A Reconciliation of Electromagnetism and Gravitation

B.G. Sidharth

Centre for Applicable Mathematics and Computer Sciences
B.M.Birla Science Centre,Hyderabad, 500063,India

ABSTRACT. It is argued that once we consider the underpinning of a Non Commutative geometry, itself symptomatic of extended particles, for example in Quantum Superstring theory, then a reconciliation between gravitation and electromagnetism is possible.

1 Introduction

Despite nearly a century of work, it has not been possible to achieve a unification of gravitation and electromagnetism. It must be borne in mind that the tools used, be it Quantum Theory or General Relativity are deeply entrenched in differentiable space time manifolds - the former with Minkowski space time and the latter with curved space time. The challenge has been, as Wheeler noted[1], the introduction of Quantum Mechanical spin half into General Relativity on the one hand and the introduction of curvature into Quantum Mechanics on the other.

More recent models including Quantum Superstrings on the contrary deal with extended and not point particles and lead to a non-differentiable spacetime and a non commutative geometry (NCG)[2, 3, 4, 5].

Indeed way back in the 1930s, Einstein himself observed[6] "...it has ben pointed out that the introduction of a space-time continuum may be considered as contrary to nature in view of the molecular structure of everything which happens on a small scale. It is maintained that perhaps the success of the Heisenberg method points to a purely algebraic method of description of nature that is to the elimination of continuous functions from physics. Then however, we must also give up, by principle the space-time continuum. It is not unimaginable that human ingenuity

will some day find methods which will make it possible to proceed along such a path."
Even at the beginning of the twentieth century several physicists including Poincare and Abraham amongst others were working unsuccessfully with the problem of the extended electron[7, 8]. The problem was that an extended electron appeared to contradict Special Relativity, while on the other hand, the limit of a point particle lead to inexplicable infinities. These infinities dogged physics for many decades. Infact the Heisenberg Uncertainity Principle straightaway leads to infinities in the limit of spacetime points. It was only through the artifice of renormalization that 't Hooft could finally circumvent this vexing problem, in the 1970s[9].
Nevertheless it has been realized that the concept of spacetime points is only approximate. We are beginning to realize that it may be more meaningful to speak in terms of spacetime foam, strings, branes, non commutative geometry, fuzzy spacetime and so on[10].
Indeed non commutativity arises if there is a minimum space time length as shown a long time ago by Snyder[11]. What we will argue below is that once the underlying non commutative nature of the geometry is recognized then it is possible to reconcile electromagnetism and gravitation.

2 NCG

It is well known that once we consider non zero minimum space time intervals or equivalently extended particles as in Quantum Superstrings, then consistent with Lorentz invariance, we have the following non commutative geometry (Cf.refs.[2]-[5],[11]):

$$[x, y] = 0(l^2), [x, p_x] = \imath\hbar[1 + l/\hbar)^2 p_x^2] etc. \tag{1}$$

(and similar equations) where l, τ are the extensions of the space time coordinates. This result of Snyder has been brought back into reckoning in recent years by several scholars.
In conventional theory the space time coordinates as also the momenta commute amongst themselves unlike in equation (1). It must be observed that the non commutative relations are self evident, in the sense that xy or yx is each of the order of l^2, and so is their difference because of the non commutativity.
The non commutative or in Witten's words[12], Fermionic feature is symptomatic of the breakdown of the concept of the spacetime points

and point particles at small scales or high energies. As has been noted by Snyder, Witten, and several other scholars, the divergences encountered in Quantum Field Theory are symptomatic of precisely such an extrapolation to spacetime points and which necessitates devices like renormalization.

Interestingly it has been shown that the commutation relations (1) lead directly to the Dirac equation, on the one hand[13]. On the other hand, it is interesting that a differential calculus over a non commutative algebra uniquely determines a gravitational field in the commutative limit and that there is a unique metric which remains as a classical "shadow" as shown by Madore[14].

