On non-equivalence of Lorentz and Coulomb gauges within classical electrodynamics.

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ABSTRACT. It is shown that the well-known procedure for proving the equivalence of the expressions for the electric field calculated using the Lorentz and Coulomb gauges is incorrect. The difference between the two gauges is due to the difference in the speed of propagation of a disturbance of the scalar potential.

1. Introduction.

Recently, Tzontchev *et al.* [1] reported on an experiment in which they detected a longitudinal component of the electric field propagating at the speed of light in the near field of a radiator. This result seems to be obvious because an electric field propagating with the speed of light can easily be calculated, for example, by using Eq. 14.14 in [2]. However, the problem is that Eq. 14.14 was derived using the Lorentz gauge in which disturbances of the scalar potential propagate with the speed c. Because it follows from the experiment described in [1] that 0.95 of the total magnitude of the **E** field is created by the scalar potential, it can be concluded that the scalar potential propagates at the speed of light. However, this contradicts conventional electrodynamics. It has been established in classical electrodynamics that the EM potential cannot be treated as a physical quantity, but as a mathematical tool for calculating EM fields [2, Ch. 6.5, 3]. Therefore, the solutions of the wave equation can be chosen so that the speed of propagation u of the scalar potential can vary from zero to infinity. By choosing such solutions, the gauge is also determined [4].

The most used gauges in electrodynamics are the Coulomb ($u = \infty$) and Lorentz ($u = c$) gauges. However, the infinite speed of propagation of the scalar potential in the Coulomb gauge seems to contradict the theory of relativity. Despite this, references [3, 2 (p 291, problem 6.20)] state that the issues of causality and of the finite speed of propagation of electromagnetic disturbances are obscured by the choice of the Coulomb gauge: the potentials $\varphi(u)$ and $\mathbf{A}(u)$ are manifestly not causal, but the fields can be shown to be. So it contradicts the conclusion given in [1] that "*The proper inference from this experiment is that the Coulomb interaction cannot be considered as so called 'instantaneous action at a distance'*"(i.e. the scalar potential in the Coulomb gauge). Actually, since we are only able to measure the EM fields experimentally, we cannot draw any conclusions about the speed of propagation of the scalar potential, if two solutions for the scalar and vector potentials $\varphi(u_1)$, $\mathbf{A}(u_1)$ and $\varphi(u_2)$, $\mathbf{A}(u_2)$, where u_i is determined by the choice of gauge, yield identical expressions for the electric field.

However, the results of the experiment described in [1] suggest that these expressions are not identical. So, one can question whether or not different gauges in electrodynamics are actually equivalent. In this paper, we consider this problem. Because existing studies of the equivalence of the gauges have dealt with the Coulomb and Lorentz gauges, we will focus our analysis on these gauges as well.

The plan of this paper is as follows. In section 2, we will calculate the electric field in both gauges for the simplest model of the experimental setup of [1]. Also we will show that these calculations must be made by means of the potentials but cannot be done by using the wave equation for the $\mathbf{E}$ field derived from the Maxwell equations directly, i.e. without introducing the potentials. Some explanation of the results of the Sec. 2 is given in the sections 3 and 4 where we will review the derivation of the equivalence of the expressions for the electric field calculated in both gauges and then we will show at what point this derivation is wrong, i.e. the difference in the shapes of the elementary classical charges calculated in both gauges is neglected. This difference follows unambiguously from the expressions for the scalar potential in the Coulomb and Lorentz gauges. Because its motion causes a change in the shape of the charge, since the size of the charge contracts along its direction of motion, one would expect that the greatest difference in the gauges should occur in this direction as well. Finally, in section 5, we will draw some conclusion about what physical effect is responsible for the difference in the gauges.

2. An example of difference between the electric field calculated in these gauges.

It is quite impossible to process complete calculations of the E field detected by the antenna in the experiment of [1]. So for analysis of the fields in this system we should simplify the latter but in such a way that its inherent features will be held. Therefore, we consider the following simplification of the real experiment: a single charge moving at the straight line which corresponds to the axis of symmetry of the experimental installation. We assume too that this charge moves uniformly which allows us to consider the most general case; i.e., when the properties of the system do not depend on their initial conditions, and, therefore, when choosing the advanced and retarded solutions as well.

Now we state the question: *is it possible that the longitudinal component of the E field calculated in different gauges has different values?.* Similar calculations have not been made in [1] so we wish to make up this gap.

It should be noted that all formulas for calculations of the longitudinal fields of the moving charge (in near non-radiative zone) are made for the Lorentz gauge. So while calculating the electric field in the Coulomb gauge, we use the method given in [5].

Thus, the equation for the vector potential in the Coulomb gauge is (the Eq. 6.46 of [2])

$$\nabla^2 \mathbf{A}_C - \frac{\partial^2 \mathbf{A}_C}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}_C + \frac{1}{c}\nabla\frac{\partial \varphi_C}{\partial t}$$

Because of independence of the current $\mathbf{J}_C$ and the scalar potential φ on each other, we are able to express the quantity $\mathbf{A}_C$ in terms of the sum of two quantities; the first of which is determined by one wave equation and the second from the other wave equation, i.e.

$$\mathbf{A}_C = \mathbf{A}_L + \mathbf{A}_\varphi \tag{2.1}$$

$$\nabla^2 \mathbf{A}_L - \frac{\partial^2 \mathbf{A}_L}{c^2 \partial t^2} = -\frac{4\pi}{c}\mathbf{J}_L \tag{2.2}$$

$$\nabla^2 \mathbf{A}_\varphi - \frac{\partial^2 \mathbf{A}_\varphi}{c^2 \partial t^2} = \frac{1}{c}\nabla\frac{\partial \varphi_C}{\partial t} \tag{2.3}$$

