A set of new localized Superluminal solutions to the Maxwell equations (†)

Michel Zamboni Rached[a] and Erasmo Recami[b]

[a]DMO–FEEC, State University at Campinas, Campinas, S.P., Brazil
[b]Facoltà di Ingegneria, Università statale di Bergamo, Dalmine (BG), Italy;
INFN—Sezione di Milano, Milan, Italy;
and
C.C.S., State University at Campinas, Campinas, S.P., Brazil

ABSTRACT. By a generalized bidirectional decomposition method, we obtain new Superluminal localized solutions to the wave equation (for the electromagnetic case, in particular) which are suitable for arbitrary frequency bands; various of them being endowed with *finite* total energy. We construct, among the others, an infinite family of generalizations of the so-called "X-shaped" waves. Results of this kind may find application in the other fields in which an essential role is played by a wave-equation (like acoustics, seismology, geophysics, etc.).

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1. – Introduction

Already in 1915 Bateman[1] showed that Maxwell equations admit (besides of the ordinary solutions, endowed in vacuum with speed c) of wavelet-type solutions, endowed in vacuum with group-velocities $0 \leq v \leq c$. But Bateman's work went practically unnoticed. Only few authors, as Barut et al.[1], followed such a research line; incidentally, Barut et al. constructed even a wavelet-type solution travelling with Superluminal group-velocity $v > c$.

In recent times, however, many authors discussed the fact that all (homogeneous) wave equations admit solutions with $0 < v < \infty$: see, e.g., Donnelly & Ziolkowski[1] or Esposito[1]. Most of those authors confined themselves to investigate (sub- or Super-luminal) *localized* non-dispersive solutions in vacuum: namely, those solutions that were called "undistorted progressive waves" by Courant & Hilbert. Among localized solutions, the most interesting appeared to be the so-called "X-shaped" waves, which —predicted to exist even by Special Relativity in its extended version*— had been mathematically constructed by Lu & Greenleaf[2] for acoustic waves, and by Ziolkowski et al.[2], and later Recami[2], for electromagnetism.

Let us recall that such "X-shaped" localized solutions are Superluminal (i.e., travel with $v > c$ in the vacuum) in the electromagnetic case; and are "super-sonic" (i.e., travel with a speed larger than the sound-speed in the medium) in the acoustic case. The first authors to produce X-shaped waves *experimentally* are known to be Lu & Greenleaf for acoustics, Saari et al. for optics, and Mugnai et al. for microwaves.

In other words, it has been known since many years that localized (non-dispersive) solutions exist to the (homogeneous) wave equation[1], endowed with subluminal or Superluminal[2] velocities.

Particular attention has been paid to the localized Superluminal solutions, which seem to propagate not only in vacuum but also in media with boundaries[3], like normal-sized metallic waveguides[4] and possibly optical fibers.

Let us repeat that such Superluminal Localized Solutions (SLS) have been *experimentally* produced in acoustics[5], in optics[6] and recently in microwave physics[7].

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*See refs.[10] below.

However, all the analytical SLSs considered till now and known to us, with one exception[8], are superposition of Bessel beams with a frequency spectrum starting with $\nu = 0$ and suitable for low frequency regions. In this paper we shall set forth a new class of SLSs with a spectrum starting at any arbitrary frequency, and therefore well suited for the construction also of high frequency (microwave, optical,...) pulses.

2. – "V-cone" variables: A generalized bidirectional expansion

Let us start from the axially symmetric solution (Bessel beam) to the wave equation in cylindrical co-ordinates:

$$\psi(\rho, z, t) \;=\; J_0(k\rho)\, \mathrm{e}^{+ik_z z}\, \mathrm{e}^{-i\omega t} \tag{1}$$

with the conditions

$$k^2 = \frac{\omega^2}{c^2} - k_z^2 \; ; \qquad k^2 \geq 0 \; , \tag{2}$$

where J_0 is the zeroth-order ordinary Bessel function, and where (as usual) k_z is the longitudinal component of the wavenumber while $k \equiv k_\perp$ is the wavenumber transverse component magnitude. The second condition (2) excludes the non-physical solutions.

It is essential to stress right now that the dispersion relation (2), with positive (but not constant, a priori) k^2, while enforcing the consideration of the truly propagating waves only (with exclusion of the evanescent ones), does allow for both subluminal and Superluminal solutions!; the latter being the ones of interest here for us. Conditions (2) correspond in the (ω, k_z) plane to confining ourselves to the sector shown in Fig.1; that is, to the region delimited by the straight lines $\omega = \pm c k_z$.

A general, axially symmetric superposition of Bessel beams (with Φ as spectral weight-function) will therefore be:

$$\Psi(\rho, z, t) \;=\; \int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}\omega \int_{-\omega/c}^{+\omega/c} \mathrm{d}k_z \; \psi(\rho, z, t)\; \delta\Big(k - \sqrt{\frac{\omega^2}{c^2} - k_z^2}\Big)\; \Phi(k, k_z, \omega) \; . \tag{3}$$

Notice that it is $k \geq 0$; $\omega \geq 0$ and $-\omega/c \leq k_z \leq +\omega/c$. The question of the negative k_z values entering expansion (3) will soon be considered below.

The base functions $\psi(\rho, z, t)$ can be however rewritten as

$$\psi(\rho, \zeta, \eta) \;=\; J_0(k\rho)\; \exp i[\alpha\zeta - \beta\eta] \;,$$

where (α, β), which will substitute in the following for the parameters (ω, k_z), are

$$\alpha \;\equiv\; \frac{1}{2V}(\omega + Vk_z)\,; \qquad \beta \;\equiv\; \frac{1}{2V}(\omega - Vk_z)\;, \tag{4}$$

in terms of the new "V-cone" variables:

$$\begin{cases} \zeta \;\equiv\; z - Vt \\ \eta \;\equiv\; z + Vt \end{cases} \tag{5}$$

The present procedure is a generalization of the so-called "bidirectional decomposition" technique[9], which was previously devised for $V = c$ only.

The "V-cone" of Fig.2a corresponds in the (ω, k_z) plane to the sector limited by the straight-lines $\omega \pm Vk_z = 0$, that is, by the lines $\alpha = 0$ and $\beta = 0$ (Fig.2b); while conditions (2) become [let us put $c = 1$ *whenever convenient,* throughout this paper]:

$$k^2 \;=\; V^2(\alpha+\beta)^2 - (\alpha-\beta)^2 \;\equiv\; (\alpha^2+\beta^2)(V^2-1) + 2(V^2+1)\alpha\beta\;; \quad k^2 \geq 0$$
(2')

Inside the allowed region shown in Fig.1, we can choose for simplicity the sector delimited by the straight-lines $\omega = \pm Vk_z$ shown in Fig.2b, *provided that* $V \geq 1$.

Let us observe that integrating over the ranges $\alpha, \beta \geq 0$ corresponds in eq.(3) to integrating over k_z between $-\omega/V$ and $+\omega/V$. But we shall choose in eq.(3) spectral weights $\Phi(k, k_z, \omega)$, and therefore spectral weights $\Phi(k, \alpha, \beta)$ in eq.(3') below, such as to either eliminate or make negligible the contribution from the negative values of k_z, that is, from the backwards moving waves: thus curing from the start the problem met by the "bidirectional decomposition" technique in connection with the so-called non-causal components. Therefore, our SLSs will all be physical solutions.

