectories for the particle (charge), being possible to define a particle velocity and a charge current similar to the classical ones. In the usual formulation of quantum theory the trajectory of the charge is a meaningless concept and it is impossible to introduce a charge current (or a velocity) for the particle in the classical sense; the charge current in this formulation is only an operator whose expectation values are calculated according to the usual rules.

Note that the charge current J_c refers to a point particle, instead j_c, the density charge current, does to a density of charge (see Eq. (31) below).

In many applications of the quantum theory of radiation for nonrelativistic particles is usual to neglect the effects of the term proportional to e^2 79 in comparison to the term of the charge current (that only depends on e).

In order to find the equation of motion of the field coordinate we express (30) in the form

$$-\frac{e^2}{m}\vec{A}^2 + J^c_\mu A^\mu = \int\left(-\frac{\rho^2}{m}\vec{A}^2 + j^c_\mu A^\mu\right)d^3\vec{x}$$

$$= \int\left(-\frac{\rho^2}{m}\vec{A}^2 - \vec{j}^c.\vec{A}\right)d^3\vec{x} \qquad (31)$$

with $j^c_\mu = (\rho, \rho\vec{v}_c)$ 81 the causal density charge current.

Finally, we apply the functional derivative $\delta/\delta A_\mu$ 82 to Eq. (24). Following the same steps of Sect. 2 we obtain

$$\_\ A_i = j^c_i - \frac{\delta Q}{\delta A_i} - \frac{2\rho^2}{m} A_i \qquad (32)$$

for i=1,2,3, and an equation identically zero for A_o (since we have assumed the Coulomb gauge A_o=0, Eqs. (19) and (24) do not depend on A_o and all the derivatives are zero).

Note that we also obtain a term of the form $\delta[(\Delta_{\bar{y}}S)^2/2m]/\delta A_\mu$ 84, but this term is zero. Effectively, operating we would have $(\Delta_{\bar{y}}S).\delta(\Delta_{\bar{y}}S)/\delta A_\mu$ 85, which changing the order of the normal and functional derivatives and using Eq. (28) gives $\Delta_{\bar{y}}(\partial A_\mu/\partial t) = 0$ 86.

When we neglect the term in ρ^2 87 we obtain the following simplified covariant equation (note that this equation is covariant, but the density charge current is nonrelativistic) for i=1,2,3.

$$\Box A_i = j_i^c - \frac{\delta Q}{\delta A_i} \tag{33}$$

We shall consider the interpretation of the causal equations in the Conclusions.

4 Conclusions

We have studied in this article the behaviour of the quantum electromagnetic fields associated to a quantum charge in the framework of the causal interpretation of Quantum Mechanics.

The analysis has been carried out in the Coulomb gauge. This particular choice breaks down the manifest covariance of the theory but allows for a simpler treatment of the problem. As it is well-known a completely covariant quantization of the electromagnetic field in the framework of the Hamiltonian formalism presents serious problems. Several solutions have been proposed for this difficulty, for instance, the introduction of new equations of motion restoring Maxwell's theory by appropiate constraints on the physical states [6]. In this paper, in order to do not obscure the physical implications of the causal interpretation, we have avoided such technicalities and we have worked in a particular gauge. The price to be paid by this simplification is that we cannot obtain the equation for A_o. However, we have derived the equations for $\vec{A}$ 89, and we can compare the classical and causal equations of motion.

The causal equation, Eq. (32), shows important similarities and differences when compared to the classical one, Eq. (18). The causal equation for the field not only depends on the density charge current, but also on the functional derivative of the quantum potential and on the term proportional to ρ^2 90 (in the case of point particles to e^2 91). The presence of the causal charge current in the equation implies the dependence of the fields on the causal trajectory. The classical fields are also a function of the classical trajectories of the charges, but different from the causal one because of the two new terms present in (32).

Equation (32) becomes nonlinear due to the term $\delta Q/\delta A_i$ 92. This nonlinearity is the main distinctive characteristic of the causal equation when compared to the linear classical one.

We must also consider the classical limit of (32). In the causal theory the classical limit is obtained taking $Q \to 0$. This limit also implies $j_\mu^c \to j_\mu^{cl}$ 96 with j_μ^{cl} 97 the classical density charge current, because the causal velocities tend to the classical ones [4]. In this limit we obtain the classical equation plus the term proportional to ρ^2 98. This term has no classical counterpart, reflecting its purely quantum origin.

Acknowledgments

This work has been partially supported by the Spanish Ministry of Education and Science under contract REN2000-1621 CLI.

References

[1] H. Goldstein, Classical Mechanics (2nd. edn, Addison-Wesley, Reading MA, 1980).

[2] J. D. Jackson, Classical Electrodynamics (John Wiley, New York, 1975).

[3] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields (Pergamon Press, Oxford, 1969).

[4] P. R. Holland, The Quantum Theory of Motion (Cambridge University Press, Cambridge, 1993).

[5] A. Galindo and P. Pascual, Quantum Mechanics (Springer-Verlag, 1990).

[6] C. Itzykson and J. B. Zuber, Quantum Field Theory (McGraw-Hill, New York, 1980).

[7] L. de Broglie, Une Tentative d'Interprétation Causale et Non-Linéaire de la Mécanique Ondulatoire (Gauthier-Villars, Paris, 1956).

[8] D. Bohm and B. J. Hiley, The Undivided Universe (Routledge, London and New York, 1993).

Manuscrit reçu le 7 mars 2001