One can see from the Eq. 2.2 that $\mathbf{A}_L$ is the vector potential in the Lorentz gauge. Now we find the difference between the electric fields calculated in the Coulomb $\mathbf{E}_C$ and Lorentz $\mathbf{E}_L$ gauges

$$\mathbf{E}_C - \mathbf{E}_L = \nabla[\varphi_L - \varphi_C] + \frac{\partial}{c\partial t}[\mathbf{A}_L - \mathbf{A}_C] =$$

$$= \nabla[\varphi_L - \varphi_C] - \frac{\partial \mathbf{A}_\varphi}{c\partial t} + \frac{\partial}{c\partial}\frac{1}{c}\int \frac{\mathbf{J}_L(r_1)}{|\mathbf{r}_1 - \mathbf{r}|}d\mathbf{r}_1 - \frac{\partial}{c\partial}\frac{1}{c}\int \frac{\mathbf{J}_C(r_1)}{|\mathbf{r}_1 - \mathbf{r}|}d\mathbf{r}_1 \tag{2.4}$$

Here we take into account that, as it will be shown in the Sec. 4, the charges have different shapes in the Coulomb and Lorentz gauges so the current densities $\mathbf{J}_C$ and $\mathbf{J}_L$ are different as well. But taking into account the arguments in Ch. 4 and Appendix I, we can prove that the sum of third and fourth terms of Eq. 2.4 is asymptotically equal to zero, so we omit them from further consideration.

We use Eq. 2.4 to calculate the difference between the electric fields in the system defined above, i.e.
the charge moves uniformly along the X-axis and the detector of the electric field is on this axis as well.

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The first term on the *rhs* of Eq. 2.4 is the difference between the *retarded* and *instantaneous* scalar potentials. The magnitude of the retarded potential of a uniformly moving point charge (if we measure this quantity at the axis of motion of the charge) is

$$\varphi_L = 1/\left|x - vt\right| \tag{2.5}$$

Eq. 2.5 is the reduced form of Eq. 21.39 in [6] for $y = 0$, $z = 0$, and the 'current' time, but not in terms of the retarded time. Eq. 2.5 coincides with the expression for the Coulomb potential of the charge, when the charge is at the point $x - vt$, where t is an instantaneous ('current') time. Therefore, the sum in the brackets on the *rhs* of Eq. 2.4 is equal to zero. For the electric fields calculated in the Coulomb and Lorentz gauges to be equivalent, it is necessary that the second term on the *rhs* of the Eq. 2.4 be equal to zero. But this is impossible if the terms, which are proportional to the gradients of the scalar potentials, eliminate each other. One term on the rhs of the Eq. 2.4 still remains and it can be expressed in terms of the equation

$$\mathbf{E}_C - \mathbf{E}_L = \frac{\partial \mathbf{A}_\varphi}{c\partial t} \tag{2.6}$$

where $\mathbf{A}_\varphi$ is the solution of the wave equation with source $\nabla[\partial\varphi_C/c\partial t]$.

To obtain solution of the Eq. 2.6, we will use the Lorentz procedure of solving the wave equation ([7] Ch 18.3); we do not refer to the original work of Lorentz because he finds the solution for the fields and not the potentials). In the Coulomb gauge, the distributed 'longitudinal current' (the term $\nabla[\partial\varphi_C/c\partial t]$ instantaneously follows the charge creating this current, therefore the Lorentz transformation of the coordinates reduces a static case and, as a result, calculating the difference in the fields belonging to different gauges reduces to solving a three dimensional integral.

Thus, the wave equation for $\mathbf{A}_\varphi$ is

$$\left(\frac{\partial^2}{c^2\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)\mathbf{A}_\varphi(x,y,z,t) = \frac{1}{c}\nabla\frac{\partial\varphi_C(x-vt,y,z)}{\partial t} \tag{2.7}$$

where we take into account that the *rhs* of the Eq. 2.7 is formed from the derivatives of scalar potential in the Coulomb gauge, where this potential 'instantaneously' follows the motion of the charge so that the x and t variables enter in the *rhs* of the Eq. 2.7 in terms of the combination $(x - vt)$.

Because the EM fields created by uniformly moving source must move with this source too, the time and spatial derivatives are not independent of each other, but are linked by the relation (the Eq. 18.10 of [7])

$$\frac{\partial}{\partial t} = -v\frac{\partial}{\partial x}$$

Therefore, the Eq 2.7 reduces to

$$\left(\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2}{\partial(x')^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathbf{A}_\varphi(x',y,z,t') = -\frac{v}{c}\frac{\partial^2\varphi_C(x',y,z)}{\partial(x')^2} \tag{2.8}$$

where $x' = x - vt$. Since the *rhs* of Eq. 2.8 does not depend on time, the *lhs* does not depend on time either, which means that Eq. 2.8 reduces to the Poisson equation in elliptic coordinates. By changing the variables $x'/\sqrt{1-\frac{v^2}{c^2}} = \chi$, Eq. 2.8 reduces to the ordinary Poisson equation with the *rhs* containing a spatially distributed source. Its solution is:

$$\mathbf{A}_{\varphi,\mathbf{X}} = \frac{v}{c(1-\frac{v^2}{c^2})}\int \frac{\partial^2\varphi(\chi,y,z)/\partial\chi^2}{|\mathbf{R}_1 - \mathbf{r}(\chi,y,z)|}d\chi dy dz \tag{2.9}$$

where

$$R_1 = \sqrt{(1-v^2/c^2)\,(X-vt)^2 + Y^2 + Z^2}$$

Inserting the expression for $\mathbf{A}_{\varphi,\mathbf{X}}$ (Eq. 2.9) into Eq. 2.6, we finally obtain

$$\mathbf{E}_C(\mathbf{R},t) - \mathbf{E}_L(\mathbf{R},t) = \frac{v}{c(1-\frac{v^2}{c^2})}\frac{\partial}{c\partial t}\int \frac{\partial^2\varphi(\chi,y,z)/\partial\chi^2}{|\mathbf{R}_1 - \mathbf{r}(\chi,y,z)|}d\chi dy dz$$

One can easily see that because the integrand is not a symmetric expression, the integral over the whole space is not equal to zero (we do not finish the calculation of this integral because its concrete form is not essential). Therefore, we find that the field is actually different in the different gauges.

