

# About Algebraic Foundations of Majorana–Oppenheimer Quantum Electrodynamics and de Broglie–Jordan Neutrino Theory of Light

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**ABSTRACT.** An algebraic description of basic physical fields (neutrino field, electron–positron field and electromagnetic field) is studied. It is shown that the electromagnetic field can be described within a quotient representation of the proper orthochronous Lorentz group. The relation of such a description with Majorana–Oppenheimer quantum electrodynamics and de Broglie–Jordan neutrino theory of light is discussed.

Many years ago Bogoliubov and Shirkov [1] pointed out that among all physical fields the electromagnetic field (beyond all shadow of doubt the main physical field) is quantized with the most difficulty. In the standard Gupta–Bleuler approach an unobservable magnitude (electromagnetic four–potential  $\mathbf{A}$ ) is quantized. At this point, the four–potential has four degrees of freedom, but in nature there are only two degrees of freedom for a photon field (left and right handed polarizations). Besides, the electromagnetic four–potential is transformed within  $(1/2, 1/2)$ –representaion of the homogeneous Lorentz group and, therefore, in accordance with a well-known Weinberg Theorem [35] the field described by  $\mathbf{A}$  has a null helicity, that also contradicts with experience. Moreover, at the present time electromagnetic field is understood as a ‘gauge field’ that gives rise to a peculiar opposition with other physical fields called by this reason as ‘matter fields’.

With the aim of overcoming this unnatural opposition all the physical fields should be considered on an equal footing. In this paper we present an algebraic construction of the most fundamental physical fields such as neutrino field, electron–positron field and electromagnetic field.

Our consideration based mainly on the relation between Clifford algebras and Lorentz group (all mathematical background contained in [30, 31, 32, 33, 34]). In [34] all the Clifford algebras are understood as ‘algebraic coverings’ of finite-dimensional representations of the proper Lorentz group  $\mathfrak{G}_+$ . In [34] it has been shown that there is a following classification:

### I. Complex representations.

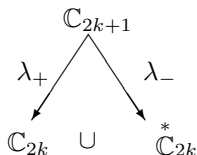
- 1) Representations  $\mathfrak{C}^{l_0+l_1-1,0} \leftrightarrow \mathbb{C}_n$  with the field  $(j, 0)$ , where  $j = \frac{l_0+l_1-1}{2}$ .
- 2) Representations  $\mathfrak{C}^{0,l'_0-l'_1+1} \leftrightarrow \mathbb{C}_n^*$  with the field  $(0, j')$ , where  $j' = \frac{l'_0-l'_1+1}{2}$ .
- 3) Representations  $\mathfrak{C}^{l_0+l_1-1,l'_0-l'_1+1} \leftrightarrow \mathbb{C}_n \otimes \mathbb{C}_n^*$  with the field  $(j, j')$ .
- 4) Representations  $\mathfrak{C}^{l_0+l_1-1,0} \oplus \mathfrak{C}^{0,l_0-l_1+1} \leftrightarrow \mathbb{C}_n \oplus \mathbb{C}_n^*$  with the field  $(j, 0) \oplus (0, j)$ ,  $j = j'$ .
- 5) Quotient representations  $\chi \mathfrak{C}^{l_0+l_1-1,0} \cup \chi \mathfrak{C}^{0,l_0-l_1+1} \leftrightarrow \epsilon \mathbb{C}_n \cup \epsilon \mathbb{C}_n^*$  with the field  $(j, 0) \cup (0, j)$ .

### II. Real representations.

- 6) Real representations  $\mathfrak{R}_{0,2}^{l_0} \leftrightarrow \mathcal{C}_{p,q}$ ,  $p - q \equiv 0, 2 \pmod{8}$ , with the field  $[j]$ , where  $j = \frac{l_0}{2}$ ,  $l_0 = \frac{p+q}{4}$ .
- 7) Quaternionic representations  $\mathfrak{H}_{4,6}^{l_0} \leftrightarrow \mathcal{C}_{p,q}$ ,  $p - q \equiv 4, 6 \pmod{8}$ , with the field  $[j]$ .
- 8) Quotient representations  $\chi \mathfrak{D}^{l_0} \cup \chi \mathfrak{D}^{l_0} \leftrightarrow \epsilon \mathcal{C}_{p,q} \cup \epsilon \mathcal{C}_{p,q}$  with the field  $[j] \cup [j]$ , where  $\chi \mathfrak{D}^{l_0} = \{\chi \mathfrak{R}_{0,2}^{l_0}, \chi \mathfrak{H}_{4,6}^{l_0}\}$ .

