Maxwell's theory on non-commutative spaces and quaternions

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ABSTRACT. The Maxwell theory on non-commutative spaces has been considered. The non-linear equations of electromagnetic fields on non-commutative spaces were obtained in the compact spin-tensor (quaternion) form. It was shown that the plane electromagnetic wave is the solution of the system of non-linear wave equations of the second order for the electric and magnetic induction fields. We have found the canonical and symmetrical energy-momentum tensors and their non-zero traces. So, the trace anomaly of the energy-momentum tensor was obtained in electrodynamics on non-commutative spaces. It was noted that the dual transformations of electromagnetic fields on non-commutative spaces are broken.

1 Introduction

The field models on non-commutative (NC) spaces are of great interest now due to the recent development of the superstring theory. It was shown that NC coordinates emerge naturally in the perturbative version of the D-brane theory (low energy excitations of a D-brane) with the presence of the external background magnetic field [1]. So, non-commutative Yang-Mills (NCYM) theories appear in the string theory and therefore they are being widely investigated. The NC field theories have the same degrees of freedom as effective commutative theories and, therefore, there exists a map (the Seiberg-Witten map between NC field theory and the corresponding commutative field theory) between them. The simplest theory with the gauge group $U(1)$ is QED and its prototype - NC quantum electrodynamics (NCQED). The investigation of gauge theories on NC geometry leads to the non-local interactions of fields due to the presence of higher derivatives in the Lagrangian.

The distinctive features of the NC theory are the appearance of the dipole moments of particles at one loop level and the violation of the CP-symmetry [2]. So, in NCQED the "electron" possesses the magnetic dipole moment which contains the spin-independent term (proportional to the non-commutative parameter θ) and the electric dipole moment violating the CP-symmetry, but it should be noted that the CPT-symmetry remains unbroken [3]. At the charge conjugation the theory transfers to the sector with $\theta \to -\theta$. At one-loop level NCQED is a renormalizable [4,5] and asymptotic free (the β-function is negative and is not θ dependent) theory [6]. Besides, the parameter θ does not acquire the quantum corrections. The non-commutative version of a standard model was considered in [2,7]. It should be mentioned that NC field theories possess unitarity at the space-like non-commutative tensor $\theta_{\mu\nu}$ ($\theta_{0j} = 0$) [8,9]. There is infrared-ultraviolet (IR/UV) mixing in NC theories, and if we remove the UV divergences at cut-off$\to \infty$, the new IR divergences appear.

The assumption that coordinates do not commute was made a long time ago [10] (see also [11]). The NC coordinates of the corresponding spaces obey the following commutation relation

$$[\widehat{x}_\mu, \widehat{x}_\nu] = i\theta_{\mu\nu}, \tag{1}$$

where the non-commutative parameter $\theta_{\mu\nu}$ possesses the dimension of (length)^2. It is implied that we have ordinary commutative relations between coordinates $\widehat{x}_\mu$ and the momentum $\widehat{p}_\mu$: $[\widehat{x}_\mu, \widehat{p}_\nu] = i\hbar\delta_{\mu\nu}$, $[\widehat{p}_\mu, \widehat{p}_\nu] = 0$. The astro-physical bounds on the NC scale Λ_{NC} [7,12] are given by

$$\theta_{\mu\nu} = \frac{1}{\Lambda_{NC}^2}\epsilon_{\mu\nu}, \qquad \Lambda_{NC} \geq 10^3 \text{ GeV}, \tag{2}$$

where $\epsilon_{\mu\nu}$ is a dimensionless antisymmetric tensor, $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$. The parameter $\theta_{\mu\nu}$ is extremely small, and, therefore, observable effects can emerge only at the cosmological scale (of the order of the Plank length), i.e. at high energy. As $\theta_{\mu\nu}$ is a constant tensor, the Lorentz symmetry is broken for field theories on NC geometry. It was noted in [7] that at the replacement

$$x_i = \widehat{x}_i + \frac{1}{2\hbar}\theta_{ij}\widehat{p}_j, \qquad p_j = \widehat{p}_j, \tag{3}$$

we arrive at the standard commutation relationships: $[x_i, x_j] = 0$, $[x_i, p_j] = i\hbar\delta_{ij}$, $[p_i, p_j] = 0$. The non-local character of field theories on NC geometry follows from Eq. (3).