Here, one can expect an objection that because the **E** field can be calculated directly from Maxwell equations, the analysis of the difference in the **E** fields calculated in both gauges loses its sense. However, it is not so. We show that for this case, i.e. the case of *longitudinal* fields, it is impossible to obtain the solution for the **E** field without using the EM potentials.

To avoid any cumbersome calculations which can be caused by necessity to describe radiation processes, we consider simplest electrodynamical system which one is given above, i.e. the charge moves uniformly along the X axis. In description of this system, we will be able to obtain the expressions for the field in explicit form which allows to compare the solutions for the **E** field obtained in two ways.

Firstly, we consider derivation of direct, i.e. made without introducing the potentials, wave equation (DWE) for **E** field. Using two Maxwell equations (second and fourth Eqs. 6.28 of [2])

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{c\partial t} \tag{2.10}$$

$$\nabla \times \mathbf{H} = +\frac{\partial \mathbf{E}}{c\partial t} + \frac{4\pi \mathbf{J}}{c} \tag{2.11}$$

Taking the curl of Eq. 2.10 and partial time derivative, divided by c, of Eq. 2.11, we obtain

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial \nabla \times \mathbf{H}}{c\partial t}$$

$$\frac{\partial \nabla \times \mathbf{H}}{c\partial t} = +\frac{\partial^2 \mathbf{E}}{c^2 \partial t^2} + \frac{4\pi \partial \mathbf{J}}{c^2 \partial t}$$

Eliminating the **H** field from the above equations, we have

$$\nabla \times \nabla \times \mathbf{E} + \frac{\partial^2 \mathbf{E}}{c^2 \partial t^2} = -\frac{4\pi \partial \mathbf{J}}{c^2 \partial t} \tag{2.12}$$

Substituting the vector identity

$$\nabla \times \nabla \times \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

to the Eq. 2.12, we obtain

$$-\nabla^2 \mathbf{E} + \frac{\partial^2 \mathbf{E}}{c^2 \partial t^2} = -\frac{4\pi \partial \mathbf{J}}{c^2 \partial t} - \nabla (\nabla \cdot \mathbf{E})$$

From the first of Eqs. 6.28 of [2], $\nabla \cdot \mathbf{E} = 4\pi\rho$, which gives

$$-\nabla^2 \mathbf{E} + \frac{\partial^2 \mathbf{E}}{c^2 \partial t^2} = -\frac{4\pi \partial \mathbf{J}}{c^2 \partial t} + 4\pi \nabla \rho \tag{2.13}$$

Now we use the Eq. 14 for calculation of the electric fields created by the elementary uniformly moving along the X axis.

Since the wave operator $-\nabla^2 + \left(\partial^2 .../c^2 \partial t^2\right)$ is a scalar, direction of the **E** vector is defined by direction of the vector of the source, i.e. of $-\left(4\pi \partial \mathbf{J}/c^2 \partial t\right) - 4\pi \nabla \rho$. Now we use the principle of superposition and present the source as four separate sources directed along the axes (x, y, z).

$$\left(-\frac{4\pi}{c^2}\frac{\partial J_x}{\partial t}; -4\pi\frac{\partial \rho}{\partial x}; -4\pi\frac{\partial \rho}{\partial y}; -4\pi\frac{\partial \rho}{\partial z}\right)$$

The total **E** field can be presented as a sum of four independent fields, each of them is a solution of the wave equation

$$-\nabla^2 E_{x,J} + \frac{\partial^2 E_{x,J}}{c^2 \partial t^2} = -\frac{4\pi \partial J_x}{c^2 \partial t} \tag{2.14a}$$

$$-\nabla^2 E_{x,\rho} + \frac{\partial^2 E_{x,\rho}}{c^2 \partial t^2} = -4\pi \frac{\partial \rho}{\partial x} \tag{2.14b}$$

$$-\nabla^2 E_{y,\rho} + \frac{\partial^2 E_{y,\rho}}{c^2 \partial t^2} = -4\pi \frac{\partial \rho}{\partial y} \tag{2.14c}$$

$$-\nabla^2 E_{z,\rho} + \frac{\partial^2 E_{z,\rho}}{c^2 \partial t^2} = -4\pi \frac{\partial \rho}{\partial z} \tag{2.14d}$$

To obtain the solution of the Eq. 2.14b, we use the Green formula (the Eq. 6.66 of [2] with the '*source*'

$$f(\mathbf{r}', t') = (\partial \rho / \partial x')$$

i.e.

$$E_{x,\rho}(\mathbf{r}, t) = \int \frac{(\partial \rho / \partial x')_{ret}}{|\mathbf{r} - \mathbf{r}'|} dr' \tag{2.15}$$

where r is the radius vector of the point of detection of the fields and note '*ret*' means that the function $(\partial \rho / \partial x')$ should be calculated at retarded time.

To calculate the integral 2.15 in the limit of point charge, we should make integration by parts of the *rhs* of the above equation.

$$\begin{aligned} E_{x,\rho}(\mathbf{r}, t) &= \int \frac{(\partial \rho / \partial x')_{ret}}{|\mathbf{r} - \mathbf{r}'|} dr' = -\int \rho_{ret} \frac{\partial}{\partial x'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} dr' \\ &= \int \rho_{ret} \frac{x - x'}{|\mathbf{r} - \mathbf{r}'|^3} dr' \end{aligned} \tag{2.16}$$

Now, while calculating the integral 2.16, we should take into account that the charge is a non-point object; so after going in integration over the volume occupied by the elementary charge to integration over the charge itself, we have (all details of transition from dr' integration to de integration are given in Ch. 18.1 of [7])

$$E_{x,\rho}\left(\mathbf{r},t\right)=\int\frac{(x-x')}{\left|\mathbf{r}-\mathbf{r}'\right|^{3}}\rho_{ret}dr'=\int\frac{(x-x')}{\left|\mathbf{r}-\mathbf{r}'\right|^{3}}\frac{de}{\left[r-(\mathbf{r}\cdot\mathbf{v})/(cr)\right]}\qquad(2.17)$$

As a result, we obtain

$$E_{x,\rho}(\mathbf{r},t)=\frac{qx}{r^{3}\left[1-\left(\mathbf{r}\cdot\mathbf{v}\right)/(cr)\right]_{ret}}$$

We will not calculate transversal terms because to show incorrectness of the DWE solutions, it is sufficient to obtain for the only component of the **E** field that this component obtained from the DWE and from the LW potential is different.