Here the numbers  $l_0$  and  $l_1$  define the finite-dimensional representation in the Gel'fand–Naimark representation theory of the Lorentz group [10, 19]. In its turn, quotient representations correspond to the type  $n \equiv 1 \pmod{2}$  of  $\mathbb{C}_n$  (or to the types  $p - q \equiv 1, 5 \pmod{8}$ ) for the real

representations). Over the field  $\mathbb{F} = \mathbb{C}$  these representations obtained in the result of the following decomposition



Here central idempotents

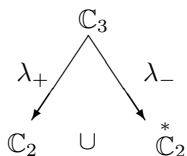
$$\lambda^+ = \frac{1 + \varepsilon \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_{2k+1}}{2}, \quad \lambda^- = \frac{1 - \varepsilon \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_{2k+1}}{2},$$

where

$$\varepsilon = \begin{cases} 1, & \text{if } k \equiv 0 \pmod{2}, \\ i, & \text{if } k \equiv 1 \pmod{2} \end{cases},$$

satisfy the relations  $(\lambda^+)^2 = \lambda^+$ ,  $(\lambda^-)^2 = \lambda^-$ ,  $\lambda^+ \lambda^- = 0$ . Thus, we have a decomposition of the initial algebra  $\mathbb{C}_{2k+1}$  into a direct sum of two mutually annihilating simple ideals:  $\mathbb{C}_{2k+1} \simeq \frac{1}{2}(1 + \varepsilon \omega) \mathbb{C}_{2k+1} \oplus \frac{1}{2}(1 - \varepsilon \omega) \mathbb{C}_{2k+1}$ . Each of the ideals  $\lambda^\pm \mathbb{C}_{2k+1}$  is isomorphic to the subalgebra  $\mathbb{C}_{2k} \subset \mathbb{C}_{2k+1}$ . In accordance with Chisholm and Farwell [4] the idempotents  $\lambda^+$  and  $\lambda^-$  can be identified with helicity projection operators which distinguish left and right handed spinors. The Chisholm–Farwell notation for  $\lambda^\pm$  we will widely use below.

The first simplest case of such a decomposition is presented by the algebra  $\mathbb{C}_3$  related with the neutrino field. Indeed,



here central idempotents

$$\lambda_- = \frac{1 - i \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3}{2}, \quad \lambda_+ = \frac{1 + i \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3}{2} \tag{1}$$

in accordance with [4] can be identified with helicity projection operators. In such a way, we have two helicity states describing by the quotient algebras  ${}^{\epsilon}\mathbb{C}_2$  and  ${}^{\epsilon}\mathbb{C}_2^*$ , and a full neutrino–antineutrino algebra is  ${}^{\epsilon}\mathbb{C}_2 \cup {}^{\epsilon}\mathbb{C}_2^*$  (cf. electron–positron algebra  $\mathbb{C}_2 \oplus \mathbb{C}_2^*$ ).

Let  $\varphi \in \mathbb{C}_3$  be an algebraic spinor of the form (sometimes called operator spinor, see [8])

$$\varphi = a^0 + a^1 \mathbf{e}_1 + a^2 \mathbf{e}_2 + a^3 \mathbf{e}_3 + a^{12} \mathbf{e}_1 \mathbf{e}_2 + a^{13} \mathbf{e}_1 \mathbf{e}_3 + a^{23} \mathbf{e}_2 \mathbf{e}_3 + a^{123} \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3. \tag{2}$$

Then it is easy to verify that spinors

$$\varphi^+ = \lambda_+ \varphi = \frac{1}{2}(1 + i\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3)\varphi, \quad \varphi^- = \lambda_- \varphi = \frac{1}{2}(1 - i\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3)\varphi \tag{3}$$

are mutually orthogonal,  $\varphi^+ \varphi^- = 0$ , since  $\lambda_+ \lambda_- = 0$ , and also  $\varphi^+ \in \mathbb{C}_2$ ,  $\varphi^- \in \mathbb{C}_2^*$ . Further, it is obvious that a spinspace of the algebra  ${}^{\epsilon}\mathbb{C}_2 \cup {}^{\epsilon}\mathbb{C}_2^*$  is  $\mathbb{S}_2 \cup \dot{\mathbb{S}}_2$ . It should be noted here that structures of the spinspaces  $\mathbb{S}_2 \cup \dot{\mathbb{S}}_2$  and  $\mathbb{S}_2 \oplus \dot{\mathbb{S}}_2$  are different. Indeed,

$$\mathbb{S}_2 \cup \dot{\mathbb{S}}_2 = \left( \begin{array}{cc} [00, \dot{0}\dot{0}] & [01, \dot{0}\dot{1}] \\ [10, \dot{1}\dot{0}] & [11, \dot{1}\dot{1}] \end{array} \right), \quad \mathbb{S}_2 \oplus \dot{\mathbb{S}}_2 = \left( \begin{array}{cc} 00 & 01 \\ 10 & 11 & & \\ & & \dot{0}\dot{0} & \dot{0}\dot{1} \\ & & \dot{1}\dot{0} & \dot{1}\dot{1} \end{array} \right).$$