The field operators $\widehat{A}(\widehat{x})$ in field theories on NC geometry are the functions of $\widehat{x}_\mu$. After the Fourier transformation we have

$$\widehat{A}(\widehat{x}) = \frac{1}{(2\pi)^4}\int d^4p \exp(ip_\mu\widehat{x}_\mu)A(p), \qquad A(p) = \int d^4x \exp(-ip_\mu x_\mu)A(x). \tag{4}$$

Then the product of two field operators $\widehat{A}(\widehat{x})$, $\widehat{B}(\widehat{x})$ can be represented as

$$\widehat{A}(\widehat{x})\widehat{B}(\widehat{x}) = \int \frac{d^4p}{(2\pi)^4}\int \frac{d^4k}{(2\pi)^4} \exp\left\{i(p_\mu + k_\mu)x_\mu - \frac{1}{2}p_\mu k_\nu\left[\widehat{x}_\mu, \widehat{x}_\nu\right]\right\} A(p)B(k) \tag{5}$$

$$= \left[\exp\left(\frac{i}{2}\theta_{\mu\nu}\partial_\mu\partial'_\nu\right) A(x)B(x')\right] |_{x=x'},$$

where $\partial_\mu = \partial/\partial x_\mu$, $\partial'_\nu = \partial/\partial x'_\nu$. Thus we come to the Weil-Moyal correspondence [13,14]:

$$\widehat{A}(\widehat{x})\widehat{B}(\widehat{x}) \longleftrightarrow A(x) \star B(x), \tag{6}$$

where the star-product ($\star$-product) is given by

$$A(x) \star B(x) = \left[\exp\left(\frac{i}{2}\theta_{\mu\nu}\partial_\mu\partial'_\nu\right) A(x)B(x')\right] |_{x=x'} . \tag{7}$$

Using Eq. (7) it is easy to check that quadratic terms in the actions of NC theories (kinetic terms) coincide with that of their commutative versions, i.e., propagators are identical. The star-product also satisfies the associative low: $(F \star G) \star H = F \star (G \star H)$.

We use the system of units $\hbar = c = 1$, $e^2/4\pi = 1/137$, $e > 0$.

2 Field equations

The free Maxwell action on NC space [5] is given by

$$S = -\frac{1}{4}\int d^4x F_{\mu\nu} \star F_{\mu\nu} = -\frac{1}{4}\int d^4x \widehat{F}^2_{\mu\nu}, \tag{8}$$

were non-commutative strength $\widehat{F}_{\mu\nu}$ reads

$$\widehat{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie\left[A_\mu, A_\nu\right]_M , \tag{9}$$

and the Moyal bracket is

$$\left[A_\mu, A_\nu\right]_M = A_\mu \star A_\nu - A_\nu \star A_\mu. \tag{10}$$

The Seiberg-Witten expansion to the first order in $\theta_{\mu\nu}$ [5] gives

$$\begin{aligned} \widehat{A}_\mu &= A_\mu - \frac{1}{2}\theta_{\alpha\beta}A_\alpha\left(\partial_\beta A_\mu + F_{\beta\mu}\right), \\ \widehat{F}_{\mu\nu} &= F_{\mu\nu} + \theta_{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \theta_{\alpha\beta}A_\alpha\partial_\beta F_{\mu\nu}, \end{aligned} \tag{11}$$

with e absorbed in $\theta_{\alpha\beta}$. The field strength tensor corresponding to the commutative Maxwell theory is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{12}$$

with the vector-potential of the electromagnetic field A_μ, the electric field $E_i = iF_{i4}$ and the magnetic induction field $B_i = \epsilon_{ijk}F_{jk}$ ($\epsilon_{123} = 1$). The Lagrangian within four-dimensional divergences (see [5]) is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{8}\theta_{\alpha\beta}F_{\alpha\beta}F_{\mu\nu}^2 - \frac{1}{2}\theta_{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}F_{\mu\nu} + \mathcal{O}(\theta^2) + A_\mu J_\mu, \tag{13}$$

where we added the external four-current J_μ and took into consideration that the term $A_\mu \star J_\mu$ in the Maxwell Lagrangian on NC spaces coincides within four-divergences with $A_\mu J_\mu$. The Lagrangian (13) can also be cast in the form of

$$\mathcal{L} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right)\left[1 + (\theta\cdot\mathbf{B})\right] - (\theta\cdot\mathbf{E})\left(\mathbf{E}\cdot\mathbf{B}\right) + \mathcal{O}(\theta^2) + A_\mu J_\mu, \tag{14}$$

where $\theta_i = (1/2)\epsilon_{ijk}\theta_{jk}$, $\theta_{i4} = 0$. It is seen from Eq. (14) that terms containing the non-commutative parameter θ violate CP - symmetry. Using the Lagrange-Euler equations

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)} - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0, \tag{15}$$

we obtain from Eq. (13) field equations (equations of motion)

$$\partial_\mu F_{\nu\mu} + \frac{1}{2}\theta_{\alpha\beta}\partial_\mu\left(F_{\mu\nu}F_{\alpha\beta}\right) + \frac{1}{4}\theta_{\mu\nu}\partial_\mu\left(F^2_{\alpha\beta}\right)$$
$$-\theta_{\nu\beta}\partial_\mu\left(F_{\alpha\beta}F_{\mu\alpha}\right) + \theta_{\mu\beta}\partial_\mu\left(F_{\alpha\beta}F_{\nu\alpha}\right) - \theta_{\alpha\beta}\partial_\mu\left(F_{\mu\alpha}F_{\nu\beta}\right) = J_\nu . \tag{16}$$