Let us recall also that each Bessel beam is associated with an ("axicone") angle θ, linked to its speed by the relations[10]:

$$\tan\theta = \sqrt{V^2-1}; \qquad \sin\theta = \frac{\sqrt{V^2-1}}{V}; \qquad \cos\theta = \frac{c}{V} \ , \tag{6}$$

where $V \to c$ when $\theta \to 0$, while $V \to \infty$ when $\theta \to \pi/2$.

Therefore, instead of eq.(3) we shall consider the (more easily integrable) Bessel beam superposition in the new variables [with $V \geq 1$]

$$\begin{aligned} \Psi(\rho,\zeta,\eta) \ = \ & \int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}\alpha \int_0^\infty \mathrm{d}\beta \ J_0(k\rho)\, \mathrm{e}^{i\alpha\zeta}\, \mathrm{e}^{-i\beta\eta} \times \\ & \times \delta\left(k - \sqrt{(\alpha^2+\beta^2)(V^2-1) + 2(V^2+1)\alpha\beta}\right) \ \Phi(k,\alpha,\beta) \end{aligned} \tag{3'}$$

where the integrations over α, β between 0 and ∞ just correspond to the dashed region of Fig.2b.

Let us now go on to constructing new Superluminal Localized Solutions for arbitrary frequencies, various of them possessing finite total energy.

3. – Some new Superluminal Localized Solutions for arbitrary frequencies and/or with finite total energy

3.1 – *The classical "X-shaped solution" and its generalizations.*

Let us start by choosing the spectrum [with $a > 0$]:

$$\Phi(\alpha,\beta) \ = \ \delta(\beta-\beta')\, \mathrm{e}^{-a\alpha} \ , \tag{7}$$

$a > 0$ and $\beta' \geq 0$ being constants (related to the transverse and longitudinal localization of the pulse).

In the simple case when $\beta' = 0$, one completely dispenses with the "non-causal" (backwards-moving) components of the bidirectional Fourier-type expansion (3'). For the sake of clarity, let us go back to examining Fig.2b: The $\delta(\beta)$ factor in spectrum (7) does actually imply the integrations over α and β in eq.(3') to run along the α-line only; i.e., along the $\beta = 0$ straight-line (where $\omega = +Vk_z$). In this case, even more than in the others, it is easy to verify that the group-velocity† of the present solution [cf. eq.(8) below] is $\partial\omega/\partial k_z = 1/\cos\theta \equiv V > 1$. Let us, then, choose $\beta' = 0$, and observe that for $\beta = 0$ all the solutions $\Psi(\rho, \zeta, \eta)$ are actually functions only of ρ and $\zeta = z - Vt$. [Let us also notice that in empty space such solutions $\Psi(\rho,\, \zeta = z - Vt)$ can be transversely localized only if $V \neq c$, because if $V = c$ the function Ψ has to obey the Laplace equation on the transverse planes. Let us recall that in this paper we always assume $V > 0$].

In the present case, eq.(3') can be easily integrated over β and k by having recourse to identity (6.611.1) of ref.[11], yielding

$$
\begin{aligned}
\Psi_{\mathrm{X}}(\rho,\zeta) \;&=\; \int_0^\infty \mathrm{d}\alpha\, J_0(\rho\alpha\sqrt{V^2-1})\, \mathrm{e}^{-\alpha(a-i\zeta)} \;= \\
&=\; \left[(a-i\zeta)^2 + \rho^2(V^2-1)\right]^{-1/2} \;,
\end{aligned}
\tag{8}
$$

which is exactly the classical X-shaped solution proposed by Lu & Greenleaf[12] in acoustics, and later on by others[12] in electromagnetism, once relations (6) are taken into account.

Many other SLSs can be easily constructed; for instance, by inserting into the weight function (7) the extra factor α^m, namely $\Phi(\alpha,\beta) = \alpha^m\,\delta(\beta)\exp[-a\alpha]$, while it is still $\beta' = 0$. Then an infinite family of new SLSs is obtained (for $m \geq 0$), by using, this time, identity (6.621.4) of the same ref.[11]:

†Let us observe that the group velocity of the solutions considered in this paper can a priori be evaluated through the ordinary, simple derivation of ω with respect to the wavenumber only for the infinite total energy solutions, as in the present case. However, for our SSP and SMPS solutions, below, and in general for the finite total energy Superluminal solutions, the group-velocity cannot be calculated through that simple relation, since in those cases it does not even exist a one-to-one function $\omega = \omega(k_z)$.

$$\Psi_{\mathrm{X},m}(\rho,\zeta) \; = \; -(-i)^m \frac{\mathrm{d}^m}{\mathrm{d}\zeta^m} \left[(a-i\zeta)^2 + \rho^2(V^2-1)\right]^{-1/2} \; , \qquad (9)$$

which generalize[13] the classical X-shaped solution, corresponding to $m=0$: namely, $\Psi_{\mathrm{X}} \equiv \Psi_{\mathrm{X},0}$. Notice that all the derivatives of the latter with respect to ζ lead to new SLSs, all of them being X-shaped.

In the particular case $m=1$, one gets the SLS

$$\Psi_{\mathrm{X},1}(\rho,\zeta) \; = \; \frac{-i\,(a-i\zeta)}{\left[(a-i\zeta)^2 + \rho^2(V^2-1)\right]^{3/2}} \qquad (10)$$

which is the first derivative of the X-shaped wave. One should notice that, by increasing m, the pulse becames more and more localized around its vertex. All such pulses travel, however, without deforming.

Solution (8) is suited for low frequencies only, since its frequency spectrum (exponentially decreasing) starts from zero. One can see this for instance by writing eq.(7) in the (ω, k_z) plane: by eqs.(4) one obtains

$$\Phi(\omega,k_z) = \delta\left(\frac{\omega - Vk_z}{2V} - \beta'\right)\, \exp[-a\,\frac{\omega + Vk_z}{2V}]$$

and can observe that $\beta' = 0$ in the delta implies $\omega = Vk_z$. So that the spectrum becomes $\Phi = \exp[-a\omega/V]$, which starts from zero and has a width given by $\Delta\omega = V/a$.

By contrast[13], when the factor α^m is present, the frequency spectrum of the solutions can be "bumped" in correspondence with any value ω_{M} of the angular frequency, provided that m is large [or a/V is small]: in fact, ω_{M} results to be $\omega_{\mathrm{M}} = mV/a$. The spectrum, then, is shifted towards higher frequencies (and decays only beyond the value ω_{M}).

Moreover, let us mention here that also in the spectra of the following pulses (considered in subsections 3.2 and 3.3 below) one can insert the α^m factor; in fact, in correspondence with the spectrum

$$\Phi(\alpha,\beta) \; = \; \alpha^m \; \Phi_0(\beta)\, \mathrm{e}^{-a\alpha} \; , \qquad (7')$$

one obtains *as further solutions* the m-th order derivatives of the basic $(m=0)$ solution below considered. This is due to the circumstance that our integrations over α (as in eq.(3')) are always Laplace-type transformations. We shall not write them down explicitly, however, for the sake of conciseness.