Let us now introduce this effect into the usual distance formula in flat space

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

Rewriting the product of the two coordinate differentials in (2) in terms of the symmetric and non symmetric combinations, we get for the right side $\frac{1}{2} g_{\mu\nu}[(dx^\mu dx^\nu + dx^\nu dx^\mu) + (dx^\mu dx^\nu - dx^\nu dx^\mu)]$, so that, we can write

$$g_{\mu\nu} = \eta_{\mu\nu} + kh_{\mu\nu} \tag{3}$$

where the first term on the right side of (3) denotes the usual flat space time and the second term denotes the effect of the non commutativity, k being a suitable constant.

It must be noted that if $l, \tau \to 0$ then equations (1) and also (3) reduce to the usual formulation. From a physical point of view, if we are dealing with time and length scales much greater than the Compton wavelength, so that the order $0(l^2)$ terms can be neglected, then the usual commutative geometry works, with the usual derivatives and more generally differential geometry. In that sense, and at such scales we can attribute the same meaning to coordinate differentials like dx^μ. However this formulation breaks down at and inside the scale (l, τ). In what follows, in order to see the effect of the non commutative geometry, we will consider scales, near the minimum (l, τ) scale, and continue to use the concept of derivatives and differentials, incorporating the effects of departure from the commutative geometry by using the equation (1).

The effect of the non commutative geometry is therefore to introduce a departure from flat space time, as can be seen from (3). Indeed, as is well known (Cf.ref.[15]), this is exactly as in the case of General Relativity and the second term on the right of (3) playing the role of the usual energy momentum tensor. However it must be borne in mind that we are

now dealing with elemnetary particles. For an elementary particle, the material density vanishes outside its Compton wavelength and therefore also the minimum scale. On the other hand it should be borne in mind that at and near the minimum scale itself we have the departure from the usual commutative geometry, as can be seen from (1).
Infact remembering that the second term of the right side of (3) is small, this can straightaway be seen to lead to a linearized theory of General Relativity[15]. Exactly as in this reference we could now deduce the General Relativistic relation

$$\partial_\lambda \partial^\lambda h^{\mu\nu} - (\partial_\lambda \partial^\nu h^{\mu\lambda} + \partial_\lambda \partial^\mu h^{\nu\lambda})$$

$$-\eta^{\mu\nu}\partial_\lambda \partial^\lambda h + \eta^{\mu\nu}\partial_\lambda \partial_\sigma h^{\lambda\sigma} = -k\bar{T}^{\mu\nu} \quad (4)$$

It must be mentioned that the energy momentum type term on the right side of (4) arises due to the fact that the derivatives ∂^λ and ∂^μ no longer commute and this leads to an additional contribution as can be verified from the left side of (4). To show this special origin of the right side term, we have used $\bar{T}$ instead of the usual T. More explicitely, it follows from the foregoing that (Cf.ref.[16])

$$\frac{\partial}{\partial x^\lambda}\frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\lambda} \quad \text{goes} \quad \text{over} \quad \text{to} \frac{\partial}{\partial x^\lambda}\Gamma^\nu_{\mu\nu} - \frac{\partial}{\partial x^\mu}\Gamma^\nu_{\lambda\nu} \quad (5)$$

Normally in conventional theory the right side of (5) would vanish. Let us designate this nonvanishing part on the right by

$$\frac{e}{c\hbar}F^{\mu\lambda} \quad (6)$$

We have shown here that the non commutativity in momentum components leads to an effect that can be identified with electromagnetism and infact from expression (6) we have

$$A^\mu = \hbar \Gamma^{\mu\nu}_\nu \quad (7)$$

where A_μ can be identified with the electromagnetic four potential (Cf. also ref.[16]). To see this in the light of the usual guage invariant minimum coupling (Cf.ref.[5]), we start with the effect of an infinitessimal parallel displacement of a vector in this non commutative geometry,

$$\delta a^\sigma = -\Gamma^\sigma_{\mu\nu} a^\mu dx^\nu \quad (8)$$

As is well known, (8) represents the effect due to the curvature and non integrable nature of space - in a flat space, the right side would vanish. Considering the partial derivatives with respect to the μ^{th} coordinate, this would mean that, due to (8)

$$\frac{\partial a^\sigma}{\partial x^\mu} \to \frac{\partial a^\sigma}{\partial x^\mu} - \Gamma^\sigma_{\mu\nu} a^\nu$$