However, the electric field is created not only by the charge but by the current density too. So we take into account the solution of Eq. 2.14a. One can see that the *rhs* of Eq. 2.14a $-\left(4\pi/c^{2}\right)\left(\partial j_{x}/\partial t\right)$ can be changed by the term $-\left(4\pi v^{2}/c^{2}\right)\left(\partial\rho/\partial x\right)$ in case of uniformly moving charge. So the total E_x solution of the DWE is

$$E_{x}(\mathbf{r},t)=\left(1-\frac{v^{2}}{c^{2}}\right)\frac{qx}{r^{3}\left(1-(\mathbf{r}\cdot\mathbf{v})\right)/(cr))_{ret}}\qquad(2.18)$$

and similar field calculated after the LW potential (in longitudinal direction there is no radiated term) is

$$E_{x}(\mathbf{r},t)=-\left(1-\frac{v^{2}}{c^{2}}\right)\frac{\partial}{\partial x}\frac{q}{\left[r-\left(\mathbf{r}\mathbf{v}\right)/c\right]_{ret}}\qquad(2.19)$$

Obviously, Eqs. 2.18 and 2.19 are *different*. So if we assume that the Eq. 15 is correct we must assume that the Eq. 16 is incorrect. However, the Eq. 16 is a part of general formula for the E fields of arbitrary moving charge. So our assumption will require radical revising the basic formulas of the classical electrodynamics, therefore, we must conclude that the DWE for the electric field gives incorrect result.

But this *strange result* of difference in the **E** fields calculated via the potentials but in different gauges must be explaned. So one can suggest that there is a some error in the proof of equivalence of the electrodynamical gauges. Below we will show that this suggestion has some ground but before we review existing proof of this equivalence within the classical electrodynamics.

3. Derivation of the expressions for the E field calculated in both gauges.

One can assume that a sufficient condition of equivalence for both gauges is the identical form of the expressions for the electric field calculated by both these gauges. Proof of this is well represented in the scientific literature (see [3], [2] 2^{nd} and 3^{rd} editions, and [5]). In spite of this, we recall that these derivations miss a critical point. Although our derivation does not coincide completely to those given in [2, 3, 5], we keep the basic ideas used in the cited works.

Thus, we consider the wave equations for the vector and scalar potentials in the Coulomb and Lorentz gauges, respectively. The wave equations for the vector potential in the Coulomb (for $\mathbf{A}_C$) and Lorentz (for $\mathbf{A}_L$) gauges are (Eqs. 3 and 6 of [5], where we use the same notation used in [5]):

$$\nabla^2\mathbf{A}_C - \frac{\partial^2\mathbf{A}_C}{c^2\partial t^2} = -\frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\nabla\frac{\partial\varphi_C}{\partial t} \tag{3.1}$$

$$\nabla^2\mathbf{A}_L - \frac{\partial^2\mathbf{A}_L}{c^2\partial t^2} = -\frac{4\pi}{c}\mathbf{J} \tag{3.2}$$

Subtracting Eq. 3.2 from Eq. 3.1, we obtain (Eq. 11 of [5]):

$$\nabla^2[\mathbf{A}_C - \mathbf{A}_L] - \frac{1}{c^2}\frac{\partial^2[\mathbf{A}_C - \mathbf{A}_L]}{\partial t^2} = \frac{1}{c}\nabla\frac{\partial\varphi_C}{\partial t} \tag{3.3}$$

The corresponding equations for the scalar potential in the Coulomb (for φ_C) and Lorentz (for φ_L) gauges are (Eqs. 4 and 7 of [5]):

$$\nabla^2\varphi_C = -4\pi\rho \tag{3.4}$$

$$\nabla^2\varphi_L - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi_L = -4\pi\rho \tag{3.5}$$

Using the above two equations, we take their difference and find that the term on the rhs of the Eqs. 3.4 and 3.5, corresponding to the charge density, is eliminated. However, another term appears, which corresponds to the second time derivative of φ_L:

$$\nabla^2[\varphi_C - \varphi_L] - \frac{1}{c^2}\frac{\partial^2[\varphi_C - \varphi_L]}{\partial t^2} = -\frac{1}{c^2}\frac{\partial^2\varphi_C}{\partial t^2} \tag{3.6}$$

Now we transform Eqs. 3.3 and 3.6 in such a way that their *rhs*' will have the identical form. To do it, we apply the gradient operator to Eq. 3.6 and operator $[\partial../c\partial t]$ to Eq. 3.3. As a result, we find, after commuting the gradient and the operator $[\partial../c\partial t]$ with the wave operator, that

$$\nabla^2[\nabla(\varphi_L - \varphi_C)] - \frac{1}{c^2}\frac{\partial^2[\nabla(\varphi_L - \varphi_C)]}{\partial t^2} = \frac{1}{c^2}\nabla\frac{\partial^2\varphi_C}{\partial t^2} \tag{3.7}$$

$$\nabla^2[\partial(\mathbf{A}_C - \mathbf{A}_L)/c\partial t] - \frac{1}{c^2}\frac{\partial^2[\partial(\mathbf{A}_C - \mathbf{A}_L)/c\partial t]}{\partial t^2} = \frac{1}{c^2}\nabla\frac{\partial^2\varphi_C}{\partial t^2} \tag{3.8}$$

which are similar to Eqs. 24 and 25 of [5]. From Equations (3.7) and (3.8), both $\nabla(\varphi_L - \varphi_C)$ and $\partial(\mathbf{A}_C - \mathbf{A}_L)/c\partial t$ satisfy the same differential equation. Therefore,

$$\nabla(\varphi_L - \varphi_C) = \frac{\partial(\mathbf{A}_C - \mathbf{A}_L)}{c\partial t} \tag{3.9}$$

