Unification of Electrodynamics and Gravity from START

Jaime Keller

Departamento de Física y Química Teórica, Facultad de Química
and
Centro de Investigaciones Teóricas, FES-C
Universidad Nacional Autónoma de México
AP 70-528, 04510, México D.F., MÉXICO

ABSTRACT. In order to unify electromagnetism and gravitation we derive the basic principles of both approaches from a common (geometrical) starting formulation we call START, from its structure as a Space–Time–Action Relativity Theory. Gravitation results from the epistemological approach of defining a "test particle" which explores the physical world in such a form that its trajectory indicates the influence of the rest of the system. Electromagnetism defines a collection of test particles, we call carriers, in interaction among themselves and with the rest of the system. Once both approaches have been formally defined we can unify gravitation and electromagnetism either within one or the other formalisms.

Contents

1 Introduction

In the development of physics in the XX century several formulations were proposed to recast either electrodynamics in the form of general relativity or of gravity in the form of, gauge theory, electrodynamics. Although these formalisms could be presented it was felt that they were unsatisfactory from different points of view, but mainly because electromagnetism could not be really recovered when presented in the form of general relativity and because the attempt to reformulate general relativity within gauge theory electrodynamics presented a series of formal and practical problems. In the present paper we analyze the basic postulates of both approaches and find how the unification is achieved once we have a common deductive approach for both theories. Three approaches to structural unification are explicitly exhibited.

A deductive approach requires a very fundamental starting point which should anyhow contain enough physical principles not to be forced to introduce them when developing the formal theories. Another constrain is that no new concepts should be introduced in one particular approach when deducing the theoretical structures in order to avoid mistranslation of the concepts introduced for one theory in the search of an equivalent concept for the other.

All this is achieved in our present paper in the following sequence:

.- Define a basic set of concepts which are to be used to represent the physical world.

.- Provide a mathematical structure for this set of concepts formulated in such a form that they can be handled within the mathematical structures that will be used for the deduced theories.

.- Define the basic physical quantities that will be described in the different approaches, in such a form that there is no problem in translating from one deduced theory to the other.

.- Deduce the corresponding theories.

.- Compare the resulting theories with the standard formulations and among themselves.

This is then the sequence followed in the present paper.

General Relativity (GR), in particular, is derived from the quadratic space geometry corresponding to the, in our approach fundamental concepts, of space, of time, and of action and from the philosophical prin-

ciples of Einstein's general relativity theory. The formulation allows a systematic generalization of the procedure.

We will conclude by stating to which extend we have fulfilled our basic purpose and mentioning both the advantages for future developments and the possible limitations.

1.1 Mass, Charges, Action, Space and Time

Action, as a fundamental variable, is distributed among a set of **carrier (of action) fields**. An action density $a(\mathbf{x}, t)$, action a per unit space–time hypervolume $\Delta x_0 \Delta x_1 \Delta x_2 \Delta x_3$ at point $(\mathbf{x}, t)$ with $x_0 = ct$, is the fundamental concept defining all three: space (parametrized by $\mathbf{x}$), time (parametrized by t) and action density (parametrized by an scalar analytical function $a(\mathbf{x}, t)$) as **primitive** concepts from which all other physical quantities will be derived or at least related directly or indirectly. The different forms of distributing the action among this carriers defines the carriers themselves. This is fundamental in the practical use of the four principles below.

Within our fundamental formulation we will have to define properties of the fields we call carriers. A carrier will have physical significance through its set of properties. The density of a carrier field can be defined through a set of scalar constants such that the integral of the product of these constants and the density gives the experimentally attributed value of a property for that carrier. We will use an example. A carrier field identified with an electron will have a density $\rho(\mathbf{x}, t)$ and if the property is Q we will obtain the definition $Q = \int q(\mathbf{x}, t) d\mathbf{x} = \int Q\rho(\mathbf{x}, t) d\mathbf{x}$ for all t, which defines that Q is a constant property (in space and time) for that field. The set of properties $\{Q\}$ characterizes a carrier field and in turn establishes the conditions for a density field to correspond to an acceptable carrier. This is for example the case of electromagnetism and its daughter theory: elementary particles theory where the set {mass, electric charge, weak charge, strong charge, spin} defines the 'elementary particle' one is dealing with. In the case of general relativity the only property which is used in the standard formulation, for the test particle, is the mass; this is the reason for the limitation for both the applicability and for the extension of the GR theory.