The non-linear equations (16) may be cast as follows [15]:

$$\frac{\partial}{\partial t}\mathbf{D} - \text{rot}\mathbf{H} = -\mathbf{J}, \qquad \text{div}\mathbf{D} = \rho, \tag{17}$$

where $(\text{rot}\mathbf{H})_i = \epsilon_{ijk}\partial_j H_k$ and $\text{div}\mathbf{D} = \partial_i D_i$; $\mathbf{J}$ is the vector of a current and ρ is a charge density, $J_\mu=(\mathbf{J}, i\rho)$. The displacement ($\mathbf{D}$) and magnetic ($\mathbf{H}$) fields are given by

$$\mathbf{D} = \mathbf{E} + \mathbf{d}, \qquad \mathbf{d} = (\theta\cdot\mathbf{B})\mathbf{E} - (\theta\cdot\mathbf{E})\mathbf{B} - (\mathbf{E}\cdot\mathbf{B})\theta, \tag{18}$$

$$\mathbf{H} = \mathbf{B} + \mathbf{h}, \qquad \mathbf{h} = (\theta\cdot\mathbf{B})\mathbf{B} + (\theta\cdot\mathbf{E})\mathbf{E} - \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right)\theta. \tag{19}$$

Here the scalar products of vectors are introduced, $(\theta\cdot\mathbf{E}) = \theta_i E_i$, and so on. The other equation following from Eq. (12) is

$$\partial_\mu \widetilde{F}_{\mu\nu} = 0, \tag{20}$$

where $\widetilde{F}_{\mu\nu} = (1/2)\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$ is the dual tensor, $\varepsilon_{\mu\nu\alpha\beta}$ is an antisymmetric tensor Levy-Civita; $\varepsilon_{1234} = -i$. Eq. (20) is rewritten as

$$\frac{\partial}{\partial t}\mathbf{B} + \text{rot}\mathbf{E} = 0, \qquad \text{div}\mathbf{B} = 0. \tag{21}$$

Let us obtain the second order equations for electric and magnetic fields. Such wave equations are convenient for studying the propagation of electromagnetic fields. Applying the operator rot to the first equation of (17) with the help of the equality rot rot$\mathbf{H}$ = grad div$\mathbf{H} - \triangle\mathbf{H}$ [$(\text{grad})_i \equiv \partial/\partial x_i$, $\triangle \equiv \partial^2/(\partial x_i)^2$], replacing in this equation rot$\mathbf{E}$ from Eq. (21) and taking into account Eqs. (18), (19), we find

$$\triangle\mathbf{B} - \frac{\partial^2}{(\partial t)^2}\mathbf{B} + \triangle\mathbf{h} - \text{grad div}\mathbf{h} + \frac{\partial}{\partial t}\text{rot}\mathbf{d} = -\text{rot}\mathbf{J}. \tag{22}$$

We repeat this procedure, starting with the first equation of (21) and taking into consideration Eqs. (17)-(19), one obtains

$$\triangle\mathbf{E}-\frac{\partial^2}{(\partial t)^2}\mathbf{E}-\frac{\partial^2}{(\partial t)^2}\mathbf{d}+\text{grad div}\mathbf{d}+\frac{\partial}{\partial t}\text{rot}\mathbf{h}=\frac{\partial}{\partial t}\mathbf{J}+\text{grad}\rho. \tag{23}$$

It is easy to verify that electric and magnetic induction fields in the form of plane electromagnetic waves, such as

$$\mathbf{E}=\mathbf{E}_0\exp(ik_\mu x_\mu), \qquad \mathbf{B}=(\mathbf{n}\times\mathbf{E}_0)\exp(ik_\mu x_\mu), \tag{24}$$

where $\mathbf{n}=\mathbf{k}/k_0$, $k_\mu=(\mathbf{k},ik_0)$ $[(\mathbf{n}\times\mathbf{E}_0)_i=\epsilon_{ijk}n_jE_{0k}$ is the vector product], are the solutions of the linear wave equations of classical electrodynamics as well non-linear Eqs. (22), (23) at $\mathbf{J}=0$, $\rho=0$. It was shown in [15] that in the theory under consideration the velocity of propagation when transverse to a background magnetic induction field differs from c. This is a consequence of the non-linearity of field equations. But this effect of the photon propagation is very small due to the smallness of the non-commutative parameter θ.