Transforming Eq. 3.9 and using the definition for the electric field, we have

$$\mathbf{E}_C = -\nabla\varphi_C - \frac{\partial \mathbf{A}_C}{c\partial t} = -\nabla\varphi_L - \frac{\partial \mathbf{A}_L}{c\partial t} = \mathbf{E}_L \tag{3.10}$$

i.e. the equivalence of the expressions for the **E** field in both gauges is proven.

We note that Eq. 3.10 is a constructive method for calculating the **E** field in the Coulomb gauge [5]: scalar Coulomb potential, entering in Eq. 3.10, is calculated as a solution of the Poisson equation and that part of the **E** field created by the vector potential is determined by using the following form of Eq. 3.10

$$\frac{\partial \mathbf{A}_C}{c\partial t} = \nabla\varphi_L - \nabla\varphi_C + \frac{\partial \mathbf{A}_L}{c\partial t} \tag{3.11}$$

Thus, the total electric field is presented as a superposition of rotational and irrotational components which is important for analysis of the fields near the radiator [5]. Equivalence of the magnetic field in both gauges follow from the equation

$$\nabla \times \mathbf{A}_C = \nabla \times \mathbf{A}_L$$

since $\mathbf{A}_L$ differs from $\mathbf{A}_C$ by only the gradient of some scalar function.

4. Analysis of the derivation presented in the Sec. 4.

It is necessary to say that despite the obviousness of the proof presented above, it contains a few mistakes. First, to prove equivalence in a mathematically strict way, one must analyze the expressions for the electric fields and not the differential equations for these fields. If one focuses on analysis of the latter, one must take into account the initial and boundary conditions, because solutions of identical equations, but for different boundary and initial conditions, are different. This point is missed in the existing proof.

The second missing point is in the procedure of proof itself, i.e. when *rhs* of equations 3.1, 3.2, 3.4 and 3.5 is eliminated, one does not consider that the functions describing the current and charge densities in the Coulomb and Lorentz gauges are *different*. This fact can be established by using the idea of Lorentz to find that the sizes of uniformly moving charge, which contract along their direction of motion (this procedure developed by Lorentz is described with more clarity in [8]). Lorentz found that the equipotential surfaces of the scalar Liennard-Wiechert potential, expressed in terms of the coordinates of the frame of reference where it is assumed that the observer is at rest and the charge is moving, i.e. $\varphi(r,t) = const$, are ellipsoids of rotation contract along the axis of motion of the charge. Since, as Lorentz concluded, the surface of the charge is defined to be an equipotential surface, this surface must have an ellipsoidal shape in this frame too.

Following Lorentz's procedure, we will show that the functions ρ and $\mathbf{J}$, describing the charge and current densities of the elementary charge, are different in these gauges. But first, we must define what an elementary charge in the classical electrodynamics is.

It is a widespread opinion in classical electrodynamics, that we are able to assume point charges only. At least, any calculations of electrodynamical quantities cannot be based on a specific distribution of the charge inside the electron. However, from a physical point of view, it is impossible to treat the classical electron as a point particle because it leads to divergences in the theory (runaway solutions, etc., see, for example, [9]). Therefore, according to the recommendations given in [7] (beginning of Ch. 18.1), we assume that the radius of classical charge is *finite* and we associate a physical meaning to those properties of the electrodynamical system which do not depend on the radius of the charge.

Thus, we have Statement I:

A surface of the elementary charge is the surface for which the condition

$$\varphi(x_0, y_0, z_0) = const \tag{4.1}$$

is fulfilled
(where, x_0, y_0, z_0 are the coordinates of the surface of the charge). So, this surface is an equipotential surface. We note that for a moving charge, the lines of **E** field are not normal to the equipotential surface (Eq. 4.1), since we must take into account not only the term

$$\mathbf{E} = -\nabla\varphi$$

but rather the entire expression

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{c\partial t} \tag{4.2}$$

Due to the last term in Eq. 4.2, the **E** field lines are no longer normal to the φ surfaces.

For the Lorentz gauge, it is proven in [7], and for the Coulomb gauge, we prove it in the Appendix II. We emphasize that the results presented Appendix II, i.e. the spherical shape of a moving charge in the Coulomb gauge changes, which contradicts to relativistic theory because according to the latter, *any* charge should contract. But our 'strange result' is caused by using the Coulomb gauge which is essentially non-relativistic so some quantities calculated in this gauge have no relativistic properties.

It is necessary to point out that the 'relativistic suggestion' that a moving charge in the Coulomb gauge must contract also cannot be checked experimentally. It results from the following:
1. the shape of the surface of the charge can be determined by measureing the fields or by calculating the potentials since direct measurement (not via field quantities) is *impossible* (there is no 'charge-charge' interaction);
2. because the lines of **E** field are not normal to the surface of the moving charge, one cannot use direct measurement of the EM fields to reconstruct the shape of the surface.

So the only way to obtain information about the shape of this surface is to do it via calculation of the φ potential, as it has been made by Lorentz, and we will using his method.

Now we have <u>Statement II</u>
for both gauges, the equipotential surfaces, i.e. those ones meeting the

condition

$$\varphi(r,t) = const$$

for different r and given instant t are the concentric surfaces converging to the limiting point, which is the center of the elementary charge.
For the Coulomb gauge this Statement follows from rotational symmetry of the expression for the scalar potential (Eq. 6.45 of [2]) and for the Lorentz gauge, it follows from the Eq. 18.20, but in the latter case, we must calculate the shapes of the surfaces separately when the point of observation is outside the charge and when this point is inside the elementary charge.