Under action of the pseudoautomorphism  $\mathcal{A} \rightarrow \overline{\mathcal{A}}$  (charge conjugation  $C$ , see [34]) spinspaces  $\mathbb{S}_2 \cup \dot{\mathbb{S}}_2$  take a form

$$\dot{\mathbb{S}}_2 \cup \mathbb{S}_2 = \left( \begin{array}{cc} [\dot{0}\dot{0}, 00] & [\dot{0}\dot{1}, 01] \\ [\dot{1}\dot{0}, 10] & [\dot{1}\dot{1}, 11] \end{array} \right).$$

Since spinor representations of the quotient algebras  ${}^{\epsilon}\mathbb{C}_2$  and  ${}^{\epsilon}\mathbb{C}_2^*$  are defined in terms of Pauli matrices  $\sigma_i$ , then the algebraic spinors  $\varphi^+ \in {}^{\epsilon}\mathbb{C}_2$  and  $\varphi^- \in {}^{\epsilon}\mathbb{C}_2^*$  correspond to spinors  $\xi^{\alpha_i} \in \mathbb{S}_2$  and  $\xi^{\dot{\alpha}_i} \in \dot{\mathbb{S}}_2$  ( $i = 0, 1$ ). Hence we have Weyl equations

$$\left( \frac{\partial}{\partial x^0} - \boldsymbol{\sigma} \frac{\partial}{\partial \mathbf{x}} \right) \xi^\alpha = 0, \quad \left( \frac{\partial}{\partial x^0} + \boldsymbol{\sigma} \frac{\partial}{\partial \mathbf{x}} \right) \xi^{\dot{\alpha}} = 0. \tag{4}$$

Therefore, two–component Weyl theory can be naturally formulated within quotient representation  ${}^{\chi}\mathfrak{C}_c^{1,0} \cup {}^{\chi}\mathfrak{C}_c^{0,-1}$  of the group  $\mathfrak{G}_+$ . Further,

in virtue of an isomorphism  $\mathbb{C}_2 \simeq \mathcal{O}_{3,0} \simeq \mathcal{O}_{1,3}^+$  ( $\mathcal{O}_{1,3}$  is the space–time algebra) the spinor field of the quotient representation  ${}^X\mathfrak{C}_c^{0,-1}$  ( ${}^X\mathfrak{C}_c^{1,0}$ ) can be expressed via the Dirac–Hestenes spinor field  $\phi(x) \in \mathcal{O}_{3,0}$  [12, 13, 16]. Indeed, the Dirac–Hestenes spinor is represented by a following biquaternion number

$$\begin{aligned} \phi = & a^0 + a^{01}\gamma_0\gamma_1 + a^{02}\gamma_0\gamma_2 + a^{03}\gamma_0\gamma_3 \\ & + a^{12}\gamma_1\gamma_2 + a^{13}\gamma_1\gamma_3 + a^{23}\gamma_2\gamma_3 + a^{0123}\gamma_0\gamma_1\gamma_2\gamma_3, \end{aligned} \tag{5}$$

or using  $\gamma$ -matrix basis

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \tag{6}$$

we can write (5) in the matrix form

$$\phi = \begin{pmatrix} \phi_1 & -\phi_2^* & \phi_3 & \phi_4^* \\ \phi_2 & \phi_1^* & \phi_4 & -\phi_3^* \\ \phi_3 & \phi_4^* & \phi_1 & -\phi_2^* \\ \phi_4 & -\phi_3^* & \phi_2 & \phi_1^* \end{pmatrix}, \tag{7}$$

where

$$\phi_1 = a^0 - ia^{12}, \quad \phi_2 = a^{13} - ia^{23}, \quad \phi_3 = a^{03} - ia^{0123}, \quad \phi_4 = a^{01} + ia^{02}.$$

From (2)–(3) and (5) it is easy to see that spinors  $\varphi^+$  and  $\varphi^-$  are algebraically equivalent to the spinor  $\phi \in \mathbb{C}_2 \simeq \mathcal{O}_{3,0}$ . Further, since  $\phi \in \mathcal{O}_{1,3}^+$ , then actions of the antiautomorphisms  $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$  and  $\mathcal{A} \rightarrow \tilde{\mathcal{A}}^*$  on the field  $\phi$  are equivalent. On the other hand, in accordance with Feynman–Stueckelberg interpretation, time reversal for the chiral field is equivalent to charge conjugation (particles reversed in time are antiparticles). Thus, for the field  $\phi \in {}^X\mathfrak{C}_c^{0,-1}$  we have  $C \sim T$  and, therefore, this field is  $CP$ -invariant.