Different SLSs can be obtained also by *modifying* (still with $\beta' = 0$) the spectrum (7). Some interesting solutions are reported in Appendix A.

Let us now construct SLSs more suited for high frequencies (always confining ourselves to pulses well localized not only longitudinally, but also transversely).

3.2 – *The Superluminal "Focus-Wave Modes"* (SFWM).

Let us go back once more to spectrum (7), but examining now the general case with $\beta' \neq 0$. After integrating over k and β, eq.3') yields $[a > 0;\ \beta' > 0;\ V > c]$:

$$\Psi(\rho,\zeta,\eta) \;=\; \mathrm{e}^{-i\beta'\eta} \int_0^\infty \mathrm{d}\alpha\, J_0\left(\rho\sqrt{V^2(\alpha+\beta')^2 - (\alpha-\beta')^2}\right)\, \mathrm{e}^{-\alpha(a-i\zeta)}\,. \tag{11}$$

When releasing the condition $\beta' = 0$ we are in need also of backwards-moving components for the construction of our pulses, since they enter superposition (3') and therefore eq.(11). In fact, the spectrum $\Phi = \delta(\beta - \beta')\,\exp[-a\alpha]$ does obviously entail that $\beta = \beta'$ and hence, by relations (4), that $\omega = Vk_z + 2V\beta'$. This means (see Fig.3) that we are now integrating along the continuous line, i.e., also over the interval $V\beta' \leq \omega < 2V\beta'$, or $-\beta' \leq k_z < 0$, corresponding to the "non-causal" components. Nevertheless, we can obtain physical solutions when making the contribution of that interval negligible, by choosing small values of $a\beta'$: so that the exponential decay of the weight Φ with respect to ω is very slow. Actually, one can go from the (α,β) space back to the (ω,k_z) space by use of eqs.(4), the weight being re-written (when $\beta' = \beta$) as $\Phi = \exp(-a\omega/V)\cdot\exp(-a\beta')$; wherefrom it is clear that‡ for $a \ll 1$ the contribution of the interval $k_z \geq 0$ (or $\omega \geq 2V\beta'$) overruns the $k_z < 0$ contribution. Notice, incidentally, that the corresponding solutions are

‡One can easily show that the condition $a \ll 1$ should be actually replaced with the condition $a\beta' \ll 1$. In fact (see Fig.3), the non-causal interval is $\Delta\omega_{\rm NC} = V\beta'$, while the total spectral band-width is $\Delta\omega = V/a$, so that the non-physical components bring a negligible contribution to the solution in the case of spectrum (7), provided that $\Delta\omega_{\rm NC}/\Delta\omega \ll 1$, which just means $a\beta' \ll 1$.

associated with large frequency bandwidths and therefore to pulses with very short extension in space and in time. Let us mention even now that the spectral weight $\Phi = \exp[-a(\omega - V\beta')/V]$ entails the frequency band-width

$$\Delta\omega \;=\; \frac{V}{a} \;,$$

a relation that we shall find to be valid (at least approximately) for all our solutions. We shall discuss this point in Sect.5 below.

An analytical expression for integral (11) can be easily found for small positive β' values, when $\beta'^2 \approx 0$. Under such a condition we obtain, by using identity (6.616.1) of ref.[11] and calling now X the classical[12] X-shaped solution (8)

$$X \;=\; X(\rho,\eta) \;\equiv\; \left[(a-i\zeta)^2+\rho^2(V^2-1)\right]^{-1/2} \;, \tag{12}$$

we obtain the new[§] SLSs $[a>0;\ \beta'\geq 0;\ V\geq c]$:

$$\Psi_{\rm SFWM}(\rho,\zeta,\eta) \;=\; {\rm e}^{-i\beta'\eta}\; X\, \exp\left[\frac{\beta'(V^2+1)}{V^2-1}\;\left((a-i\zeta)-X^{-1}\right)\right] \tag{13}$$

which for $V\to c^+$ reduce to the well known FWM (focus-wave mode) solutions[15], traveling with speed c:

$$\Psi_{\rm FWM}(\rho,\zeta,\eta) \;=\; \frac{{\rm e}^{-i\beta'\eta}}{a-i\zeta}\;\exp\left[-\frac{\beta'\rho^2}{a-i\zeta}\right] \;. \tag{14}$$

Our solutions (13) are a generalization of them for $V>c$; we shall call eqs.(13) the *Superluminal focus wave modes* (SFWM). See Fig.4. Such modes travel without deforming.

Let us emphasize that, when setting $\beta'>0$, the spectrum (7) results to be constituted (cf. Fig.3) by angular frequencies $\omega\geq V\beta'$. Thus, our new solutions can be used to construct high frequency pulses (e.g., in the microwave or in the optical regions): cf. also subsect.5B below.

[§]Notice that another, slightly different solution —called the FXW— appeared however in ref.[14]

We are going now to build up suitable superpositions of $\Psi_{\rm SFWM}(\rho,\zeta,\eta)$ in order to get *finite* total energy pulses, in analogy with what is currently attempted[16] for the c-speed FWMs.

3.3 – *The Superluminal "Splash Pulses"* (SSP).

In the case of the c-speed FWMs, in ref.[16] suitable superpositions of them were proposed (the SPs and the "MPS pulses") which possess finite *total* energy (even without truncating them).

Let us analogously go on from our solutions (13) to finite total energy solutions, by integrating our SFWMs (13) over β' :

$$\Psi(\rho,\zeta,\eta) \equiv \int_0^\infty {\rm d}\beta' \; B(\beta') \, {\rm e}^{-i\beta'\eta} \; X \; \exp\left[\frac{\beta'(V^2+1)}{V^2-1} \; \left((a-i\zeta) - X^{-1}\right)\right] . \qquad (15)$$

where it must be still $a \ll 1$, while the weight-functions $B(\beta')$ must be bumped in correspondence with *small* positive values of β' since eq.(13) was obtained under the condition $\beta'^2 \approx 0$. In the following, for simplicity, we shall call β, instead of β', the integration variable.