It follows from the Statement II that
Consequence I:
While $r \to 0$ the set $\varphi(r,t) = const$ forms a geometric sequence (the sequence of converging surfaces).
Consequence II:
For different gauges, these surfaces are different and for any r

$$\varphi_C(r,t) = C_1 \tag{4.3}$$

$$\varphi_L(r,t) = C_2 \tag{4.4}$$

where C_1 and C_2 are constants; *and for any C_1 and C_2*

$$\varphi_C(r,t) \neq \varphi_L(r,t) \tag{4.5}$$

The Eq. 4.5 can be easily proven. Because $\varphi_C(r,t)$ and $\varphi_L(r,t)$ are solutions of *different equations* (Eqs. 3.5 and 3.5), (?) they must be different too. Strictly speaking, the intersection of the two surfaces the Eqs. 4.3 and 4.4, yields some curve but the coincidence of these surfaces is never possible.
Consequence III:
For all gauges, the limiting point of converging sequences is unique, it is the point of center of the elementary charge

Now we choose the parameter R_0 as a 'radius' of moving elementary charge. In the Coulomb gauge, $R_{C,0} = \sqrt{(x_0)^2 + (y_0)^2 + (z_0)^2}$, and in the Lorentz gauge $R_{L,0} = \sqrt{(x_0)^2/(1 - v^2/c^2) + (y_0)^2 + (z_0)^2}$, where x_0... are defined above. Actually, φ_L depends on the coordinates x_0... not via $R_{L,0}$ but rahter via some other combination of the these variables, but since this specific dependence is not important for our procedure, we schematically write this dependence via $R_{L,0}$. We do not know the exact values for $R_{C,0}$ and $R_{L,0}$, we only know that both $R_0 \to 0$ but both $R_0 \neq 0$, which corresponds to the definition for the radius of the elementary charge given in [7].

It follows from Statement I that the shape of the charge in the Coulomb gauge is described by

$$\varphi_C(R_{C,0}, t) = C_3 \tag{4.6}$$

and the shape of the charge in the Lorentz gauge is described by

$$\varphi_L(R_{L,0}, t) = C_4 \tag{4.7}$$

where C_3 and C_4 are some constants; i.e. Eq. 4.6 belongs to the sequence in Eq. 4.3 and Eq. 4.7 to the sequence in Eq. 4.4, respectively. However, due to the inequality in Eq. 4.5, the equation

$$\varphi_C(R_{C,0}, t) = \varphi_L(R_{L,0}, t)$$

cannot be fulfilled *for any* $R_{C,0}$ *and* $R_{L,0}$. Physically it means that the shapes of the charge in different gauges are different too and, therefore, $\rho_L(r)$ and $\rho_C(r)$ are not identical and the mathematical operation of subtracting one function from the other yields a non-zero result. Because the above proof does not depend on specific values of R_0, it is correct in the limit of a point charge too.

A further consideration is trivial. Taking into account that the functions $\mathbf{J}$ and ρ are different in different gauges, we obtain

$$\nabla^2 \mathbf{A_C} - \frac{\partial^2 \mathbf{A}_C}{c^2 \partial t^2} = -\frac{4\pi}{c}\mathbf{J}_C + \frac{1}{c}\nabla \frac{\partial \varphi_C}{\partial t} \tag{4.8}$$

$$\nabla^2 \mathbf{A}_L - \frac{\partial^2 \mathbf{A}_L}{c^2 \partial t^2} = -\frac{4\pi}{c}\mathbf{J}_L \tag{4.9}$$

The analogue of Eq 3.3 is

$$\nabla^2[\mathbf{A}_C - \mathbf{A}_L] - \frac{1}{c^2}\frac{\partial^2[\mathbf{A}_C - \mathbf{A}_L]}{\partial t^2} = \frac{1}{c}\nabla \frac{\partial \varphi_C}{\partial t} + \frac{4\pi}{c}[\mathbf{J}_L - \mathbf{J}_C] \tag{4.10}$$

Applying the wave equation for the scalar potential in a similar way

$$\nabla^2 \varphi_C = -4\pi \rho_C \tag{4.11}$$

$$\nabla^2\varphi_L - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi_L = -4\pi\rho_L \tag{4.12}$$

we obtain

$$\nabla^2[\varphi_C - \varphi_L] - \frac{1}{c^2}\frac{\partial^2[\varphi_C - \varphi_L]}{\partial t^2} = -\frac{1}{c^2}\frac{\partial^2\varphi_C}{\partial t^2} - 4\pi[\rho_C - \rho_L] \tag{4.13}$$

Thus, we find that the wave equations for the quantities $\nabla(\varphi_L - \varphi_C)$ and $\partial(\mathbf{A}_C - \mathbf{A}_L)/c\partial t$ coincide, provided the condition

$$\frac{\partial(\mathbf{J}_C - \mathbf{J}_L)}{c^2\partial t} - \nabla(\rho_C - \rho_L) = 0 \tag{4.14}$$

is satisfied. But in general, this is not the case and for uniformly moving charge, the *lhs* of the Eq 4.14 reduces to

$$(1 - v^2/c^2)\nabla(\rho_C - \rho_L) \tag{4.15}$$

It seems, however, that there is one more way to prove the equivalence of the expressions for the electric field, because the non-zero term on the *rhs* of the wave equation is not equal to zero only in the area occupied by the charge itself. So we can expect that, after integration of the wave equation, the non-compensated term in Eq. 4.15 will tend to zero, while receding the point of observation from the charge.