It is easy to see that free NC Maxwell's equations (16), (20) at $J_\mu=0$ are not invariant under the dual transformations of electromagnetic fields [16]

$$\begin{aligned} F'_{\mu\nu}&=F_{\mu\nu}\cos\alpha-\widetilde{F}_{\mu\nu}\sin\alpha,\\ \widetilde{F}'_{\mu\nu}&=\widetilde{F}_{\mu\nu}\cos\alpha+F_{\mu\nu}\sin\alpha. \end{aligned}\tag{25}$$

The terms containing the parameter $\theta_{\mu\nu}$ in Eq. (16) violate the dual symmetry (25), and the condition $\theta_{\mu\nu}=0$ (at $J_\mu=0$) recovers the dual symmetry of standard Maxwell's equations. It should be noted that dual transformations (25) describe the symmetry of the polarization space. For example, in the case of the plane electromagnetic wave, Eqs. (24), these transformations rotate the polarization axes around the wave vector.

3 Energy and momentum of electromagnetic field on NC spaces

Multiplying Eq. (17) by $\mathbf{E}$, Eq. (21) by $\mathbf{H}$, and adding them, we find

$$\mathbf{E}\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\frac{\partial\mathbf{B}}{\partial t}=-(\mathbf{J}\cdot\mathbf{E})-\text{div}(\mathbf{E}\times\mathbf{H}), \tag{26}$$

Using Eq. (18), (19) one can represent

$$\mathbf{E}\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\frac{\partial \mathbf{B}}{\partial t} = \frac{\partial \mathcal{E}}{\partial t}, \tag{27}$$

where the energy density of the electromagnetic field is given by

$$\mathcal{E} = \frac{\mathbf{E}^2 + \mathbf{B}^2}{2}\left[1 + (\theta\cdot\mathbf{B})\right] - (\mathbf{E}\cdot\mathbf{B})\,(\theta\cdot\mathbf{E})\,. \tag{28}$$

With the help of Eqs. (26)-(28), the conservation law of the energy-momentum reads [17]

$$\frac{\partial \mathcal{E}}{\partial t} = -(\mathbf{J}\cdot\mathbf{E}) - \mathrm{div}\mathbf{P}, \qquad \mathbf{P} = \mathbf{E}\times\mathbf{H}, \tag{29}$$

where $\mathbf{P}$ is the vector of the momentum density of the electromagnetic field, so that the four-vector of the energy-momentum is $P_\mu = (\mathbf{P}, i\mathcal{E})$. It follows from Eq. (29) that at $\mathbf{J} = 0$ the continuity equation $\partial_\mu P_\mu = 0$ holds. The energy density (28) and the momentum density of the electromagnetic field with the accuracy of $O(\theta^2)$ and using Eqs. (18),(19) may be represented as

$$\begin{gathered}\mathcal{E} = \frac{\mathbf{D}^2 + \mathbf{H}^2}{2} - (\theta\cdot\mathbf{B})\,\mathbf{B}^2, \\ \mathbf{P} = \left[1 + (\theta\cdot\mathbf{B})\right](\mathbf{E}\times\mathbf{B}) + \frac{1}{2}\left(\mathbf{B}^2 - \mathbf{E}^2\right)(\mathbf{E}\times\theta).\end{gathered} \tag{30}$$

Now we use the general expression for the canonical conservative energy-momentum tensor [17]:

$$T^{can}_{\mu\nu} = (\partial_\nu A_\alpha)\,\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\alpha\right)} - \delta_{\mu\nu}\mathcal{L}, \tag{31}$$

so that $\partial_\mu T^{can}_{\mu\nu} = 0$. To get the gauge-invariant energy-momentum tensor, we can explore the transformation [17]:

$$T_{\mu\nu} = T^{can}_{\mu\nu} + \Lambda_{\mu\nu}, \tag{32}$$

where the function $\Lambda_{\mu\nu}$ obeys the equation $\partial_\mu\Lambda_{\mu\nu} = 0$ due to equations of motion. Indeed, it follows from Eq. (32) that the tensor $T_{\mu\nu}$ is conservative, i.e., $\partial_\mu T_{\mu\nu} = \partial_\mu T^{can}_{\mu\nu} = 0$. It should be noted that in classical linear electrodynamics this transformation (32) is used to obtain the symmetric energy-momentum tensor. We may choose the following function

$$\Lambda_{\mu\nu} = -\left(\partial_\alpha A_\nu\right)\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\alpha\right)}. \tag{33}$$

It is easy to verify that the function $\Lambda_{\mu\nu}$, (33), meets the requirement $\partial_\mu\Lambda_{\mu\nu} = 0$ due to the Lagrange-Euler equation (15) (at $J_\mu = 0$ the equality $\partial\mathcal{L}/\partial A_\nu = 0$ holds) and the fact that $\partial\mathcal{L}/\partial\left(\partial_\mu A_\alpha\right)$ is anti-symmetric in indexes μ and α. With the help of Eqs. (13), (31)-(33) we obtain the gauge-invariant conservative energy-momentum tensor of electromagnetic fields (at $J_\mu = 0$) on NC spaces

$$\begin{gathered} T_{\mu\nu} = -F_{\mu\alpha}F_{\nu\alpha}\left(1-\frac{1}{2}\theta_{\gamma\beta}F_{\gamma\beta}\right)+\frac{1}{4}\theta_{\mu\alpha}F_{\nu\alpha}F^2_{\rho\beta} \\ -\theta_{\mu\beta}F_{\gamma\nu}F_{\rho\beta}F_{\gamma\rho}-\left(F_{\mu\alpha}F_{\nu\gamma}+F_{\nu\alpha}F_{\mu\gamma}\right)\theta_{\alpha\beta}F_{\gamma\beta}-\delta_{\mu\nu}\mathcal{L}. \end{gathered} \tag{34}$$