First of all, let us choose in eq.(15) the simple weight-function $[\beta' \equiv \beta]$:

$$B(\beta) \; = \; {\rm e}^{-b\beta} \qquad (16)$$

with $b \gg 0$ for the above-named reasons. Let us recall that such weight (16) is the one yielding in the $V \to c^+$ case the ordinary (c-speed) Splash Pulses[16]; and notice that this choice is equivalent to inserting into eq.(3') the spectral weight

$$\Phi(k,\alpha,\beta) \; \equiv \; {\rm e}^{-a\alpha} \; {\rm e}^{-b\beta} \; . \qquad (7'')$$

Our *Superluminal Splash Pulses* (SSP) will therefore be:

$$\Psi_{\rm SSP}(\rho,\zeta,\eta) \; = \; X \int_0^\infty {\rm d}\beta \; {\rm e}^{-\beta(b+i\eta)} \; {\rm e}^{\beta Y} \; = \; \frac{X}{b+i\eta-Y} \; , \qquad (17)$$

with

$$Y \equiv \frac{V^2+1}{V^2-1} \left((a - i\zeta) - X^{-1} \right) .$$

Let us repeat that our SSPs have *finite* total energy, as one can easily verify; we shall come back to this result also from a geometric point of view. They however get deformed while traveling, and their amplitude decreases with time: see Figs.5a and 5b. It is worth mentioning that, due to the form (7") of the SSP spectrum, our solution (17) can be regarded as *the finite energy version of the classical X-shaped solution.*

3.4 – *The Superluminal "Modified Power Spectrum"* (SMPS) *pulses.*

In connection with eq.(15), let us now go on to a more general choice for the weight-function:

$$\begin{cases} B(\beta) \;=\; \mathrm{e}^{-b(\beta-\beta_0)} & \text{for } \beta \geq \beta_0 \\ B(\beta) \;=\; 0 & \text{for } 0 \leq \beta < \beta_0 \end{cases} \tag{16'}$$

which for $V \rightarrow c^+$ yields the ordinary (c-speed) "Modified Power Spectrum" (MPS) pulses[16]. Such a choice is now equivalent to inserting into eq.(3') for $\beta \geq \beta_0$ the spectrum

$$\Phi \;=\; \mathrm{e}^{-a\alpha}\, \mathrm{e}^{-b(\beta-\beta_0)} \qquad \text{for } \beta \geq \beta_0 \; . \tag{7' ' '}$$

We then obtain the *Superluminal Modified Power Spectrum (*SMPS*) pulses* as follows [for $\beta_0 \ll 1$]:

$$\Psi_{\rm SMPS}(\rho,\zeta,\eta) \;=\; \mathrm{e}^{b\beta_0}\, X \int_{\beta_0}^{\infty} \mathrm{d}\beta\, \mathrm{e}^{-(b+i\eta-Y)\beta} \;=\; X\, \frac{\exp[(Y-i\eta)\beta_0]}{b-(Y-i\eta)}$$

(18)

in which the integration over β runs *now* from β_0 (no longer from zero) to infinity.

It is worthwhile to emphasize that our solutions (18), like solutions (17), possess a *finite total energy.*¶ Even if this is easily verified, let us address the question from an illuminating geometric point of view. Let us here only add that their amplitude too (as for the SSPs) decreases with time.

With reference to Fig.6, let us observe that the *infinite* total energy solutions X, in eq.(12), and SFWM, in eq.(13), correspond to integrations along the $\beta = 0$ axis (i.e., the α-axis) and the $\beta = \beta_0$ straight-line, respectively; that is to say, correspond to a delta factor, $\delta(\beta - \beta_0)$, in the spectrum (7), where $\beta' \equiv \beta_0$.

In order to go on to the *finite* total energy solutions (SMPS), eq.(18), we replaced the delta factor with the function (16'), which is zero in the region above the $\beta = \beta_0$ line, while it decays[17] in the region below (as well as along) such a line. The same procedure was followed by us for the solutions SSP, eq.(17), which correspond to the particular case $\beta_0 = 0$. The faster the spectrum decay takes place in the region below the $\beta = \beta_0$ line [i.e. $b \gg 1$], the larger the field depth‖ of the corresponding pulse results to be: as we shall see in Sect.4.2C. Let us add that, since $b \gg 1$, even in the present case the non-causal components contribution becomes negligible provided that one chooses $a\beta_0 \ll 1$; in analogy with what we obtained in the previous SFWM case.

It seems important to stress also that, while the X and SSP solutions, eqs.(12) and (18), mainly consist in low-frequency (Bessel) beams, on the contrary our solutions SFWM and SMPS, eqs.(13) and (18), can be constituted by higher frequency beams (corresponding, namely, to $\omega \geq V\beta_0$). This property can be exploited for constructing SLSs in the microwave or optics fields, by suitable choices of the V and β_0 values.

4. – Geometric description of the new pulses in the (ω, k_z) plane

4.1 – *A preliminary analysis of the localized pulses.*

¶One should recall that the first finite energy solution, the MFXW, different from but analogous to our one, appeared in ref.[14].

‖The "depth of field" is the distance along which the pulse (approximately) keeps its shape, besides its group-velocity; cf. refs.[16,2].

Let us add some intuitive considerations about the *localized* solutions Ψ to the wave equation, which by our definition[18] must possess the property

$$\Psi(x, y, z; t) \;=\; \Psi(x, y,\, z + \Delta z_0;\, t + \frac{\Delta z_0}{v}) \tag{19}$$

v being the pulse propagation speed, that here can assume a priori any[1,2] value: $0 \leq v < \infty$. Such a definition entails that the pulse "oscillates" while propagating, it being required that it resumes (periodically) its shape only after each space interval Δz_0, that is, with the time interval $\Delta t_0 = \Delta z_0 / v$ (cf. refs.[18,19]).

Let us write the Fourier-expansion of Ψ

$$\Psi(x, y, z; t) \;=\; \int_{-\infty}^{\infty} \mathrm{d}\omega \int_{-\infty}^{\infty} \mathrm{d}k_z \; \overline{\Psi}(x, y, k_z; \omega)\, \mathrm{e}^{ik_z z}\, \mathrm{e}^{-i\omega t} \;, \tag{19a}$$

functions $\overline{\Psi}(x, y, k_z; \omega)$ and $\overline{\Psi}(x, y, k_z; \omega)\; \exp[i(k_z\, \Delta z_0 - \omega\, \Delta z_0/v)]$ being the Fourier transforms (with respect to the variables z, t) of the l.h.s. and r.h.s. functions in eq.(19), respectively; where we used the translation property

$$\mathcal{T}[f(x+a)] \;=\; \mathrm{e}^{ika}\; \mathcal{T}[f(x)]$$

of the Fourier transformations. From condition (19), we then get[18] the fundamental constraint

$$\omega \;=\; vk_z \pm 2n\pi\, \frac{v}{\Delta z_0} \tag{20}$$

linking ω with k_z. Let us explicitly mention that constraint (20) does *not* imply any breakdown of the wave-equation validity. In fact, when inserting expression (19a) into the wave equation, one gets —in cylindrical plane coordinates (ρ, ϕ)— the physical base-solution

$$\Psi(\rho, \phi, k_z; \omega) \;=\; J_\mu(k\rho)\, \cos(\mu\phi) \tag{19b}$$

with μ an integer and

$$k^2 \ = \ \omega^2 - k_z^2 \geq 0 \ . \tag{19c}$$

Therefore, our constraint (20) is consistent with relations (19b), (19c), which followed from the wave equation.

Relation (20) is important, since it clarifies the "spectral origin" of the various localized solutions introduced in the past literature (e.g., for $v = c$), which originated from superpositions performed either by running "along" the straight-lines (20) themselves, or in terms of spectral weights favouring ω, k_z values not far from lines (20). In particular, in our case, in which $v \equiv V > c$, relation (20) brings in a formal further support of our procedures, as stated in Figs.2, 3 and 6. One may also notice that, when the pulse spectrum does strictly obey eq.(20), the pulse depth of field is *infinite* (for instance, the classical X-shaped wave and the SFWM can be regarded as corresponding to eq.(20) with $n = 0$ and $n = 1$, respectively.** While, when the spectrum is *only* (well) localized in the (ω, k_z) plane, *near* one of the lines (20), the corresponding pulse has a *finite* field depth (as it is the case for our SSP and SMPS solutions). The more "localized" the pulse spectrum is, in the (ω, k_z) plane, in the vicinity of a line (20), the longer the pulse field depth will be. We shall investigate all these points more in detail, in the next subsection.