The spinor (5) (or (7)) satisfies the Dirac–Hestenes equation

$$\partial\phi\gamma_2\gamma_1 - \frac{mc}{\hbar}\phi\gamma_0 = 0, \tag{8}$$

where  $\partial = \gamma^\mu \frac{\partial}{\partial x^\mu}$  is the Dirac operator. Let us show that a massless Dirac–Hestenes equation

$$\partial\phi\gamma_2\gamma_1 = 0 \tag{9}$$

describes the neutrino field. Indeed, the matrix  $\gamma_0\gamma_1\gamma_2\gamma_3$  commutes with all the elements of the biquaternion (5) and, therefore,  $\gamma_0\gamma_1\gamma_2\gamma_3$  is equivalent to the volume element  $\omega = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$  of the biquaternion algebra  $\mathcal{O}_{3,0}$ . In such a way, we see that idempotents

$$P_+ = \frac{1 + \gamma_5}{2}, \quad P_- = \frac{1 - \gamma_5}{2}$$

cover the central idempotents (1), where  $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$ . Further, from (9) we obtain

$$P_{\pm}\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}\phi\gamma_2\gamma_1 = \gamma^{\mu}P_{\mp}p\frac{\partial}{\partial x^{\mu}}\phi\gamma_2\gamma_1 = 0,$$

that is, there are two separated equations for  $\phi^{\pm} = P_{\pm}\phi\gamma_2\gamma_1$ :

$$\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}\phi^{\pm} = 0, \quad (10)$$

where

$$\phi^{\pm} = \frac{1}{2}(1 \pm \gamma_5)\phi\gamma_2\gamma_1 = \frac{i}{2} \begin{pmatrix} \phi_1 \mp \phi_3 & \phi_2^* \pm \phi_4^* & \phi_3 \mp \phi_1 & -\phi_4^* \mp \phi_2^* \\ \phi_2 \mp \phi_4 & -\phi_1^* \mp \phi_3^* & \phi_4 \mp \phi_2 & \phi_3^* \pm \phi_1^* \\ \mp\phi_1 + \phi_2 & \mp\phi_2^* - \phi_4^* & \mp\phi_3 + \phi_1 & \pm\phi_4^* + \phi_2^* \\ \mp\phi_2 + \phi_4 & \pm\phi_1^* + \phi_3^* & \mp\phi_4 + \phi_2 & \mp\phi_3^* - \phi_1^* \end{pmatrix}$$

Therefore, each of the functions  $\phi^+$  and  $\phi^-$  contains only four independent components and in the split form we have

$$\phi^+ = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ -\psi_1 & -\psi_2 & -\psi_3 & -\psi_4 \end{pmatrix}, \quad \phi^- = \begin{pmatrix} \psi_5 & \psi_6 & \psi_7 & \psi_8 \\ \psi_5 & \psi_6 & \psi_7 & \psi_8 \end{pmatrix},$$

where

$$\begin{aligned} \psi_1 &= \frac{i}{2} \begin{pmatrix} \phi_1 - \phi_3 \\ \phi_2 - \phi_4 \end{pmatrix}, & \psi_2 &= \frac{i}{2} \begin{pmatrix} \phi_2^* + \phi_4^* \\ -\phi_1^* - \phi_3^* \end{pmatrix}, \\ \psi_3 &= \frac{i}{2} \begin{pmatrix} \phi_3 - \phi_1 \\ \phi_4 - \phi_2 \end{pmatrix}, & \psi_4 &= \frac{i}{2} \begin{pmatrix} -\phi_4^* - \phi_2^* \\ \phi_3^* + \phi_1^* \end{pmatrix}, \\ \psi_5 &= \frac{i}{2} \begin{pmatrix} \phi_1 + \phi_3 \\ \phi_2 + \phi_4 \end{pmatrix}, & \psi_6 &= \frac{i}{2} \begin{pmatrix} \phi_2^* - \phi_4^* \\ -\phi_1^* + \phi_3^* \end{pmatrix}, \\ \psi_7 &= \frac{i}{2} \begin{pmatrix} \phi_3 + \phi_1 \\ \phi_4 + \phi_2 \end{pmatrix}, & \psi_8 &= \frac{i}{2} \begin{pmatrix} -\phi_4^* + \phi_2^* \\ \phi_3^* - \phi_1^* \end{pmatrix}. \end{aligned}$$

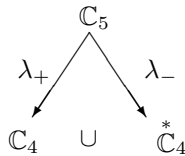
Thus, in the  $\gamma$ -matrix basis we obtain from (10)

$$\left( \frac{\partial}{\partial x^0} - \boldsymbol{\sigma} \frac{\partial}{\partial \mathbf{x}} \right) \psi_i = 0, \quad \left( \frac{\partial}{\partial x^0} + \boldsymbol{\sigma} \frac{\partial}{\partial \mathbf{x}} \right) \psi_{i+4} = 0, \quad (i = 1, 2, 3, 4)$$

These equations are equivalent to Weyl equations (4) for neutrino field and, therefore, by analogy with the Dirac-Hestenes equations for  $m \neq 0$  should be called *Weyl-Hestenes equations for neutrino field*.