The tensor (34) is still non-symmetric, however in the limit $\theta \to 0$, for classical electrodynamics, the energy-momentum tensor (34) becomes symmetric. From Eq. (34) we obtain the components of the energy-momentum tensor

$$T_{44} = \mathcal{E}, \qquad T_{m4} = -iP_m,$$

$$T_{4m} = -i\epsilon_{mnk}\left\{E_nB_k\left[1+(\theta\cdot\mathbf{B})\right]+(\mathbf{E}\cdot\mathbf{B})B_n\theta_k\right\},$$

$$\begin{gathered} T_{mn} = E_mE_n + B_mB_n - \frac{1}{2}\delta_{mn}\left(\mathbf{E}^2+\mathbf{B}^2\right)+(\theta\cdot\mathbf{B})\left(2E_mE_n+B_mB_n\right) \\ +\frac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right)\left[B_m\theta_n-2\delta_{mn}\left(\theta\cdot\mathbf{B}\right)\right]-\left(\theta\cdot\mathbf{E}\right)E_nB_m \\ -\left(\mathbf{E}\cdot\mathbf{B}\right)\left[\theta_mE_n+\theta_nE_m-\delta_{mn}\left(\theta\cdot\mathbf{E}\right)\right]-\left(\mathbf{E}\times\theta\right)_m\left(\mathbf{B}\times\mathbf{E}\right)_n. \end{gathered} \tag{35}$$

Eqs. (35) clearly show that the trace of the energy-momentum tensor does not equal zero, and is given by

$$T_{\mu\mu} = (\theta\cdot\mathbf{B})\left(\mathbf{E}^2-\mathbf{B}^2\right)-2\left(\theta\cdot\mathbf{E}\right)\left(\mathbf{E}\cdot\mathbf{B}\right). \tag{36}$$

Non-zero value of the energy-momentum tensor trace (36) indicates the anomaly in the case of the NC electrodynamics at the classical level. This reflects the fact that the classical conformal invariance is broken. The violation of conformal invariance, or the trace anomaly, relates to the violation of the Lorentz invariance in NC space.

In order to find the symmetric tensor of the energy-momentum, we will explore the general procedure of the curve coordinate system usage [17]. In the curve space-time the Lagrangian (13) with the accuracy of $\mathcal{O}(\theta^2)$, and at $J_\mu = 0$, reads

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}\left(1+\frac{1}{2}\theta_{\gamma\delta}F_{\sigma\rho}g^{\gamma\sigma}g^{\delta\rho}\right)\\ &-\frac{1}{2}\theta_{\alpha\beta}F_{\mu\gamma}F_{\nu\delta}F_{\rho\sigma}g^{\alpha\gamma}g^{\beta\delta}g^{\mu\rho}g^{\nu\sigma}.\end{aligned} \tag{37}$$

We notice also that Eq. (37) looks like the covariant expression, but the covariance is broken because the variable $\theta_{\mu\nu}$ is not transformed as the second rank tensor at the Lorentz transformations. Therefore the action corresponding to the Lagrangian (37) is not a scalar at the transformations of the metric $g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$. Consequently, the conservation of the symmetrical energy-momentum tensor obtained by variation of the Lagrangian (37) on the metric tensor is questionable. Using the general formula for the symmetric energy-momentum tensor (in the case when $\mathcal{L}$ does not depend on $\partial_\alpha g^{\mu\nu}$) [17]

$$T^{sym}_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\partial\sqrt{-g}\mathcal{L}}{\partial g^{\mu\nu}},$$

and varying the Lagrangian (37) on the metric tensor $g^{\mu\nu}$, with the help of equation $g = -1$ for the Minkowski space, we arrive at the symmetric energy-momentum tensor:

$$T^{sym}_{\mu\nu} = T_{\mu\nu} + \frac{1}{4}\theta_{\nu\alpha}F_{\mu\alpha}F^2_{\rho\beta} - \theta_{\nu\beta}F_{\gamma\mu}F_{\rho\beta}F_{\gamma\rho}, \tag{38}$$

where the conservative tensor $T_{\mu\nu}$ is given by Eq. (34). It is easy to check with the help of Eqs. (28), (29), (38) that the equations $T^{sym}_{44} = \mathcal{E}$, $T^{sym}_{m4} = -iP_m$ are valid. From Eq. (38) we also find the spacial components of the symmetric energy-momentum tensor (the stress tensor)