4.2 – *Spectral analysis of the new pulses.*

Let us first recall that throughout this paper it is $\omega \geq 0$, and that, whenever we deal with Superluminal or luminal speeds $V \geq c$, we are confining ourselves (cf. Fig.2b) to the region

$$-\frac{\omega}{V} \leq k_z \leq \frac{\omega}{V} \ ; \qquad [\omega \geq 0] \ . \tag{21}$$

We are going now to generalize, among the others, what performed in ref.[18] for the $V = c$.

**On a more rigorous ground, the classical X-shaped solution does actually correspond to eq.(20) with $\Delta z_0 \to \infty$. For such a reason, it does not oscillate while propagating, and travels rigidly. Analogously, the SSPs will not oscillate: cf. subsect.4.2.

A) *Generalized* X-*shaped waves* — In the case of the classical X-shaped wave, *the spectrum* $\Phi(\alpha, \beta) = \delta(\beta) \; \exp[-a\alpha]$ corresponds, because of eqs.(4), to $\Phi(\omega, k_z) = \delta(\omega - Vk_z) \cdot \exp[-a(\omega + Vk_z)/(2V)]$, which imposes the linear constraint

$$\omega \; = \; Vk_z \; ; \tag{20a}$$

starts from $\omega = 0$; possesses the (frequency) width

$$\Delta\omega \; = \; \frac{V}{a} \; ,$$

and results to be bumped for low frequencies.

Notice that this spectrum does exactly lies *along* one of the straight-lines in Fig.3. Actually, eq.(20a) agrees with eq.(20) for $\Delta z_0 \rightarrow \infty$, in accord with the known fact that the pulse moves rigidly.

In the case of the *generalized* X-pulses, while the straight-line (20a) remains unchanged and the pulse go on being non-oscillating, the spectrum bump moves towards higher frequencies with increasing m or/and V/a (cf. subsect.3.1).

B) *Superluminal Focus Wave Modes* — In the case of the SFWMs, the spectrum $\Phi(\alpha, \beta) = \delta(\beta - \beta') \; \exp[-a\alpha]$ corresponds (because of eqs.(4)) to $\Phi(\omega, k_z) = \delta(\omega - Vk_z - 2V\beta')) \cdot \exp[-a(\omega + Vk_z)/(2V)]$, which imposes the linear constraint

$$\omega \; = \; Vk_z + 2V\beta' \; . \tag{20b}$$

The minimum value of ω is given (see Fig.3 and relation (21)) by the intersection of the straight-lines (20b) and $\omega = -Vk_z$. This spectrum starts from $\omega_{\rm min} = V\beta'$ and possesses the (frequency) width

$$\Delta\omega \; = \; \frac{V}{a} \; .$$

Notice that, once more, the spectrum runs exactly along the line (20b). By comparing eq.(20b) with eq.(20), one gets that for these oscillating solutions the *periodicity* space and time intervals are

$$\Delta z_0 \;=\; \frac{\pi}{\beta'} \; ; \qquad \Delta t_0 \;=\; \frac{\pi}{V\beta'} \; .$$

Let us recall from subsect.3.2 and Fig.3 that it must be $a\beta' \ll 1$ in order to make negligible the non-causal component contribution (in the two-dimensional expansion). As mentioned in subsect.3.2, the relation $\omega \geq V\beta'$ can be exploited for obtaining high frequency SLSs.

C) *Superluminal Splash Pulses* — In the case of the SSPs, the spectrum $\Phi(\alpha,\beta) = \exp[-b\beta]\ \exp[-a\alpha]$ corresponds (because of eqs.(4)) to $\Phi(\omega,k_z) = \exp[-b(\omega - Vk_z)/(2V)] \cdot \exp[-a(\omega + Vk_z)/(2V)]$. This time the spectrum is *no longer* exactly localized over one of the lines (20); however, if we choose $b \gg 1$ and $a \ll 1$, such a choice together with condition (21) implies $\Phi(\omega,k_z)$ to be well localized in the neighborhood of the line

$$\omega \;=\; Vk_z \; , \tag{20c}$$

besides being almost exclusively composed of causal components. All this can be directly inferred also from the form of $\Phi(\alpha,\beta)$, in connection with Fig.6. The spectrum starts from $\omega_{\rm min} = 0$, with the frequency width

$$\Delta\omega \simeq \frac{V}{a} \; .$$

Equation (20) can be compared with eq.(20c) only when $b \gg 1$; under such a condition, we obtain that $\Delta z_0 \to \infty$. However, since b can be large but not infinite, the pulse is expected to be endowed in reality with a slowly decaying amplitude, as shown below in subsect.5.2.

D) *Superluminal Modified Power Spectrum Pulses* — In the case of the SMPS pulses, the spectrum is $\Phi(\alpha,\beta) = 0$ for $0 \leq \beta < \beta_0$, and $\Phi(\alpha,\beta) = \exp[b(\beta - \beta_0)]\ \exp[-a\alpha]$ for $\beta \geq \beta_0$. Under the condition $b \gg 1$ it is $\beta \simeq \beta_0$, that is to say, the spectrum is well localized (as it follows from eqs.(4)) in the vicinity of the straight-line

$$\omega \;=\; Vk_z + 2V\beta_0 \; . \tag{20d}$$

To enforce causality, we choose (as before) also $a\beta_0 \ll 1$. Like in the SFDW pulse case, the spectrum starts from $\omega_{\rm min} = V\beta_0$, with the frequency width

$$\Delta\omega \simeq \frac{V}{a} \; .$$

Once more, in the case when $b \gg 1$, one can compare eq.(20) with eq.(20d), obtaining $\Delta z_0 \simeq \pi/\beta_0$ and $\Delta t_0 \simeq \pi/(V\beta_0)$. Under the condition $b \gg 1$, the pulse is expected to possess a long depth of field, and propagate along it (in an oscillating way) with a maximum amplitude almost constant: we shall look more in detail at this behaviour in subsect.5.3.

5. – Some exact (Superluminal localized) solutions, and their field depth

To inquiring more in detail into the field depth of our SLSs, we can confine ourselves to the propagation straight-line $\rho = 0$. Then, we can find *exact analytic* solutions holding for any value of β', without having to assume β' to be small, as we had on the contrary to assume for the SFWM, the SST and the SMPS solutions (see Sect.3, subsections 1, 2, 3). In fact, one is confronted with a simple integration of the type

$$\Psi(\rho = 0, \zeta, \eta) \; = \; \int_0^\infty {\rm d}\alpha \int_0^\infty {\rm d}\beta \; {\rm e}^{-i\beta\eta} \; {\rm e}^{i\alpha\zeta} \; \Phi(\alpha, \beta) \; . \qquad (3'\,')$$

Let us first study the infinite total energy solutions: namely, our SFWMs (skipping the generalized X-type solutions).