We already stated that in our theory space and time are fundamental, primordial, concepts. The geometrical unification of these concepts into a space–time coordinate $X = (\mathbf{x}, ct)$ and an interval ds^2 requires the introduction of a universal constant: the speed of light c. As we

will also use action as a fundamental concept we need another universal constant $\kappa = d_0/h$ which we will construct from a fundamental distance d_0 and a fundamental unit of action we will choose to be Planck's action constant h. In this form we will have a five dimensional, START, manifold $3+1+1$ with all dimensions in units of length. There will be no problem to include quantum concepts in the formulation as far as Planck's constant is already a basic part of the geometrical formulation. The concept of **charges** appears in the theory first of all from the necessity to define the objects which exchange action among them in order to give a formal meaning to the principle that action will be exchanged in integer units of the Planck constant, second to relate the carriers among themselves defining the interaction through the bilinear form containing the products of pairs of charges, one type of charge for each one of the desired interactions. In this context for an electron-like carrier both mass and electric charge belong to the generic name of 'charges'. This program can obviously not be achieved if the formulation is not suitable to contain the deduction of the theory of elementary particles itself, giving a geometric meaning to this theory. This will be mentioned in the paper but the reader is referred to our previous publications on this matter. To agree with standard formulations energy density $\mathcal{E} = \Delta x_0 \partial a/\partial t$ and momentum density $p_i = \Delta x_0 \partial a/\partial x^i$ are the fundamental rates of change of the primitive concept of action (considering a unit time-like interval $\Delta x_0 = 1$ by definition of a as the action per unit space-time hypervolume, it can be omitted numerically from many formulas).

Our approach presents a natural transition from special relativity to general relativity and a natural extension of electrodynamics, both theories considered as basic formal structures (in our case deduced from START and the basic definitions of each approach). A third possibility would be to recast one or both of those theories in a new geometrical form where some formal role is given to the concept of 'structure of space-time', although this is possible we will not enter into this new formalism in the present paper. We are presenting ideas which not only unify existing theories and transform the theories from inductive to deductive class but also we suggest a well defined procedure to create future structures of theoretical physics.

Hermann Minkowski in his 1908 Address to the 80th Assembly of German Natural Scientists and Physicians, at Cologne, presented his mathematical formulation of Special Relativity in a talk he called *Space and Time*, introducing a fundamental axiom:

.- The substance of any world-point may always, with the appropriate determination of space and time, be looked upon as at rest.

In our present formulation the quantity which was called "substance" by Minkowski is identified with the concept of "action density a" with a well defined physical and mathematical formulation, that is we include "substance" in the list of formal terms of physics. It is also appropriate to say that the concept of **matter**, hitherto formally undefined, will acquire proper formal definition in the context of the different structural theories: it is different in GR (where matter is the source of the gravitational curvature) or in Electromagnetism (where it is the source of the electrical and magnetic fields, but it does not contains the electromagnetic fields themselves, even if the concept of radiation were added to be more precise). Then the START theory presented here corresponds to a geometrization of Minkowski's fundamental axiom. In fact that author, introducing the space-time interval squared ds^2, adds: "the axiom signifies that at any world-point the expression

$$c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

always has a positive value, or, what comes to the same thing, that any velocity v always proves less than c.". In our full geometrization scheme the positive semi-definite energy–momentum expression

$$E^2/c^2 - p_x^2 - p_y^2 - p_z^2 = P^2 \tag{2}$$

will also be considered, besides the ds^2 above, when action change $dK = P \cdot dX$ is introduced through a series $|dK|^2$ of quadratic terms

$$dS^2 - ds^2 = -|dK|^2 = -\kappa_0^2 \left\{(E^2/c^2)c^2dt^2 - p_x^2dx^2 - p_y^2dy^2 - p_z^2dz^2\right\}, \tag{3}$$

joined in one unified geometrical quadratic form dS^2. The dK vector, the directional in space-time change of action, is a new theoretical quantity formally defined by (3). It acquires additional relevance because action density will be described as a sum of contributions over carriers, $a = \sum_i \{\sum_c a_c\}_i$ and then the contributions to energy-momentum, that is the rates of change of action over space-time directions, will also be sums over equivalent carriers c or over collections $\{c\}_i$ of carriers of type i, (whenever convenient the choice can be made to consider them as interacting carriers).

Notice that the generalization $ds^2 \Longrightarrow dS^2$ also corresponds, from (1) and (2), to a generalization of the metric

$$dS^2 = (1 - \kappa_0^2 \frac{E^2}{c^2}) c^2 dt^2 - (1 - \kappa_0^2 p_x^2) dx^2 - (1 - \kappa_0^2 p_y^2) dy^2 - (1 - \kappa_0^2 p_z^2) dz^2. \tag{4}$$

It has been brought to our attention that the concept of action as a fifth coordinate, even if it has a new context in our approach [9], can be traced back to at least the pioneering work of Rumer in the 1940-1960 period of time. Rumer shows that many concepts can be cast in the framework of a five dimensional geometry [15], even our novel concept of action density can, in a sense, be extracted from his papers. The fundamental difference is that we are constructing a systematic deductive approach to Physics and it is essential that we derive many of the basic useful structures which have been used.