$$T^{sym}_{mn} = E_mE_n + B_mB_n - \frac{1}{2}\delta_{mn}\left(\mathbf{E}^2+\mathbf{B}^2\right) + (\theta\cdot\mathbf{B})\left(3E_mE_n + B_mB_n\right)$$

$$+\frac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right)\left[B_m\theta_n+\theta_m B_n-3\delta_{mn}\left(\theta\cdot\mathbf{B}\right)\right]$$

$$(39)$$

$$-\left(\mathbf{E}\cdot\mathbf{B}\right)\left[\theta_m E_n+\theta_n E_m-\delta_{mn}\left(\theta\cdot\mathbf{E}\right)\right]-\left(\theta\cdot\mathbf{E}\right)\left(E_m B_n+E_n B_m\right)$$

$$-\left(\mathbf{E}\times\theta\right)_m\left(\mathbf{B}\times\mathbf{E}\right)_n-\left(\mathbf{E}\times\theta\right)_n\left(\mathbf{B}\times\mathbf{E}\right)_m.$$

¿From Eqs. (38), (39) we find the trace of the symmetric energy-momentum tensor:

$$T^{sym}_{\mu\mu}=2\left(\theta\cdot\mathbf{B}\right)\left(\mathbf{E}^2-\mathbf{B}^2\right)-4\left(\theta\cdot\mathbf{E}\right)\left(\mathbf{E}\cdot\mathbf{B}\right). \qquad (40)$$

The trace of the symmetric energy-momentum tensor (40) is two times greater than the trace of the conservative energy-momentum tensor (36), i.e. $T^{sym}_{\mu\mu}=2T_{\mu\mu}$. It should be noted that the trace anomaly contributes to the cosmological constant. As a result the trace anomaly might be the source of significant period of inflation in the early universe.

When two Lorentz invariants - $I_1\equiv\mathbf{E}^2-\mathbf{B}^2$ and $I_2\equiv(\mathbf{E}\cdot\mathbf{B})$ equal zero, i.e., $I_1=I_2=0$, in the case of the electromagnetic waves, Eq. (24), the trace anomaly vanishes, $T_{\mu\mu}=T^{sym}_{\mu\mu}=0$. For classical electrodynamics at $\theta=0$ we arrive at the known result that the trace anomaly is absent for any electromagnetic fields, $T^{(\theta=0)}_{\mu\mu}=0$.

4 Spin-tensor form of equations

Sometimes algebraic methods allow us to obtain results in the simplest way. Multiplying Eqs. (21) by i, and adding Eqs. (17), we can write

$$\partial_k f_k=\rho, \qquad (41)$$

$$-i\epsilon_{kmn}\partial_m k_n-\frac{\partial f_k}{\partial t}=J_k, \qquad (42)$$

where $f_k=D_k+iB_k$, $k_n=E_n+iH_n$. Multiplying Eq. (42) by τ_k, and taking into account the properties of Pauli's matrices, τ_k (see Appendix), one arrives at

$$-\tau_p\tau_k\partial_p k_k-i\partial_4\tau_l f_l=\tau_k J_k-\partial_m k_m, \qquad (43)$$

In accordance with Cartan's ideas [18], for every vector we can construct a $2 \times 2-$matrix X (or $\overline{X}$) as follows

$$X = x_\mu \tau_\mu, \qquad \tau_\mu = (\tau_k, \tau_4)\,,$$
$$\overline{X} = x_\mu \overline{\tau}_\mu, \qquad \overline{\tau}_\mu = (-\tau_k, \tau_4)\,, \tag{44}$$

where $\tau_4 = i\tau_0$. With the help of Eqs. (17)-(21), Eq. (43) can be cast in the form of

$$\nabla F + \frac{1}{2}\left(\nabla G + G^+ \overleftarrow{\overline{\nabla}}\right) = -J, \tag{45}$$

where $\nabla = \tau_\mu \partial_\mu$, $G = g_m \tau_m$, $g_m = d_m + ih_m$, $J = J_\mu \tau_\mu$, $J_4 = i\rho$, $F = f_m \tau_m$, $f_4 = 0$; the matrix-differential operator $\overleftarrow{\overline{\nabla}}$ acts on the left standing function, G^+ is Hermitian conjugated matrix. We notice that the complex vector $\mathbf{g} \equiv \mathbf{d} + i\mathbf{h}$ can be represented in the compact form:

$$\mathbf{g} = i\left(\theta \cdot \mathbf{v}^*\right)\mathbf{v} - \frac{i}{2}\theta\left(\mathbf{v}^*\right)^2,$$

where $\mathbf{v} = \mathbf{E} + i\mathbf{B}$, $\mathbf{v}^* = \mathbf{E} - i\mathbf{B}$. The spin-tensor $F = f_\mu \tau_\mu$ may be defined through the potential as follows

$$F = -\overline{\nabla} A, \tag{46}$$

where $A = A_\mu \tau_\mu$, $\overline{\nabla} = \overline{\tau}_\mu \partial_\mu$. From Eq. (46) we arrive at Eq. (12).