5.1 – *The case of the Superluminal Focus Wave Modes.*

In the case of the SFWMs, solution (11) may be integrated for $\rho = 0$, without imposing the small $\beta_0 \equiv \beta'$ approximation.†† In fact, by choosing Φ like in eq.(7), one obtains

††Also in the case of the SMPS pulses, below, we shall arrive at analytical solutions without any need of imposing the condition that $\beta_0 \equiv \beta'$ be small.

$$\Psi_{\rm SFWM}(\rho=0,\zeta,\eta) \;=\; {\rm e}^{-i\beta_0\eta} \int_0^\infty {\rm d}\alpha \; {\rm e}^{i\alpha\zeta} \; {\rm e}^{-a\alpha} \;=\; {\rm e}^{-i\beta_0\eta} \; (a-i\zeta)^{-1}$$
(11a)

whose square magnitude $|\Psi|^2 = (a^2+\zeta^2)^{-1}$ reveals that $\Psi_{\rm SFWM}$ is endowed with an infinite depth of field.

Due to the linearity of the wave equation, both the real and the imaginary part of eq.(11a), as well as of all our (complex) solutions, are themselves *solutions* of the wave equation. In the following we shall confine ourselves to investigating the behaviour of the real part.

In the case of eq.(11a) it is

$${\rm Re}\,[\Psi_{\rm SFWM}(\rho=0,\zeta,\eta)] \;=\; \frac{a\,\cos(\beta_0\eta)+\zeta\,\sin(\beta_0\eta)}{a^2+\zeta^2}\;. \qquad (11b)$$

The center C of such a pulse (where the pulse reaches its maximum value, M, *oscillating* in space and time) corresponds to $z=Vt$, that is, to $\zeta=0$ and $\eta=2z$; its value being

$$M_{\rm SFWM} \;=\; \frac{\cos(2\beta_0 z)}{a}\;. \qquad (11c)$$

Notice that: (i) at C one meets the maximum value M of the whole three-dimensional pulse: (ii) quantity M is a periodic function of z (and t), with “wavelength” Δz_0 (and oscillation period Δt_0) given by

$$\Delta z_0 \;=\; \frac{\pi}{\beta_0}\;; \qquad \Delta t_0 \;=\; \frac{\pi}{V\beta_0}\;, \qquad (11d)$$

respectively: in agreement with what anticipated in subsect.4.2-B.

The delta function entering our spectrum (7), entailing that $\beta=\beta_0$, requires that

$$\omega \;=\; Vk_z + 2V\beta_0 \qquad (22)$$

which is nothing but the straight-line $\beta = \beta_0$ of Fig.6; this fact implying by the way (as we already saw) and infinite field depth, in accordance with the previous considerations in subsect.3.4.

By comparing eq.(22) with the important "localization constraint" (20), with $n = 1$, we just obtain the value Δz_0 of eq.(11d). In other words, the previously got relations (11d) are exactly what needed for the localization properties (non-dispersiveness) of our SFWMs.

Finally, let us examine the longitudinal localization of our oscillating beams. For simplicity, let us analyse the "dispersion" of the beam when its amplitude is maximal; let us therefore skip considering the oscillations and go on to the pulse *magnitude*: one gets for the pulse half-height full-width the value $D = 2\sqrt{3}a$ in the case of the magnitude itself, and

$$D \;=\; 2a \tag{23}$$

in the case of the *square* magnitude. Let us adhere to the latter choice in the following, due to a widespread use.

5.2 – *The finite total energy solutions.*

Let us now go on to the *finite* total energy solutions:

a) *The case of the Superluminal Splash Pulses* — In the case of the SSPs with $\rho = 0$, one has to insert into eq.(3' ') the spectrum (7' '), namely $\Phi = \exp[-a\alpha]\ \exp[-b\beta]$. By integrating, we obtain

$$\Psi_{\rm SSP}(\rho = 0, \zeta, \eta) \;=\; \left[(a - i\zeta)(b + i\eta)\right]^{-1}\ , \tag{17a}$$

whose *real part* is

$$\mathrm{Re}\left[\Psi_{\rm SSP}(\rho = 0, \zeta, \eta)\right] \;=\; \frac{ab + \eta\zeta}{(ab + \eta\zeta)^2 + (a\eta - b\zeta)^2}\ . \tag{17b}$$

Let us explicitly observe that the chosen spectrum, by virtue of eqs.(4), entails that these solutions (17a,b) do *not* oscillate, which correspond to $\Delta z_0 \to \infty$ and $\Delta t_0 \to \infty$ in eqs.(20): in agreement with what anticipated in subsect.4.2-C. Actually, the SSPs are the *finite energy* version of the classical X-shaped pulses.

The maximum value M of eq.(17b) (a not oscillating, but slowly decaying only, solution) still corresponds to putting $z = Vt$, that is, to setting $\zeta = 0$ and $\eta = 2z$:

$$M_{\rm SSP} \;=\; \frac{b}{a} \cdot \frac{1}{b^2 + 4z^2} \; . \tag{17c}$$

Initially, for $z = 0, \; t = 0$, we have $M = (ab)^{-1}$. If we now *define the field-depth Z as the distance over which the pulse's amplitude is 90% at least of its initial value*, then we obtain the depth of field

$$Z_{\rm SSP} \;=\; \frac{b}{6} \tag{24}$$

which shows the dependence of Z on b, namely, the dependence of Z on the spectrum localization in the surroundings of the straight-line $\omega = Vk_z$: Cf. also subsect.3.3.

At last, the longitudinal localization will be approximately given by

$$D \approx 2a \; ; \tag{25}$$

namely, it is still given (for $a \ll 1$ and $b \gg 1$) by eq.(23). Notice that, since solution (17a) does not oscillate, the same will be true for its real part, eq.(17b), as well as for the square *magnitude* of eq.(17a): as it can be straightforwardly verified. Of course, equation (25) holds for $t = 0$. During the pulse propagation, the longitudinal localization D increases, while the amplitude M decreases. By simple but lengthy calculations, one can verify that the D-increase rate is approximately equal to the M-decrease rate; so much so we obtain (practically) the same field depth, eq.(24), when requesting the longitudinal localization to suffer a limited increase (e.g., by 10% only).

b) *The case of the Superluminal Modified Power Spectrum pulses* — In the case of the SMPS pulses with $\rho = 0$, one has to insert into eq.(3' ') the spectrum (7' ' '), namely $\Phi = {\rm e}^{-a\alpha}\, {\rm e}^{-b(\beta-\beta_0)}$, with $\beta \geq \beta_0$. By integration, one gets

$$\Psi_{\rm SMPS}(\rho=0,\zeta,\eta) \;=\; {\rm e}^{-i\beta_0\eta}\;\left[(a-i\zeta)(b+i\eta)\right]^{-1}\;, \tag{18a}$$

whose real part is easily evaluated. These pulses do oscillate while traveling. Their field depth, then calculated by having recourse to the pulse square magnitude, happens still to be

$$Z_{\rm SMPS} \;=\; \frac{b}{6} \tag{26}$$

like in the SSP case. Even the longitudinal localization of the square amplitude results approximately given, for $t=0$, by

$$D \approx 2a \tag{27}$$

as in the previous cases.

