Cornelius Lanczos – Discoveries in the Quantum and General Relativity Theories

MENDEL SACHS

Department of Physics, University at Buffalo, State University of New York

Cornelius Lanczos (1893 – 1974) was indeed one of the greatly inspired scholars in theoretical physics and applied mathematics in the twentieth century. It is not only his voluminous outpouring of excellent and challenging research on diverse subjects in these fields; it was perhaps more importantly his open spirit that must influence his peers and generations of physicists and mathematicians. Thus, it is highly commendable that Lanczos' collected papers and commentaries on them have been published in *Cornelius Lanczos: Collected Published Papers and Commentaries*, editors: W.R. Davis, M. Chu, P. Dolan, J. R. McConnell, L. K. Norris, E. Ortiz, R. J. Plemmons, D. Ridgeway, B. K. P. Scaife, W. J. Stewart, J. W. York, Jr., W. O. Doggett, B. M. Gellai, A. A. Gsponer, C. A. Prioli (North Carolina State University, Raleigh, 1998). Henceforth this collection will be referred to as CLCPPC.

Because of our mutual research interests in the quantum and general relativity theories, and questions concerned with (nonrelativistic and relativistic) quantum theory, I started a correspondence with Cornelius when he was in Dublin, in the 1960s. He kindly invited me to spend some time with him at the Dublin Institute for Advanced Studies. I enjoyed the opportunity to visit (in 1964 and 1973), to discuss questions of interest of both of us and to have discussions with his colleagues at the Institute, J. Singh, J. McConnell and L. O'Raifertaigh.

In this note, I would like to briefly discuss two of Lanczos' investigations that are not generally known about in the physics community and to comment on them.

1 A Discovery in the Quantum Theory

In 1926, at the onset of the development of the formal expressions of quantum mechanics, Lanczos discovered that Heisenberg's (discrete) matrix representation and Schrodinger's (continuous) wave representation are mathematically equivalent – since each can be transformed into the other. [Schrodinger also discovered this fact independently, in the same year.]

What it was that Lanczos saw was that the equations of motion and the quantum condition could be expressed in the form of integral equations. Thus he said: "A conception of the continuum exists side by side with equal validity with the conception of the discrete, because there is a unique relationship between them."

The form of the Schrodinger wave equation is :

$$H\psi \equiv i\partial\psi/\partial t = -\nabla^2\psi + V\psi = E\psi \tag{1}$$

where H is the Hamiltonian operator (as defined by Schrodinger in terms of the operator equivalents of the momentum and position variables of a particle of matter), and E is the particle's energy eigenvalue of H (units chosen above with $h/2\pi = 1$, where h is Planck's constant).

To derive the integral form of this equation, Lanczos considered the solution of the mathematical problem

$$-\nabla^2\psi + V\psi = u, \quad \psi(\text{boundary}) = 0$$

He found that the boundary value problem may be solved by means of the Green's function K(P, Q) for this inhomogeneous differential equation, with the solution

$$\psi(P) = \int K(P, Q)u(Q)\, dQ$$

Replacing u(Q) with $E\psi(Q)$, and dividing by E, one has the integral equation:

$$\int K(P, Q)\, \psi(Q)\, dQ = (1/E)\, \psi(P) \tag{2}$$

Thus, the eigenvalues of the kernel K(P, Q) are the reciprocals of the energy values, 1/E.

The correspondence between equations (1) and (2) then leads to the Heisenberg equation of motion for quantum mechanics,

$$[H,O]\psi \equiv (HO - OH)\psi = i(\partial O/\partial t)\psi \tag{3}$$

This equation, in turn, yields the matrix representation of quantum mechanics, where H is the Hamiltonian operator for the dynamical system and O is an operator that corresponds with some particular observable physical property of the microsystem.

Thus Lanczos showed, unequivocally, that Heisenberg's equation of motion for quantum mechanics, (3), is equivalent to Schrodinger's wave mechanical form (1). The details of this correspondence are spelled out in Lanczos' paper: "Uber eine maissige Darstellung der neuen Quantenmechanik" ("On a Field Theoretical Representation of the New Quantum Mechanics"), Zeitschrift fur Physik **35,** 812 (1926). Its English translation is in CLCPPC, Volume III, p. 2-858.

There has been a controversy on the question of who first discovered this equivalence of the Heisenberg and Schrodinger representations of quantum mechanics – was it Schrodinger, Pauli or Lanczos? (from their respective points of view). There is an interesting dialogue and discussion on this by B. L. van der Waerden, "From Matrix Mechanics and Wave Mechanics to a Unified Quantum Mechanics", CLCPPC, Volume III, p. 2-896, including a letter written by Pauli to Jordan on this subject. Another interesting article on this subject is by J. R. McConnell, "Commentary on Lanczos' "On a Field Theoretical Representation of the New Quantum Mechanics""CLCPPC, Volume III, p. 2-950. Van der Waerden, in his dialogue, favors Lanczos as the actual discoverer of the equivalence of these two representations of quantum mechanics.