The spin-tensor form of NC Maxwell's equations (45) is equivalent to the quaternion form as the quaternion algebra can be realized through the Pauli matrices (see Appendix). At $\theta_{\mu\nu} = 0$ we arrive from Eq. (45) to the quaternion form of the standard Maxwell's equations [16,19].

Under the Lorentz transformations, the matrix X is transformed as

$$X' = L^+ X L, \qquad L \in SL(2, c), \tag{47}$$

where L^+ is Hermitian conjugated matrix. The matrices ∇, $\overline{\nabla}$, F, A and J are transformed as follows

$$\nabla' = L^+ \nabla L, \qquad F' = L^{-1} F L, \qquad J' = L^+ J L, \tag{48}$$

$$\overline{\nabla}' = L^{-1}\overline{\nabla}\left(L^{+}\right)^{-1}, \qquad A' = L^{+}AL.$$

The terms in Eq. (45), including the parameter θ violate the Lorentz symmetry. The Lorentz-invariants of the transformations (47), (48) are the determinants of matrices. The spin-tensor formulation of the NC Maxwell's equations in the form of Eq. (45) is convenient for considering the symmetric properties of fields.

Let us consider some spin-tensor expressions in classical electrodynamics when $\theta = 0$. The energy-momentum tensor in the case $\theta = 0$ can be represented as follows

$$\tau_\beta T^{(\theta=0)}_{\beta\gamma} = \frac{1}{2}F^{+}\tau_\gamma F, \tag{49}$$

where $F = (E_m + iB_m)\tau_m$. Using the field equation (45), it is easy to prove that $\partial_\gamma T^{(\theta=0)}_{\beta\gamma} = 0$ at $J = 0$. From Eq. (49) we verify that

$$\begin{gathered} T^{(\theta=0)}_{44} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right), \qquad T^{(\theta=0)}_{k4} = i\epsilon_{kmn}B_m E_n, \\ T^{(\theta=0)}_{kn} = E_k E_n + B_k B_n - \frac{1}{2}\delta_{kn}\left(\mathbf{E}^2 + \mathbf{B}^2\right), \end{gathered} \tag{50}$$

so, the expression $T^{(\theta=0)}_{\beta\gamma}$ corresponds to Eq. (34) at $\theta = 0$. We may define the density of the Lorentz force as

$$K_\alpha = \partial_\beta T^{(\theta=0)}_{\alpha\beta}. \tag{51}$$

Then with the help of the matrix equation (45) at $\theta = 0$ we arrive at the relation

$$\tau_\alpha K_\alpha = -\frac{1}{2}\left(J^{+}F + F^{+}J\right). \tag{52}$$

Using Eq. (52) one can verify that the four-force K_α coincides with the known definitions:

$$\begin{gathered} \mathbf{K} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B}, \\ K_4 = i(\mathbf{J}\cdot\mathbf{E}). \end{gathered} \tag{53}$$

The expressions (49), (52) are form-covariant under the Lorentz transformations due to Eq. (47).

5 Conclusion

We have just considered the Maxwell theory on NC spaces which is described by the non-linear equations of the electromagnetic fields. This means that the vacuum of electrodynamics on NC spaces is similar to a medium with complicated (non-linear) properties. The system of non-linear wave equations found, (22), (23), possesses the solutions in the form of plane electromagnetic waves. It was noted that the dual transformations of electromagnetic fields under consideration are broken.

We have found the density of the energy and momentum of the electromagnetic fields on NC spaces, and the canonical and symmetric energy-momentum tensors. The canonical energy-momentum tensor (34) is conservative, but the symmetric energy-momentum tensor (38), found by varying the action on the metric tensor, is non-conservative because the action of electromagnetic fields in the case of NC spaces is not a Lorentz-scalar. It was shown that the traces of the canonical and symmetric energy-momentum tensors do not equal zero, i.e., there is a trace anomaly. The trace anomaly is related with the violation of the conformal invariance, and is a consequence of the breaking of the Lorentz invariance in NC spaces. This anomaly is absent in the case of the plane electromagnetic waves. The field equations are also obtained in the compact spin-tensor (quaternion) form. The Lorentz transformations of fields in the matrix form have been considered. The spin-tensor formulation of the NC Maxwell's equations is useful for different applications, especially for studying the symmetric properties of fields.

There are various phenomenological effects of the non-commutativity of coordinates (1) (see [2,3,12,20]). If space-time is indeed non-commutative on short distances, it may effect cosmology and early universe physics. It is important because cosmology can verify the theories which are beyond the standard model of particle physics. Probably, the consideration of the field theories on NC spaces may solve the problem of dark energy from trans-Plankian physics.