The Dirac electron-positron field  $(1/2, 0) \oplus (0, 1/2)$  corresponds to the algebra  $\mathbb{C}_2 \oplus \overset{*}{\mathbb{C}}_2$ . It should be noted that the Dirac algebra  $\mathbb{C}_4$  considered as a tensor product  $\mathbb{C}_2 \otimes \mathbb{C}_2$  (or  $\mathbb{C}_2 \otimes \overset{*}{\mathbb{C}}_2$ ) gives rise to spintensors  $\xi^{\alpha_1 \alpha_2}$  (or  $\xi^{\alpha_1 \dot{\alpha}_1}$ ), but it contradicts with the usual definition of the Dirac bispinor as a pair  $(\xi^{\alpha_1}, \xi^{\dot{\alpha}_1})$ . Therefore, the Clifford algebra associated with the Dirac field is  $\mathbb{C}_2 \oplus \overset{*}{\mathbb{C}}_2$ , and a spinspace of this sum in virtue of unique decomposition  $\mathbb{S}_2 \oplus \overset{*}{\mathbb{S}}_2 = \mathbb{S}_4$  ( $\mathbb{S}_4$  is a spinspace of  $\mathbb{C}_4$ ) allows to define  $\gamma$ -matrices in the Weyl basis.

In common with other massless fields (such as the neutrino field  $(1/2, 0) \cup (0, 1/2)$ ) the Maxwell electromagnetic field is also described within the quotient representations of the Lorentz group. In accordance with Theorem 4 in [34] the photon field can be described by a quotient representation of the class  ${}^{\chi} \mathfrak{C}_{a_1}^{2,0} \cup {}^{\chi} \mathfrak{C}_{a_1}^{0,-2}$ . This representation admits time reversal  $T$  and an identical charge conjugation  $C \sim I$  that corresponds to truly neutral particles (see Theorem 3 in [34]). The quotient algebra  ${}^{\epsilon} \mathbb{C}_4 \cup {}^{\epsilon} \overset{*}{\mathbb{C}}_4$  associated with the Maxwell field  $(1, 0) \cup (0, 1)$  is obtained in the result of an homomorphic mapping  $\epsilon : \mathbb{C}_5 \rightarrow \mathbb{C}_4$ . Indeed, for the algebra  $\mathbb{C}_5$  we have a decomposition



where the central idempotents

$$\lambda_+ = \frac{1 + \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_4 \mathbf{e}_5}{2}, \quad \lambda_- = \frac{1 - \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_4 \mathbf{e}_5}{2}$$

correspond to the helicity projection operators of the Maxwell field. As known, for the photon there are two helicity states: left and right handed polarizations.

Let  $\varphi \in \mathbb{C}_5$  be an algebraic spinor of the form

$$\begin{aligned} \varphi = & a^0 + \sum_{i=1}^5 a^i \mathbf{e}_i + \sum_{i,j=1}^5 a^{ij} \mathbf{e}_i \mathbf{e}_j + \sum_{i,j,k=1}^5 a^{ijk} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \\ & + \sum_{i,j,k,l=1}^5 a^{ijkl} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l + a^{12345} \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_4 \mathbf{e}_5, \end{aligned}$$

then the spinors

$$\varphi^+ = \lambda_+ \varphi = \frac{1}{2}(1 + \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_4 \mathbf{e}_5) \varphi, \quad \varphi^- = \lambda_- \varphi = \frac{1}{2}(1 - \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_4 \mathbf{e}_5) \varphi$$

are mutually orthogonal,  $\varphi^+ \varphi^- = 0$ , and  $\varphi^+ \in \mathbb{C}_4$ ,  $\varphi^- \in \overset{*}{\mathbb{C}}_4$ . The spinspace of the algebra  ${}^e\mathbb{C}_4 \cup \overset{*}{e}\mathbb{C}_4$  has a form

$$\mathbb{S}_4 \cup \dot{\mathbb{S}}_4 = \begin{pmatrix} [0000, \acute{0}\acute{0}\acute{0}\acute{0}] & [0001, \acute{0}\acute{0}\acute{0}1] & [0010, \acute{0}\acute{0}1\acute{0}] & [0011, \acute{0}\acute{0}11] \\ [0100, \acute{0}1\acute{0}\acute{0}] & [0101, \acute{0}1\acute{0}1] & [0110, \acute{0}11\acute{0}] & [0111, \acute{0}111] \\ [1000, 1\acute{0}\acute{0}\acute{0}] & [1001, 1\acute{0}\acute{0}1] & [1010, 1\acute{0}1\acute{0}] & [1011, 1\acute{0}11] \\ [1100, 11\acute{0}\acute{0}] & [1101, 11\acute{0}1] & [1110, 111\acute{0}] & [1111, 1111] \end{pmatrix}$$