I might add the following comment: While the differential form of Schrodinger's wave mechanics (1) is equivalent to Lanczos' integral equation (2) – which in turn leads to the Heisenberg representation (3) - the latter integral equation form cannot be extended to an expression of wave mechanics in a curved spacetime, as required in general relativity. This is because the Green's function is *defined* in terms of a linear mathematical formalism at the outset. Thus, in the (necessarily) nonlinear, curved spacetime of general relativity, a Green's function for this problem does not exist. On the other hand, Schrodinger's differential equation form (1) for wave mechanics can be extended to the nonlinear, curved spacetime. This is achieved by going (smoothly) to a spinor-quaternion formalism, where the solutions of the equations are now *spinor variables* in a curved spacetime rather than the scalar functions of Schrodinger's nonrelativistic wave mechanics. In this extended formalism, ordinary derivatives are replaced with covariant derivatives. [This extension is indicated by the irreducible representations of the *Einstein group* – the symmetry group that underlies the general covariance requirement of general relativity theory.] The covariant derivative of a spinor field is the sum of the ordinary derivative and a spin-affine connection term. The spinor solutions themselves, for the matter fields, are then the basis functions of quaternion differential operators. This generalized nonlinear expression of wave mechanics in general relativity cannot then be interpreted in terms of a probability calculus, since the latter, *by definition*, requires a linear calculus.

This advantage of the differential equation form of quantum mechanics over the integral equation form is a significant point since the integral equation form of nonrelativistic quantum mechanics, which Lanczos addresses in this problem, does not extend to the nonlinear curved spacetime. But the latter extension is necessary since nonrelativistic quantum mechanics is supposed to be not more than an *approximation* for a generally relativistic theory of matter in the microscopic domain. The latter generalization of quantum mechanics to a nonlinear form cannot then be interpreted as the Copenhagen school does in terms of a probability calculus.

Lanczos himself indicated in his writings since the 1920s that a nonlinear extension of wave mechanics must follow.

I have spelled out the details of this extension of the formal expression of quantum mechanics to general relativity in my book: M. Sachs, *General Relativity and Matter* (Reidel, 1982) and its sequel, M. Sachs, *Quantum Mechanics from General Relativity* (Reidel, 1986). I have shown in the latter book that the formal expression of quantum mechanics, in terms of the linear Hilbert space, emerges as a linear approximation for a generally covariant, nonlinear field theory of the inertia of matter.

2 A Discovery in General Relativity

A second very important discovery of Lanczos had to do with the problem of equations of motion of material particles in the theory of general relativity. Without resorting to approximation methods [as in A. Einstein, L. Infeld and B. Hoffmann, "Gravitational Equation and the Problem of Motion", *Annals of Mathematics* **39**, 65 (1938)] Lanczos discovered that the equation of motion of a gravitational body is implicit in Einstein's tensor field equation itself. That is, there is no need to add extra equations of motion to the field equations. This is in contrast with the standard electromagnetic field theory. In the latter case, the field equations are in terms of Maxwell's equations while the equations of motion of a charged body, subjected to an electromagnetic field, must be added, in the form of the Lorentz equation of motion. The latter difference is due, in part, to the fact that the Einstein field theory is explicitly nonlinear while the Maxwell field theory is explicitly linear. Thus, the Einstein field theory of gravitation is more complete than the Maxwell field theory of electromagnetism. This is because Einstein's theory is a 'closed form' theory of matter while the Maxwell theory is not so.

In his paper, "The Dynamics of a Particle in General Relativity", *Physical Review* **59**, 813 (1941), duplicated in CLCPPC, Volume IV, p. 2-1650, Lanczos discusses this problem. He starts out by explaining that, in his view, "the moving force [that accelerates a material body] comes out in terms of a volume integral, extended over the matter-occupied central field of the particle." He points out that no matter how one might modify the spacetime metric *inside of the particle*, no motion is predicted. [This is a difficulty emphasized by Einstein in his correspondences with Lanczos.] Thus, one needs a force *external* to the body acted upon that would cause it to accelerate – as one has with Newton's second law of motion.