APPENDIX: quaternion algebra

Pauli's 2×2-matrices τ_k ($k = 1, 2, 3$) obey the following relations

$$\tau_m \tau_n = i\epsilon_{mnk}\tau_k + \delta_{mn},$$

$$\tau_\mu \overline{\tau}_\nu + \tau_\nu \overline{\tau}_\mu = -2\delta_{\mu\nu}, \tag{54}$$

$$\tau_\mu = (\tau_k, \tau_4)\,, \qquad \overline{\tau}_\mu = (-\tau_k, \tau_4)\,,$$

where $\tau_4 = i\tau_0$, $\tau_0 \equiv I_2$ is the unit 2×2-matrix. The quaternion algebra can be realized with the help of the Pauli matrices; setting $e_4 = \tau_0$, $e_k = i\tau_k$ and using the properties (54) we obtain the quaternion algebra which is defined by four basis elements $e_\mu = (e_k, e_4)$ [19]) with the multiplication properties:

$$e_4^2 = 1, \qquad e_1^2 = e_2^2 = e_3^2 = -1, \qquad e_1e_2 = e_3,$$

$$e_2e_1 = -e_3, \qquad e_2e_3 = e_1, \qquad e_3e_2 = -e_1, \tag{55}$$

$$e_3e_1 = e_2, \qquad e_1e_3 = -e_2, \qquad e_4e_m = e_me_4 = e_m,$$

where $m = 1$, 2, 3, and $e_4 = 1$ is the unit element.

The complex quaternion (or biquaternion) q is

$$q = q_\mu e_\mu = q_m e_m + q_4 e_4, \tag{56}$$

where the q_μ are complex numbers. Using the laws of multiplication (55), we find that the product of two arbitrary quaternions, q, q', is defined by:

$$qq' = \left(q_4q_4' - q_mq_m'\right)e_4 + \left(q_4'q_m + q_4q_m' + \epsilon_{mnk}q_nq_k'\right)e_m. \tag{57}$$

It is convenient to represent the arbitrary quaternion as $q = q_4 + \mathbf{q}$ (so $q_4e_4 \to q_4$, $q_me_m \to \mathbf{q}$), where q_4 and $\mathbf{q}$ are the scalar and vector parts of the quaternion, respectively. With the help of this notation, Eq. (57) can be rewritten as

$$qq' = q_4q_4' - (\mathbf{q}\cdot\mathbf{q}') + q_4'\mathbf{q} + q_4\mathbf{q}' + \mathbf{q}\times\mathbf{q}'. \tag{58}$$

Thus the scalar $(\mathbf{q}\cdot\mathbf{q}') = q_mq_m'$, and vector $\mathbf{q}\times\mathbf{q}'$ products are parts of the quaternion multiplication. It is easy to verify the combined law for three quaternions: $(q_1q_2)\,q_3 = q_1\,(q_2q_3)$.

The operation of quaternion conjugation (hyperconjugation) denotes the transition to

$$\overline{q} = q_4e_4 - q_me_m \equiv q_4 - \mathbf{q}, \tag{59}$$

so that the equalities $\overline{q_1 + q_2} = \overline{q}_1 + \overline{q}_2$, $\overline{q_1 q_2} = \overline{q}_2 \overline{q}_1$ are valid for two arbitrary quaternions q_1 and q_2. The modulus of the quaternion $\mid q \mid$ is defined by $\mid q \mid = \sqrt{q\overline{q}} = \sqrt{q_\mu^2}$. This formula allows us to divide one quaternions by another, and thus the quaternion algebra includes this division.

Quaternions are a generalization of the complex numbers and we can consider quaternions as a doubling of the complex numbers. They are convenient for investigating the symmetry of fields and relativistic kinematics. In particular, the finite transformations of the Lorentz eigengroup are given by [19]:

$$x' = Lx\overline{L}^*, \tag{60}$$

where $x = x_4 + \mathbf{x}$ is the quaternion of the coordinates ($x_4 = it$, t is the time, x_m are the spatial coordinates), L is the quaternion of the Lorentz group with the constraint $L\overline{L} = 1$, $\overline{L}^* = L_4^* - \mathbf{L}^*$ and $*$ means the complex conjugation. The biquaternion L with the constraint $L\overline{L} = 1$ is defined by six independent parameters which characterize the Lorentz transformations. The squared four-vector of coordinates, x_μ^2, is invariant under the transformations (60): $x_\mu'^2 = x'\overline{x}' = Lx\overline{L}^* L^* \overline{x}\overline{L} = x\overline{x} = x_\mu^2$, as $\overline{L}^* L^* = L^* \overline{L}^* = 1$. This shows that the 6−parameter transformations (60) belong to the Lorentz group $SO(3, 1)$.

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