Let us consider now an explicit construction of the Maxwell field  $(1, 0) \cup (0, 1)$  within the quotient algebra  ${}^e\mathbb{C}_4 \cup \overset{*}{e}\mathbb{C}_4$ . First of all, let us define a spinor representation of the field  $(1, 0) \cup (0, 1)$ . As a rule, a spinor field of the quotient algebra  ${}^e\mathbb{C}_4 \simeq \mathcal{C}_{4,1}$  is appeared at the extraction of the minimal left ideal [8, 9, 26]:

$$I_{4,1} = \mathcal{C}_{4,1} e_{41} \simeq \mathcal{C}_{1,3}^+ e_{13} \frac{1}{2}(1 + i\gamma_1 \gamma_2) = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix}, \quad (11)$$

where  $e_{13} \frac{1}{2}(1 + \gamma_0)$  and  $e_{41} = \frac{1}{2}(1 + \gamma_0) \frac{1}{2}(1 + i\gamma_1 \gamma_2)$  are primitive idempotents of the algebras  $\mathcal{C}_{1,3}$  and  $\mathbb{C}_4$ . Further, since  $\mathcal{C}_{1,3}^+ \simeq \mathcal{C}_{3,0} \simeq \mathbb{C}_2$ ,



then the spinor field  $\psi$  in (11) can be expressed via the Dirac–Hestenes spinor field  $\phi \in \mathcal{C}_{3,0} \simeq \mathbb{C}_2$ . Let

$$\nabla = \partial^0 \mathbf{e}_0 + \partial^1 \mathbf{e}_1 + \partial^2 \mathbf{e}_2 + \partial^3 \mathbf{e}_3, \quad A = A^0 \mathbf{e}_0 + A^1 \mathbf{e}_1 + A^2 \mathbf{e}_2 + A^3 \mathbf{e}_3$$

be linear elements of the algebra  $\mathcal{C}_{3,0}$ , where  $A_i$  are the components of the electromagnetic four–potential. Then

$$\begin{aligned} \nabla A &= (\partial^0 \mathbf{e}_0 + \partial^1 \mathbf{e}_1 + \partial^2 \mathbf{e}_2 + \partial^3 \mathbf{e}_3)(A^0 \mathbf{e}_0 + A^1 \mathbf{e}_1 + A^2 \mathbf{e}_2 + A^3 \mathbf{e}_3) = \\ &\quad \underbrace{(\partial^0 A^0 + \partial^1 A^1 + \partial^2 A^2 + \partial^3 A^3)}_{E^0} \mathbf{e}_0 + \underbrace{(\partial^0 A^1 + \partial^1 A^0)}_{E^1} \mathbf{e}_0 \mathbf{e}_1 + \\ &\quad \underbrace{(\partial^0 A^2 + \partial^2 A^0)}_{E^2} \mathbf{e}_0 \mathbf{e}_2 + \underbrace{(\partial^0 A^3 + \partial^3 A^0)}_{E^3} \mathbf{e}_0 \mathbf{e}_3 + \underbrace{(\partial^2 A^3 - \partial^3 A^2)}_{H^1} \mathbf{e}_2 \mathbf{e}_3 + \\ &\quad \underbrace{(\partial^3 A^1 - \partial^1 A^3)}_{H^2} \mathbf{e}_3 \mathbf{e}_1 + \underbrace{(\partial^1 A^2 - \partial^2 A^1)}_{H^3} \mathbf{e}_1 \mathbf{e}_2. \quad (12) \end{aligned}$$

The scalar part  $E_0 \equiv 0$ , since the first bracket in (12) is a Lorentz condition  $\partial^0 A^0 + \operatorname{div} \mathbf{A} = 0$ . It is easy to see that other brackets are components of electric and magnetic fields:  $-E^i = -(\partial^i A^0 + \partial^0 A^i)$ ,  $H^i = (\operatorname{curl} \mathbf{A})^i$ .

Since  $\omega = \mathbf{e}_{123}$  belongs to a center of  $\mathcal{C}_{3,0}$ , then

$$\omega \mathbf{e}_1 = \mathbf{e}_1 \omega = \mathbf{e}_2 \mathbf{e}_3, \quad \omega \mathbf{e}_2 = \mathbf{e}_2 \omega = \mathbf{e}_3 \mathbf{e}_1, \quad \omega \mathbf{e}_3 = \mathbf{e}_3 \omega = \mathbf{e}_1 \mathbf{e}_2.$$

In accordance with these relations we can write (12) as follows

$$\nabla A = (E^1 + \omega H^1) \mathbf{e}_1 + (E^2 + \omega H^2) \mathbf{e}_2 + (E^3 + \omega H^3) \mathbf{e}_3 \quad (13)$$