It is expressed in explicit form as (for simplicity we consider the case of a uniformly moving charge, where all details of the calculations are given in Appendix II):

$$\mathbf{E}_C(R) - \mathbf{E}_C(R) = 4\pi(1 - v^2/c^2)\int\frac{\nabla_r[\rho_C(r) - \rho_L(r)]}{|\mathbf{R} - \mathbf{r}|}d\mathbf{r} \tag{4.16}$$

where the integral is calculated for the retarded time. Since the integral of the charge density over the whole space is equal to the total charge in both gauges, it is easy to show that the above expression rapidly tends to zero when $R >> a$, where a is the radius of the elementary classical charge (Appendix II), i.e. the expressions for the electric field are asymptotically equivalent in both gauges, and if one takes into account the radius of the classical charge, the limiting area of integration in the Eq. 4.16 should be set to zero, the gauges are equivalent in classical electrodynamics.

But one must take into account that the term $-[\partial^2\varphi_C/c^2\partial t^2]$ is artificially added to the *rhs* and *lhs* of the wave equation (Eq. 3.4). Since this term is added to the both the right and the left sides of the equations, it is mathematically correct. But if this term is used on the *lhs* of the equation to construct the Green's function and the same term in the rhs of the equation is used as a source for the Green's function, which means that the *lhs* and *rhs* of the equation are being treated differently, it is absolutely incorrect. As a consequence of this incorrect procedure, it leads to a difference in the expressions for the calculated fields.

5. Conclusion.

It would be interesting to do an analysis as to why such a microscopic effect as the changing the shape of electric charge (for a uniformly moving electron, its surface becomes elliptical in the Lorentz gauge and remains spherical in the Coulomb gauge) causes a macroscopic effect (difference in the fields). It is especially strange since formally we are able to decrease the radius of the charge to zero. So from our point of view, the macroscopic effect is not caused by the changing shape of the elementary classical charge, but rather by properties of the aether: finiteness (of infiniteness) of the speed of the scalar EM interaction determines the magnitude of both the EM fields and the shape of the charge creating these fields.

Thus, just the act of defining the speed of propagation of the scalar EM interaction in a medium (aether or vacuum) defines the correct gauge for this system, as well as the shape of the elementary classical charge. Because we have an example of the reverse influence of the medium on the charge (in the case of uniform motion of the charge we have some equilibrium process for converging and diverging EM waves), this influence unambiguously determines the equilibrium shape of the moving charge. The mechanism by which the medium influences the charge is still unexplained, but within the framework of this effort, it is impossible to find an explanation. It should be noted that in relativistic theory, the term 'medium' is not used, so we use the term 'aether' but we do not make any claims about its reality.

It would be noted that one of the aims of this work is to turn the scientific community's attention to the fact that until now, some problems of electrodynamics, which seemed to be absolutely irrefutable, cannot be conclusively solved. So it would be interesting to re-examine some of the ideas of Whittaker([11], also see [12]), especially regarding the formation of the Coulomb (or scalar) potential from convergent and divergent EM waves. Finally, the difference in the properties of the scalar potential calculated in the Coulomb and Lorentz gauges gives differences for all other field quantities.

Thus, the final conclusion of this work is that *in classical electrodynamics*, the uniqueness of the description of some systems requires setting not only the initial and boundary conditions but also *the speed of propagation* of the scalar potential as well, where the latter unambiguously determines the gauge which we must use while obtaining solutions for the EM fields.

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Appendix I. Obtaining of the shape of the elementary charge in the Coulomb gauge.

Here we analyze the following statement: if the moving charge acquires a shape of contracted ellipsoid in the Lorentz gauge, will we observe the same effect for the moving charge in the Coulomb gauge. It seems it must be so because any physical quantity must transform according to the Lorentz transformations while going from one inertial frame to the other one. But without possibility to verify experimentally how the shape of the charge actually changes, in the given gauge, the above statement can be treated only as assumption.

However, a problem is just in this experimental verification since we are not able to reconstruct the shape of the elementary charge directly from the experimental data, i.e. from measured EM fields.

- For the moving charge, the lines of the E field are not normal to the surface, which these lines outcome from (the example of such a configuration of the lines of the E field and the moving charge is given in the Fig. 26.4 of [6]), so we cannot use geometric methods

- Formally the shape of the uniformly moving charge may be determined as a solution of the integral equation for the electric and magnetic fields, where the function r is a source for the Green function, but because we cannot fix the radius of the elementary classical charge, the problem cannot have unambiguous solution.

It follows from the pp. 1 and 2, that the only way to determine the shape of the elementary charge is after equipotential surfaces of φ, i.e. the way used by Lorentz. One can object that because the EM potentials are treated, within the classical electrodynamics, as some abstract but not physical quantities, unambiguous determination of the shape of the charge via the scalar potential is impossible. However, in the above problem, just the properties of the EM potentials are under investigation, therefore, for us it is not so important what is an origin of the potentials. But what is important is the fact that φ and $\mathbf{A}$ are unambiguously defined after the EM fields and the condition on a type of the gauge. Therefore, the shape of the charge will be determined unambiguously too because there is no ambiguity in the gauge condition.

Now we show by *reductio ad absurdum* that the uniformly moving elementary charge cannot have, in the Coulomb gauge, a shape of contracted ellipsoid.
Thus, in the frame with the charge at rest, the Poisson equation for the elementary charge is (in Gauss units)

$$\left(\frac{\partial^2}{\partial(x')^2}+\frac{\partial^2}{\partial(y')^2}+\frac{\partial^2}{\partial(z')^2}\right)\varphi' = 4\pi\rho(x',y',z') \qquad \text{(AI.1)}$$

and its solution is

$$\varphi'(\mathbf{R}') = \int \frac{\rho(x',y',z')}{|\mathbf{R}'-\mathbf{r}'|}dx'dy'dz' \qquad \text{(AI.2)}$$

We don't know what is the shape of the uniformly moving elementary charge but we exactly know the shape of the charge while it is at rest. Due to rotational symmetry, its shape *must be spherical.*
For the 'point-like' charge, i.e. for $R' >> r'$ the Eq. 2 reduces to

$$\varphi'(\mathbf{R}') = \frac{q}{R'} \qquad \text{(AI.3)}$$

Now we go to the second frame where the charge moves uniformly. According to the Lorentz transformations, the function ρ must transform and the shape of the elementary charge becomes elliptic.