The second term on the right side can be written as:

$$-\Gamma^\lambda_{\mu\nu} g^\nu_\lambda a^\sigma = -\Gamma^\nu_{\mu\nu} a^\sigma$$

where we have utilised equation (3). That is we have

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x^\mu} - \Gamma^\nu_{\mu\nu}$$

Comparison with (7) establishes the required identification.
It is quite remarkable that equation (7) is mathematically identical to Weyl's unified formulation, though this was not originally acceptable because of the adhoc insertion of the electromagnetic potential. Here in our case it is a consequence of the non commutative geometry (1) (Cf.refs.[5] and [16] for a detailed discussion).
We can see this in greater detail as follows. The gravitational field equations can be written as[15]

$$\bar{\Box}\phi^{\mu\nu} = -k\bar{T}^{\mu\nu} \tag{9}$$

where

$$\phi^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h \tag{10}$$

It also follows, if we use the usual gauge and equation (7) that

$$\partial_\mu h^{\mu\nu} = A^\nu \tag{11}$$

in this linearised theory.
Whence, remembering that we have (3), operating on both sides of equation (9) with ∂_μ we get Maxwell's equations of electromagnetism.
This is not surprising because as is well known if equation (7) holds as in the Weyl formulation, then in the absence of matter the general relativistic field equations (4) reduce to Maxwell equations[17]. In any case, all this provides a rationale for the fact that from (9) we get the equation for spin 2 gravitons (Cf.ref.[15]) while from the Maxwell equations, we have Spin 1 (vector) photons.

3 Discussion

1. The characterization of the metric in equations (2) and (3) in terms of symmetric and non symmetric components is similar to the torsional formulation of General Relativity[18]. However in this latter case, there is no contribution to the differential interval from the torsional (that is non-commutative) effects. The non-commutative contribution is given by (1) and herein comes the extended, rather than point like particle.
In any case the above attempt at unification of electromagnetism and gravitation had made part headway, but unless the underpinning of a non commutative geometry is recognised, the full significance does not manifest itself.
2. We now make the following remarks:
We know that the minimum space time intervals are at or below the Compton scale where the momentum p equals mc. For a Planck mass $\sim 10^{-5} gms$, this is also the Planck scale, as in Quantum Superstring theory.
In Snyder's original work, the commutation relations like (1) hold good outside the minimum space time intervals, and are Lorentz invariant. This is quite pleasing because in any case, even in Quantum Field Theory, we use Minkowski space time.
3. The above non commutative geometry also holds the key to the mysterious extra dimensions of Quantum Superstrings. This has been discussed in detail in references[5, 19]. But to see in a simple way, we note that equation (1) shows that the coordinates y and z show up as some sort of a momenta, though with a different multiplying constant as the analogue of the Planck constant. That is instead of the single x momentum, p_x, we have two extra momenta, this being the same for the y and z momenta also. This leads to the well known $9+1$ dimensions of Quantum superstrings, though because for all these extra "momenta", the multiplying factor, the analogue of the Planck constant is different, so these extra dimensions are supressed or curled up in the Kaluza-Klein sense.
4. A concept which one encounters in Quantum SuperString theory and more generally in the presence of the Non commutative geometry (1) is that of Duality. We will briefly examine this now and see its significance in relation to electrodynamic theory. Infact the relation (1) leads to[3, 19],

$$\Delta x \sim \frac{\hbar}{\Delta p} + \alpha' \frac{\Delta p}{\hbar} \tag{12}$$

where $\alpha' = l^2$, which in Quantum SuperStrings Theory $\sim 10^{-66} cm^2$. This is an expression of the duality relation,