Further, let us compose the product  $\nabla \mathbf{F}$ , where  $\mathbf{F}$  is an expression of the type (12):

$$\begin{aligned} \nabla \mathbf{F} &= \operatorname{div} \mathbf{E} \mathbf{e}_0 - ((\operatorname{curl} \mathbf{H})^1 - \partial^0 E_1) \mathbf{e}_1 - ((\operatorname{curl} \mathbf{H})^2 - \partial^0 E^2) \mathbf{e}_2 - \\ &- ((\operatorname{curl} \mathbf{H})^3 - \partial^0 E^3) \mathbf{e}_3 + ((\operatorname{curl} \mathbf{E})^1 + \partial^0 H^1) \mathbf{e}_2 \mathbf{e}_3 + ((\operatorname{curl} \mathbf{E})^2 + \partial^0 H^2) \mathbf{e}_3 \mathbf{e}_1 + \\ &\quad ((\operatorname{curl} \mathbf{E})^3 + \partial^0 H^3) \mathbf{e}_1 \mathbf{e}_2 + \operatorname{div} \mathbf{H} \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3. \quad (14) \end{aligned}$$

It is easy to see that the first coefficient of the product  $\nabla \mathbf{F}$  is a left part of the equation  $\operatorname{div} \mathbf{E} = \varrho$ . The following three coefficients compose a left part of the equation  $\operatorname{curl} \mathbf{H} - \partial^0 \mathbf{E} = \mathbf{j}$ , other coefficients compose the equations  $\operatorname{curl} \mathbf{E} + \partial^0 \mathbf{H} = 0$  and  $\operatorname{div} \mathbf{H} = 0$ , respectively.

Further, since the element  $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$  commutes with all other elements of the biquaternion (5) and  $\gamma_5^2 = -1$ , then we can rewrite (5) as follows

$$\phi = (a^0 + \gamma_{0123}a^{0123}) + (a^{01} + \gamma_{0123}a^{23})\gamma_{01} + (a^{02} + \gamma_{0123}a^{31})\gamma_{02} + (a^{03} + \gamma_{0123}a^{12})\gamma_{03}$$

or taking into account (13) we obtain

$$\begin{aligned} \phi &= (E^1 + iH^1)\gamma_{01} + (E^2 + iH^2)\gamma_{02} + (E^3 + iH^3)\gamma_{03} \\ &= F_1\gamma_{01} + F_2\gamma_{02} + F_3\gamma_{03}. \end{aligned}$$

In the matrix form we have

$$\phi = \begin{pmatrix} 0 & 0 & F_3 & F_1 - iF_2 \\ 0 & 0 & F_1 + iF_2 & -F_3 \\ F_3 & F_1 - iF_2 & 0 & 0 \\ F_1 + iF_2 & -F_3 & 0 & 0 \end{pmatrix}$$

Then from (11) the relation immediately follows between spinors  $\psi \in \mathbb{C}_4$  and  $\phi \in \mathcal{O}_{3,0}$

$$\psi = \phi \frac{1}{2}(1 + \gamma_0) \frac{1}{2}(1 + i\gamma_1\gamma_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ F_3 & 0 & 0 & 0 \\ F_1 + iF_2 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ F_1 \\ F_2 \\ F_3 \end{pmatrix} \quad (15)$$

Let us consider now an action of the antiautomorphism  $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$  on the arbitrary element of  $\mathcal{O}_{3,0}$  represented by a formula

$$\mathcal{A} = (a^0 + \omega a^{123})\mathbf{e}_0 + (a^1 + \omega a^{23})\mathbf{e}_1 + (a^2 + \omega a^{31})\mathbf{e}_2 + (a^3 + \omega a^{12})\mathbf{e}_3. \quad (16)$$

The action of the antiautomorphism  $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$  on the homogeneous element  $\mathcal{A}$  of a degree  $k$  is defined by a formula  $\tilde{\mathcal{A}} = (-1)^{\frac{k(k-1)}{2}}\mathcal{A}$ . Thus, for the element (16) we obtain

$$\mathcal{A} \longrightarrow \tilde{\mathcal{A}} = (a^0 - \omega a^{123})\mathbf{e}_0 + (a^1 - \omega a^{23})\mathbf{e}_1 + (a^2 - \omega a^{31})\mathbf{e}_2 + (a^3 - \omega a^{12})\mathbf{e}_3.$$

Therefore, under action of  $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$  the element (13) takes a form

$$\widetilde{(\nabla \mathcal{A})} = (E^1 - iH^1)\mathbf{e}_1 + (E^2 - iH^2)\mathbf{e}_2 + (E^3 - iH^3)\mathbf{e}_3$$

and

$$\tilde{\phi} = \begin{pmatrix} 0 & 0 & \overset{*}{F}_3 & \overset{*}{F}_1 - i\overset{*}{F}_2 \\ 0 & 0 & \overset{*}{F}_1 + i\overset{*}{F}_2 & -\overset{*}{F}_3 \\ \overset{*}{F}_3 & \overset{*}{F}_1 - i\overset{*}{F}_2 & 0 & 0 \\ \overset{*}{F}_1 + i\overset{*}{F}_2 & -\overset{*}{F}_3 & 0 & 0 \end{pmatrix}.$$