The field depth (26) depends only on b. However, the *behaviour* of the propagating pulse *changes* with the β_0-value change, besides with b's. Let us examine the maximum amplitude of the real part of eq.(18a), which for $z=Vt$ writes (when $\zeta=0$ and $\eta=2z$):

$$M_{\rm SFWM} \;=\; \frac{1}{ab}\;\frac{\cos(2\beta_0 z)+2[z/b]\;\sin(2\beta_0 z)}{1+4[z/b]^2}\;. \tag{18b}$$

Initially, for $z=0,\; t=0$, one has $M=(ab)^{-1}$ like in the SSP case.

From eq.(18b) one can infer that:

(i) when $z/b \ll 1$, namely, when $z<Z$, eq.(18b) becomes

$$M_{\rm SMPS} \;\simeq\; \frac{\cos(2\beta_0 z)}{ab}\;, \qquad\qquad [\text{for } z<Z] \tag{28}$$

and the pulse does actually oscillate *harmonically* with wavelength $\Delta z_0 = \pi/\beta_0$ and period $\Delta t_0 = \pi/(V\beta_0)$, all along its field depth: In agreement with what anticipated in subsect.4.2-D.

(ii) when $z/b > 1$, namely, when $z>Z$, eq.(18b) becomes

$$M_{\rm SMPS} \simeq \frac{\sin(2\beta_0 z)}{ab} \, \frac{1}{2\,[z/b]} \qquad [\text{for } z > Z] \qquad (28')$$

Therefore, *beyond* its depth of field, the pulse go on oscillating with the same Δz_0, but *its maximum amplitude decays* proportionally to z (the decay coefficient being $b/2$).

Last but not least, let us add the observation that results of this kind may find application in the other fields in which an essential role is played by a wave-equation (like acoustics, seismology, geophysics, relativistic quantum physics, and gravitational waves, possibly).

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APPENDIX A

Further families of "X-type" Superluminal localized solutions

As announced in subsect.3.1, let us mention in this Appendix that one can obtain new SLSs by considering *for instance* the following modifications (still with $\beta' = 0$ of the spectrum (7), with a, d arbitrary constants:

$$\Phi(k, \alpha, \beta) \;=\; \delta(\beta)\; J_0(2d\sqrt{\alpha})\; \mathrm{e}^{-a\alpha} \tag{A.1a}$$

$$\Phi(k, \alpha, \beta) \;=\; \delta(\beta)\; \sinh(\alpha d)\; \mathrm{e}^{-a\alpha} \tag{A.1b}$$

$$\Phi(k, \alpha, \beta) \;=\; \delta(\beta)\; \cos(\alpha d)\; \mathrm{e}^{-a\alpha} \tag{A.1c}$$

$$\Phi(k, \alpha, \beta) \;=\; \delta(\beta)\; \frac{\sin \alpha d}{\alpha}\; \mathrm{e}^{-a\alpha} \tag{A.1d}$$

Let us call X, as in eq.(8), the classical X-shaped solution

$$X \;\equiv\; \left[(a - i\zeta)^2 + \rho^2(V^2 - 1)\right]^{\frac{1}{2}} \;.$$

One can obtain from those spectra the new, different Superluminal localized solutions, respectively:

$$\begin{aligned} \Psi(\rho, \zeta) \;=\; & X \cdot J_0(\rho d^2 \sqrt{V^2 - 1}\; X^2) \times \\ & \times \exp\left[-(a - i\zeta)\, d^2\, X^2\right] \;, \end{aligned} \tag{A.2a}$$

got by using identity (6.6444) in ref.[11];

$$\Psi(\rho, \zeta) \;=\; \frac{2d(a - i\zeta)\sqrt{2(X^{-2} + d^2)}}{(X^{-2} + d^2) - 4d^2(a - i\zeta)^2} \;, \tag{A.2b}$$

for $a > |d|$, by using identity (6.668.1) of ref.[11];

$$\Psi(\rho,\zeta) \;=\; \left[\frac{X^{-2}-d^2+\sqrt{(X^{-2}-d^2)^2+4d^2(a-i\zeta)^2}}{2[(X^{-2}-d^2)^2+4d^2(a-i\zeta)^2]}\right]^{\frac{1}{2}} \tag{A.2c}$$

by using identity (6.751.3) of ref.[11]; and

$$\begin{aligned}\Psi(\rho,\zeta) \;=\; \sin^{-1} 2d \Bigg[&\sqrt{X^{-2}+d^2+2\rho d\sqrt{V^2-1}}\;+ \\ &+\;\sqrt{X^{-2}+d^2-2\rho d\sqrt{V^2-1}}\,\Bigg]\;,\end{aligned} \tag{A.2d}$$

for $a > 0$ and $d > 0$, by using identity (6.752.1) of ref.[11].

Let us recall that, due to the choice $\beta' = 0$ and the consequent presence of a $\delta(\beta)$ factor in the weight, all such solutions are completely physical, in the sense that they e don't get *any* contribution from the non-causal components (i.e., from waves moving backwards). In fact, these new solutions are functions of ρ, ζ only (and not of η). In particular, solutions (A.2b), (A.2c), (A.2d), as well as others easily obtainable, are functions of ρ via quantity X only. This may suggest to go on from the variables (ρ,ζ) to the variables (X,ζ) and write down em the wave equation itself in the new variables. Some *preliminary* results are the following:

$$\left[F(X,\zeta)\,\frac{\partial}{\partial X} + G(X,\zeta)\,\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}\right]\;\Psi(X,\zeta) \;=\; 0 \tag{A.3}$$

where $Z \equiv a - i\zeta$ and

$$F(X,\zeta) \;\equiv\; -2X^3 - 6X^5Z^2 \tag{A.4a}$$

$$G(X,\zeta) \;\equiv\; (X^4 - Z^2X^6)\,\sqrt{V^2-1}\;. \tag{A.4b}$$

The related results and consequences will be better exploited elsewhere.

FIGURES

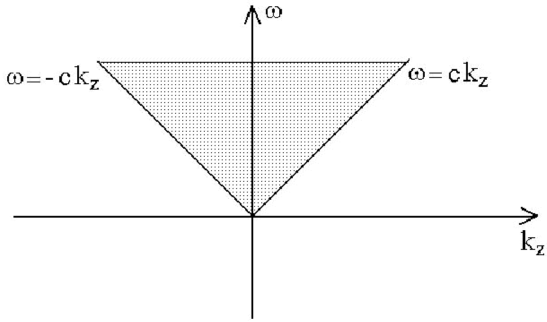


Fig.1

Fig.1 – Geometrical representation, in the plane (ω, k_z), of our conditions (2): see the text. It is essential to notice that the dispersion relation (2), with positive (but not constant, a priori) k^2, while enforcing the consideration of the truly propagating waves only (with exclusion of the evanescent ones), does allow for both subluminal and Superluminal solutions; the latter being the ones of interest for us. Conditions (2) correspond to confining ourselves to the sector delimited by the straight lines $\omega = \pm ck_z$.