What Lanczos did to overcome this difficulty was to show that "the volume integral of the moving force [can be transformed] into a boundary integral, extended over the border of the particle, or any closed surface that includes this particle." He then concluded that the motion law is established rigorously, without the need for approximation methods. The law that he derived had the form of Newton's second law of motion, *when one treats the static condition only* (that is, for a particle initially at rest). Lanczos' law of motion in general relativity then takes the form:

$$d^2\xi^i/dx_4^2 = -\Gamma^{(e)}{}_{44,i} \tag{4}$$

where i = 1, 2 or 3 denote the three spatial directions, x_4 is the time coordinate , ξ^i are the spatial coordinates of the particle and $\Gamma^{(e)}{}_{44,i}$ are the ith derivatives of the '44' component of the affine connection of the curved spacetime field *external to the particle.*

It is important to note that the particle's mass *m* does not appear in this equation of motion of a body in a gravitational field. (It is a generalization of Galileo's discovery that the gravitational acceleration of a body is independent of its inertial mass.) Lanczos calls this "equivalent to the law of the geodesic line".

Lanczos' imposed "static condition" to derive the law of motion (4) is perhaps too restrictive to conclude from it a general law of motion. In my own analysis of this problem in general relativity, I have concluded that there can be no discrete particle of matter in the first place, in the context of this theory. The geodesics of the spacetime are the solutions of the dynamical problem for a 'field concentration' that we identify empirically with a 'thing' – e.g. an electron, a planet or a galaxy. The metric solutions of general relativity are the regular (i.e. nonsingular and analytic *everywhere*) functions of the space and time coordinates. Thus, there is no 'inside' and 'outside' of a discrete material particle, in the context of the continuous field theory implicit in general relativity. What is "seen" as a particle is, rather, a continuous (though peaked) mode of a (regular) matter field. Thus, Lanczos had no need, in the framework of the theory of general relativity, to transform the continuous volume integral – a natural expression of a mode of the continuum – into a surface integral. For the volume integral itself is intimately related to the entire continuum; there is in reality no surface within the continuum, separated from it. This conclusion of the holism (essential connectivity) of a material system expresses the essence of Mach's principle, which I have found in my studies is an implication of Einstein's theory of general relativity, as a general theory of matter.

These conclusions are in accord with Lanczos' assertion that (real) singularities are to be excluded from the solutions of the generally covariant field equations of matter. His position, rather, is that "matter is no more a singularity of the field, but an *eigensolution* of the field equations." ["Die neue Feldtheorie Einsteins"(The New Field Theory of Einstein), *Eigebnisse der exakten Naturwissenschaften* **10** (1931), 97 – 132. Duplicated in: CLCPPC, Volume IV, p. 2-1443].

While I agree with the exclusion of singularities from the solutions of a generally covariant field theory of matter, it is my position that the eigenvalue structure of the field equations is an asymptotic, but *not* an exact feature of the field laws. One may see this difference, for example, on the one hand in Lanczos' comments that it is "all the same whether we work with the homogeneous equations and admit singularities, or with the inhomogeneous one excluding singularities" (*ibid*., CLCPPC, Volume IV, p. 2-1439). On the other hand, it is my view, that, physically, there are no meaningful homogeneous equations in the first place. It is because we start in general relativity with a closed system at the outset, wherein the left hand side of the field equations (where the field intensities appear) is a representation of the right hand side (sources), *and vice versa*. Still, one may approximate the right hand side, *under special circumstances*, to be asymptotically small, replacing these terms with zero for practical purposes of calculation.

The difference between Lanczos' and my position here is related to the features of nonlinear differential equations. The solutions of the corresponding inhomogeneous and homogeneous nonlinear equations do not go smoothly into one another, under any conditions. That is, it is my position that, even asymptotically, so long as the right hand side is close to, but not exactly equal to zero, the solutions of one of the respective nonhomogeneous or homogeneous differential equations have features, related to physical properties, that are not duplicated in the solutions of the other.

One may compare this, as Lanczos does, with Newton's theory of gravitation, wherein we have Laplace's equation $\nabla^2\varphi = 0$ outside of the boundary of a gravitational body, and Poisson's equation, $\nabla^2\varphi = \rho$ inside of this boundary, where φ is the gravitational potential of the body and ρ is its density. If these would be nonlinear equations, as they would in a curved spacetime, then the solutions of Poisson's equation would not be a linear superposition of the solutions of Laplace's equation. Indeed, in general relativity there could not be any discrete boundary; there is only a (variable) continuum in space and time. Only in the linear limit (of a flat spacetime), as in Newton's theory of gravitation, can one assume a discrete boundary for a quantity of matter. But the latter limit is not true in an exact sense in general relativity theory.

Cornelius Lanczos was a good friend and colleague of mine. From our personal correspondences and contacts I know that whatever criticism I have mentioned in this note would have been accepted in the spirit of the essential role of controversy in science, to ensure its progress. For this attitude as well as his contributions, he was indeed one of the great scientists of the twentieth century.

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