But when we apply the Lorentz transformation to the physical quantities, even they are treated as auxiliary ones, in some frame, we must transform *all* quantities of this frame. So in the Eqs. AI.1, AI.2 and AI.3, we must transform not only the potentials and the charge densities but the coordinates too.
As a result, we have

$$\varphi'(R) = \frac{q}{[(x-vt)^2 + (1 - v^2/c^2)(y^2+z^2)]}$$

But this solution does not coincide to well known solution for the scalar given in [2] (Eq. 6.45).

$$\varphi(\mathbf{R}) = \int \frac{\rho(x,y,z,t)}{|\mathbf{R}-\mathbf{r}|} dxdydz$$

that is correct for any law of motion of the charge. So our suggestion about the Lorentz contraction of moving elementary charge leads to incorrect expression for the φ potentials and, therefore, any analysis of the equivalence of the Coulomb and Lorentz gauges, which is presented in Refs. 2, 3 and 4 too, loses its significance.
Thus, we are not able to conclude that the shape of the moving charge in the Coulomb gauge is elliptic.

At the end, we prove that in the Coulomb gauge, the uniformly moving charge has spherical shape.

It is easily to see that the equation for the scalar potential in the Coulomb gauge obeys the 'Lorentz transformations' in the limiting case when the speed of the scalar EM interaction tends to infinity ($c \to \infty$) so the Lorentz transformations reduce to

$$x' = (x - vt) / \sqrt{1 - v^2/c^2} \to x - vt$$

Then the formula for transformation of the charge density becomes

$$\rho' = \rho / \sqrt{1 - v^2/c^2} \to \rho$$

and the spherical shape of the uniformly moving charge remains to be spherical too

Appendix II. Derivation of the Eq. 3.15

Taking the gradient of both sides of the Eq. 3.12 and partial time derivative (divided on c) of both sides of the Eq. 3.9., we obtain

$$\nabla^2[\nabla(\varphi_C-\varphi_L)]-\frac{1}{c^2}\frac{\partial^2[\nabla(\varphi_C-\varphi_L)]}{\partial t^2}=-\frac{1}{c^2}\nabla\frac{\partial^2\varphi_C}{\partial t^2}+\nabla 4\pi[\rho_C-\rho_L] \tag{AII.1}$$

$$\begin{aligned}\nabla^2[\partial(\mathbf{A_C}-\mathbf{A_L})/\partial t]-\frac{1}{\mathbf{A_L})/\partial t]}\partial t^2 \\ =\frac{1}{c^2}\nabla\frac{\partial^2\varphi_C}{\partial t^2}-\frac{\partial}{c\partial t}\frac{4\pi}{c}[\mathbf{J_C}-\mathbf{J_L}]\end{aligned} \tag{AII.2}$$

Now we are able to form, using the sum of the Eqs. AII.1 and AII.2, the wave equation for difference between the electric field $\mathbf{E_C}$ and $\mathbf{E_L}$:

$$\nabla^2[\mathbf{E_C}-\mathbf{E_L}]-\frac{1}{c^2}\frac{\partial^2[\mathbf{E_C}-\mathbf{E_L}]}{\partial t^2}=\nabla 4\pi[\rho_C-\rho_L]-\frac{\partial}{c\partial t}\frac{4\pi}{c}[\mathbf{J_C}-\mathbf{J_L}] \tag{AII.3}$$

Using the expression 3.14, we have the solution of the wave equation AII.3.

$$\mathbf{E_C}(R)-\mathbf{E_L}(R)=4\pi(1-v^2/c^2)\int\frac{\nabla_r[\rho_C(r)-\rho_L(r)]}{|\mathbf{R}-\mathbf{r}|}d\mathbf{r} \tag{AII.4}$$

which coincides to the Eq. 3.15.

Now we prove that the rhs of Eq. AII.4 asymptotically tends to zero. Because for two arbitrary functions it is fulfilled relation

$$\int F(\mathbf{R}-\mathbf{r})\nabla_r f(\mathbf{r})d\mathbf{r}=\nabla_R\int F(\mathbf{R}-\mathbf{r})f(\mathbf{r})d\mathbf{r}$$

we have for Eq. AII.4

$$\mathbf{E_C}(R)-\mathbf{E_L}(R)=4\pi(1-v^2/c^2)\nabla_R\int\frac{[\rho_C(r)-\rho_L(r)]}{|\mathbf{R}-\mathbf{r}|}d\mathbf{r} \tag{AII.5}$$

Using the Eqs. 4.8 and 4.10 of [2], we obtain for Eq. AII.5

$$\begin{aligned}\mathbf{E}_C(\mathbf{R},t)-\mathbf{E}_L(\mathbf{R},t)= \\ 4\pi\left(1-\frac{v_2}{c_2}\right)\nabla_R\left[\frac{q_C}{R}+\frac{(\mathbf{p}_C\mathbf{R})}{R^3}-\frac{q_L}{R}-\frac{(\mathbf{p}_L\mathbf{R})}{R^3}+O(1/R^3)\right]\end{aligned} \tag{AII.6}$$

where q_C and q_L, $\mathbf{p}_C$ and $\mathbf{p}_L$ are the charges and electric dipole moments in both gauges. Since the charges are identical in both gauges and absolute value of dipole moment of the charge cannot be greater aq , we have for Eq. AII.6

$$|\mathbf{E}_C(\mathbf{R},t) - \mathbf{E}_L(\mathbf{R},t)| < 4\pi \left(1 - \frac{v_2}{c_2}\right) \frac{aq}{R^3}$$

i.e. this term rapidly tends to zero from distances some times greater the classical radius of the charge.

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