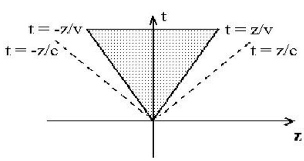


Fig.2a

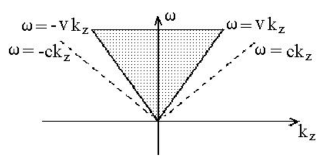


Fig.2b

Figs.2 – *Inside* the region shown in Fig.1 (which excludes the evanescent regime), we can choose the sector shown in Fig.2a, provided that V is Superluminal ($V \geq c$). Fig.2b shows the same sector, chosen by us, in the (ω, k_z) plane.

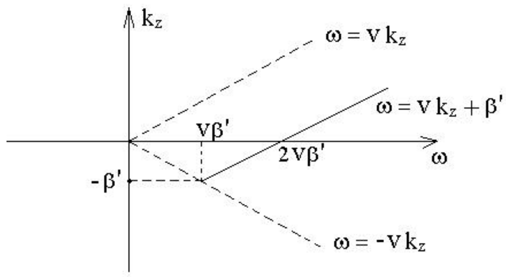


Fig.3 – When releasing the condition $\beta' = 0$ (see the text), which excluded the "backwards-traveling" components, one has to integrate in eq.(11) along the half-line $\omega = V k_z + \beta'$, namely, also along the "non-causal" interval $V\beta' < \omega < 2V\beta'$. We can obtain physical solutions, however, by making negligible the contribution of the unwanted interval, i.e., by choosing small values of a. This can be even more easily seen in the (ω, k_z) plane.

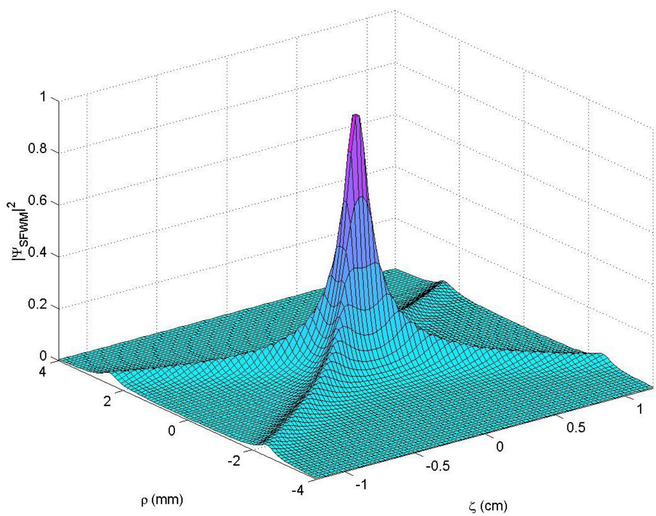
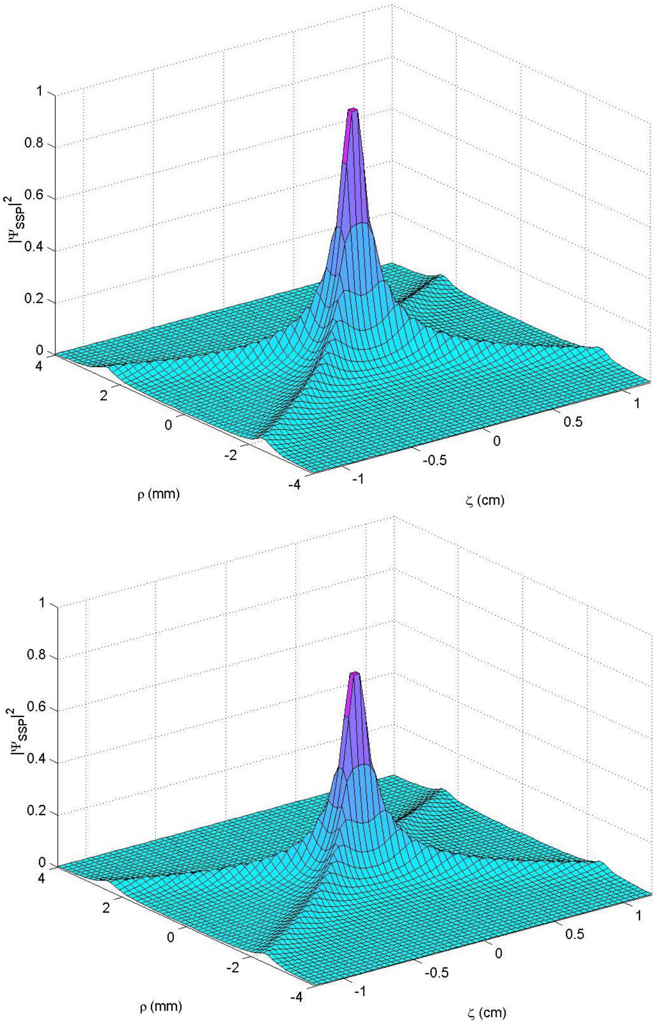
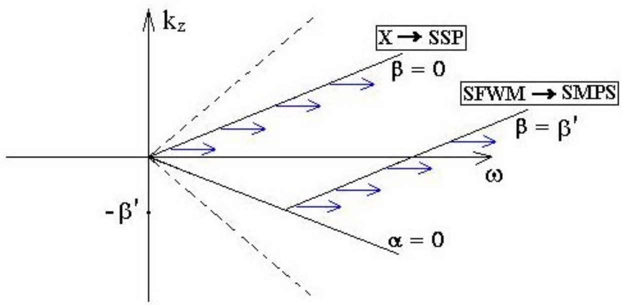


Fig.4 – Representation of our Superluminal Focus Wave Modes (SFWM), eq.(13), which are a generalization of the ordinary FWMs. The depicted pulse corresponds to $V = 5c$, $a = 0.001$ m; $\beta' = 1/(100\,\mathrm{m})$, and to arbitrary time t (since these solutions too travel without deforming). Such solutions correspond to high frequency (microwave, optical,...) pulses: see the text. The meaning of ρ, ζ, etc., is given in the caption of Fig.3.



Figs.5 – Representation of our Superluminal Splash Pulses (SSP), eq.(17). They are suitable superpositions of SFWMs (cf. Fig.4), so that their total energy is *finite* (even without any truncation). They however get deformed while propagating, since their amplitude decreases with time. In Fig.5a we represent, for $t = 0$, the pulse corresponding to $V = 5c$, $a = 0.001$ m, and $b = 200$ m. In Fig.5b it is depicted the same pulse after having traveled 50 meters.



Figs.6 – From a geometric point of view, our infinite total energy SLSs, i.e., the X-solutions, eq.(12), and the SFWMs, eq.(13), correspond —see the text— to integrations along the $\beta = 0$ axis, or α-axis, and the $\beta = \beta'$ straight-line, respectively. In order to go on to the *finite* total-energy SLSs, we had to replace the $\delta(\beta - \beta')$ factor in the spectrum (7) with the function (16'), which is different from 0 in the region along and below the $\beta = \beta'$ line and suitably *decays* therein. The faster the spectrum decays (below and along the $\beta = \beta'$ line), the larger the field depth of the pulse results to be. In such a manner we obtained the SMPSs, eq.(18), as well as the SSPs, which just correspond to the particular case $\beta' = 0$.

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