1.2 Action Carriers in START

The basic set of concepts we will use are then: that of a frame of reference SPACE, an evolution parameter to be identified with TIME, a description of the existence of physical objects which we will find useful to present as a field of ACTION over space–time (or a field of energy over space) and, finally, the carriers of that action (energy) in the form which is customary used in the different theories. We need afterwards to partition the different distributions of action over space–time by considering a set of contributions by a collection of scalar fields of 'carriers' in such a form that the total action density in space–time is the sum of the action attributed to the carriers which, in that given point of space–time, have a non null field. Some properties arise from the START geometry itself, others from the description of a physical system as a time evolving energy distribution. For a given observer the carrier field *c is defined* to have an energy density $\mathcal{E}_c \rho_c(\mathbf{x}, t)$ with $\mathcal{E}_c$ a constant in space, at time t, and N_c the integer number of carrier units of type c. The density $\rho_c(\mathbf{x}, t)$ is then required to obey $\int_V \rho_c(\mathbf{x}, t) d\mathbf{x} = N_c$ in the system volume V.

In order to make direct contact with standard formulations of physics we assume that the concepts of space, time and action have the same meaning and units as they have in those formulations. We also make a sharp distinction between action density and Lagrangian density. The Lagrangian containing prescriptions for the description of the system. A fundamental, immediate, result is that energy and momentum as the rate

of change of action with respect to time or to space are natural, primary, geometric concepts. Also the second derivatives of the action distribution are natural candidates to be the field strengths when expressed per unit carrier charges. This is particularly important in the gauge theory approach, where the rate of change of the action distribution, described through an auxiliary amplitude function, has contributions difficult to describe, as is often the case, as averages of the energy-momentum of some hypothetical point carriers .

Action is one of the properties of a distribution describing, in relation to an observer, the contents of the physical world in space–time. The concept of Physical Phenomena refers to the existence and change of this distribution. Physics corresponds to the description of the action distribution and its changes in relation to a given observer.

The action density function in space–time $a(X)$, or energy density in space for a given observer, can be considered as a density of action trajectories in space–time. In this more geometrical approach 1) action is used as a coordinate expressed in units of distance, 2) action distribution as a density of action like trajectories associated to a physical object and 3) energy-momentum as the rate of change of this density with respect to time-space, for that object. For elementary carriers the trajectories would be elementary trajectories with null space-time-action intervals.

Both the action density function $a(X)$ and the splitting among carrier fields will be considered analytically well behaved functions.

A **description** is introduced when we consider the energy $\mathcal{E}(t)$ of a system as a sum among the different carrier types $\{c\}$such that $\mathcal{E}(t) = \sum_c \mathcal{E}_c(t)$, a sum of constants $\mathcal{E}_c(t)$ in space for a given observer. For a particular purpose the energy per carrier can furthermore be described as a sum of space dependent contributions per carrier. The simplest, almost universal, type of distribution per carrier type is into the constitution or "mass" energy $\mathcal{E}_0$, the position dependent kinetic energy $\mathcal{E}_k(X)$, and the position dependent sum of potential energies $\mathcal{E}_v(X)$, then

$$\mathcal{E}_c = \mathcal{E}_0^c + \mathcal{E}_k^c(X) + \mathcal{E}_v^c(X) + \mathcal{E}_\Delta^c(X). \tag{5}$$

It is precisely this distribution (5) which defines the carrier for a given observer. $\mathcal{E}_0^c$ defines the basic carrier ("isolated" and at rest), $\mathcal{E}_k^c(X)$ the state of motion relative to the observer, and $\mathcal{E}_v^c(X)$ the relation between that carrier and the rest of the system as defined by the observer. The $\mathcal{E}_\Delta^c(X)$ term is required to make $\mathcal{E}_c$ a position independent constant, this

is needed to have a meaningful definition of the carriers of type c. For a given carrier charge q_c of type Q an interaction field intensity $\mathbf{E}^i_{Q,c}(X)$ can be defined from the second derivatives of the action, per unit charge, contributing to the energy $\left(\mathcal{E}^c_v(X)\right)_{Q,c}$ per carrier defined as arising from that interaction for that carrier.

This second derivatives in turn may have further derivatives, with respect to space or time coordinates. The invariant definition, given that the action A is invariant under a space–time Lorentz transformation, would be $A = \sum_c A_c$, the energy $\mathcal{E}_c = \partial A_c / \partial t$ is not invariant but observer dependent and this imposes structural mathematical restrictions on the energy $\mathcal{E}_c$ and, consequently, on $\mathcal{E}^c_0$, $\mathcal{E}^c_k$ and $\mathcal{E}^c_v$. Not only because energy is a component of an invariant four–vector but also because the partition of energy among carriers has to be observer dependent.