Hence it immediately follows

$$\tilde{\psi} = \tilde{\phi} \frac{1}{2}(1 + \gamma_0) \frac{1}{2}(1 + i\gamma_1\gamma_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \overset{*}{F}_3 & 0 & 0 & 0 \\ \overset{*}{F}_1 + i\overset{*}{F}_2 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ \overset{*}{F}_1 \\ \overset{*}{F}_2 \\ \overset{*}{F}_3 \end{pmatrix} \tag{17}$$

From (15) and (17) it follows that the full representation space  $\mathbb{S}_4 \cup \dot{\mathbb{S}}_4$  is reduced to a 3-dimensional symmetric space  $\text{Sym}_{(2,0)} \cup \text{Sym}_{(0,2)}$ . The transition from operator spinors to  $SO(3)$  vectors has been done by several authors (see, for example [21]). The ‘vectors’ (spintensors) of the spaces  $\text{Sym}_{(2,0)}$  and  $\text{Sym}_{(0,2)}$  are

$$\begin{aligned} f^{00} &= \xi^0 \otimes \xi^0, & f^{01} &= f^{10} = \xi^0 \otimes \xi^1 = \xi^1 \otimes \xi^0, & f^{11} &= \xi^1 \otimes \xi^1, \\ f^{\dot{0}\dot{0}} &= \xi^{\dot{0}} \otimes \xi^{\dot{0}}, & f^{\dot{0}\dot{1}} &= f^{\dot{1}\dot{0}} = \xi^{\dot{0}} \otimes \xi^{\dot{1}} = \xi^{\dot{1}} \otimes \xi^{\dot{0}}, & f^{\dot{1}\dot{1}} &= \xi^{\dot{1}} \otimes \xi^{\dot{1}}. \end{aligned} \tag{18}$$

It is well-known [27, 28, 29] that spintensors (18) correspond to a Helmholtz–Silberstein representation  $\mathbf{F} = \mathbf{E} + i\mathbf{H}$  and form a basis of the Majorana–Oppenheimer quantum electrodynamics [17, 20, 18, 11, 25, 7, 5] in which the electromagnetism has to be considered as the wave mechanics of the photon. Moreover, the field  $(1, 0) \cup (0, 1)$  satisfies the Weinberg Theorem.

In such a way, it is easy to see that

$$\begin{aligned} \text{Weyl–Hestenes neutrino field} & \quad (1/2, 0) \cup (0, 1/2), \\ \text{Dirac electron–positron field} & \quad (1/2, 0) \oplus (0, 1/2), \\ \text{Maxwell electromagnetic field} & \quad (1, 0) \cup (0, 1) \sim \tag{19} \\ & \quad (1/2, 0) \otimes (1/2, 0) \cup (0, 1/2) \otimes (0, 1/2) \end{aligned}$$

have the same mathematical structure. Namely, all these fields are obtained from the fundamental field  $(1/2, 0)$   $((0, 1/2))$  by means of unification, direct sum and tensor product. From this point of view a division of all the physical fields into ‘gauge fields’ and ‘matter fields’ has an artificial character. It is obvious that an origin of such a division takes its beginning from the invalid description (from group theoretical viewpoint) of electromagnetic field in quantum field theory (a la Gupta–Bleuler approach) and a groundless extension of the Yang–Mills idea (as known, a long derivative is equivalent to a minimal interaction in the lagrangian). Further, as follows from (20) the Maxwell field has a composite structure that gives rise to a de Broglie–Jordan neutrino theory of light [2, 14], in which the electromagnetic field is constructed via the two neutrino fields.

As known, the neutrino theory of light has a long and dramatic history (see excellent review [6, 22]). The main obstacle that decelerates the development of the neutrino theory of light is a Pryce’s Theorem [24] (see also a Berezinskii modification of this Theorem [3]). In 1938, Pryce claimed the incompatibility of obtaining transversely–polarized photons and Bose statistics from neutrinos. However, as it has been recently shown by Perkins [22] the Pryce’s Theorem contains an assumption which is unsupported and probably incorrect. The Perkins arguments based on the experimental similarities between bosons and quasi–bosons allow to overcome this obstacle (Pryce’s Theorem). In the present paper we show that transversely–polarized photons follow naturally from composite neutrinos (it is a direct consequence of the group theoretical consideration). It is obvious that such photons are quasi–bosons [23] that do not satisfy the Bose statistics, but they satisfy the Lipkin statistics of a composite particle such as the deuteron and Cooper pairs [15]. In conclusion, one can say that a correct description of electromagnetic field is a synthesis of the Majorana–Oppenheimer quantum electrodynamics and de Broglie–Jordan neutrino theory of light.

### Acknowledgments

I am grateful to Prof. E. Recami for sending me unpublished Majorana works.

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(Manuscrit reçu le 11 septembre 2001)