$$R \to \alpha'/R$$

This is symptomatic of the fact that we cannot go down to arbitrarily small spacetime intervals, below the Planck scale in this case but that the macro universe is connected with the micro universe or in Witten's words, "when one accelerates past the string scale - instead of probing short distances one just watches the propagation of large strings." (Cf.ref.[3]).
In this light, an interesting meaning to the duality relation arising from (12) has been discussed in [19, 20].
We will now see a curious connection between the foregoing micro-macro link with the apparently disparate concept of the Feynman-Wheeler action at a distance theory, which had been quite successful.
Our starting point is the so called Lorentz-Dirac equation[8]:

$$ma^{\mu} = F^{\mu}_{in} + F^{\mu}_{ext} + \Gamma^{\mu} \tag{13}$$

where

$$F^{\mu}_{in} = \frac{e}{c} F^{\mu v}_{in} v_v$$

and similarly

$$F^{\mu}_{ext} = \frac{e}{c} F^{\mu\nu}_{ext} v_{\nu}$$

and Γ^{μ} is the Abraham radiation reaction four vector related to the self force and, given by

$$\Gamma^{\mu} = \frac{2}{3}\frac{e^2}{c^3}(\dot{a}^{\mu} - \frac{1}{c^2} a^{\lambda} a_{\lambda} v^{\mu}) \tag{14}$$

Equation (13) is the relativistic generalisation for a point electron of an earlier equation proposed by Lorentz, while equation (14) is the relativisitic generalisation of the original radiation reaction term due to energy loss by radiation. It must be mentioned that the mass m in equation (13) consists of a neutral mass and the original electromagnetic mass of Lorentz, which latter does tend to infinity as the electron shrinks to a point, but, this is absorbed into the neutral mass. Thus we have the forerunner of renormalisation in quantum theory.
There are three unsatisfactory features of the Lorentz-Dirac equation (13).

Firstly the third derivative of the position coordinate in (13) through Γ^μ gives a whole family of solutions. Except one, the rest of the solutions are run away - that is the velocity of the electron increases with time to the velocity of light, even in the absence of any forces. This energy can be thought to come from the infinite self energy we get when the size of the electron shrinks to zero. If we assume adhoc an asymptotically vanishing acceleration then we get a physically meaningful solution, though this leads to a second difficulty, that of violation of causality of even the physically meaningful solutions.

It has been shown in detail elsewhere[7] that these acausal, non local effects take place within the Compton time.

We now come to the Feynman-Wheeler action at a distance theory[21, 22]. They showed that the apparent acausality of the theory would disappear if the interaction of a charge with all other charges in the universe, such that the remaining charges would absorb all local electromagnetic influences was considered. The rationale behind this was that in an action at a distance context, the motion of a charge would instantaneously affect other charges, whose motion in turn would instantaneously affect the original charge. Thus considering a small interval in the neighbourhood of the point charge, they deduced,

$$F^\mu_{ret} = \frac{1}{2}\{F^\mu_{ret} + F^\mu_{adv}\} + \frac{1}{2}\{F^\mu_{ret} - F^\mu_{adv}\} \tag{15}$$

The left side of (15) is the usual causal field, while the right side has two terms. The first of these is the time symmetric field while the second can easily be identified with the Dirac field above and represents the sum of the responses of the remaining charges calculated in the vicinity of the said charge. Also here we encounter effects within the Compton scale (Cf. ref.[7]) of the rest of the universe. We thus return to the concept from Quantum Superstring theory, or more generally a theory based on relations like (1) of extended particles and duality, a manifestation of holism.

5. One could argue that the non commutative relations (1) are an expression of Quantum Mechanical spin. To put it briefly, for a spinning particle the non commutativity arises when we go from canonical to covariant position variables. Zakrzewsk[23] has shown that we have the Poisson bracket relation

$$\{x^j, x^k\} = \frac{1}{m^2} R^{jk}, (c = 1),$$

where R^{jk} is the spin. The passage to Quantum Theory then leads us back to the relation (1).

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