Because the rate of change of momenta is related to the classical concept of force, these geometric features are the geometrical representation of potentials and their derivatives, the second derivatives of the action are then correctly termed strength fields or force fields.

Notice that within the concept of interaction the rates of change of action are to be expressed per unit charge and represent those contributions arising from sums over carriers other than the one under consideration. The action and the energy and momentum are considered to be distributed among interacting carriers.

1.2.1 Definitions and Notation

.-Definitions

a) We use the name **space** for, indistinctly, denote the 3-D space of our perception of the distribution of physical objects in Nature and for the mathematical representation of it as a $\mathbb{R}^3$ manifold with a quadratic form. Its points are denoted by the Bold-face letter $\mathbf{x}$ and represented as a vectorial quantity $\mathbf{x} = x^i \mathbf{e}_i$. We use the traditional indices i =1,2,3.

b) We use the name **time** for, indistinctly, denote the 1-D space of our perception of the evolution of physical phenomena in Nature and for the mathematical representation of it as a $\mathbb{R}^1$ manifold with a quadratic form. Its points are denoted by the normal-face letter t.

c) We use the name **space-time** for, indistinctly, denote the 4-D Minkowski space, of our perception of the physical world in the sense of

relativity theory, and for the mathematical representation of it as a $\mathbb{R}^4$ manifold with a quadratic form :

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \ (\mu, \nu = 0, 1, 2, 3). \tag{6}$$

Its points are denoted by the Normal-face letter X and represented as a vectorial quantity $X = X^\mu e_\mu$. We use the traditional indices $\mu = 0, 1, 2, 3$. The vectors e_μ in the geometry of space–time generate $G_{\rm ST}$ a 16 dimensional space–time geometry of multivectors. The basis vectors $\{e_0, e_1, e_2, e_3\}$, with $e_0^2 = -e_1^2 = -e_{2_3}^2 = -e_3^2 = 1$ and the definition property $e_\mu e_\nu = -e_\nu e_\mu$ generate a Clifford group $Cl_{1,3}$. We also use the notation $e_{0j} = e_0 e_j = \mathbf{e}_j$ (j=1,2,3) and $e_5 = e_0 e_1 e_2 e_3 = e_{0123}$. A special property of the pseudo-scalar (and also hypervolume and inverse hypervolume) in space–time e_5 is that $e_5 e_\mu = -e_\mu e_5$ (from $e_\mu e_\nu = -e_\nu e_\mu$, $\mu \neq \nu$) and then it has the same commuting properties with the generating vectors of $G_{\rm ST}$ as the generating vectors among themselves. A vector e_4 can be used to introduce an additional basis vector, adding one more dimension and, through its use, obtain the five dimensional carrier space spanned by the basic vectors e_u, $u = 0, 1, 2, 3, 4$ (identified as $e_0 \Rightarrow e_0$, $e_1 \Rightarrow e_1$, $e_2 \Rightarrow e_2$, $e_3 \Rightarrow e_3$ and e_4) with metric $\mathbf{g}_{uv} = \text{diag}(+1, -1, -1, -1, -1)$. Its use allows the construction of a geometrical framework for the description of physical processes: a unified space–time–action geometry $G_{\rm STA}$, mathematically a vector space with a quadratic form. An auxiliary element $\mathbf{j}$ anti-commutes with all $e_\mu : e_\mu \mathbf{j} = -\mathbf{j} e_\mu$ and $\mathbf{j}^2 = +1$.

For a given observer with time vector e_0 the space–time d'Alembertian operator $\Box$ has the property

$$e_0 \Box = \frac{1}{c}\partial_t + \nabla = \frac{1}{c}\partial_t + \mathbf{e}_i \partial_i \quad ,$$

with ∇ the (ordinary space) Laplacian operator for that observer.

c) We use the name **action** for, indistinctly, denote the 1-D space of our perception of the objects of physical phenomena in Nature and for the mathematical representation of it as a $\mathbb{R}^1$ manifold with a quadratic form da^2. Its points are represented by the normal-face letter a.

d) We use the name **space-time-action** for, indistinctly, denote the 5-D space of our perception of physical phenomena in Nature and for the mathematical representation of it as a $\mathbb{R}^5$ manifold with a quadratic

form

$$dS^2 = ds^2 - \kappa_0^2 da^2 = g_{\mu\nu} dx^\mu dx^\nu - \kappa_0^2 da^2 = g_{AB} dx^A dx^B, \quad (7)$$
$$(A, B = 0, 1, 2, 3, 4),\ (\mu, \nu = 0, 1, 2, 3)$$

. Its points are represented by the set of normal-face letters $(X, \kappa_0 a)$.

e) We use the name **description** to denote the partitioning into carriers c and into action (or energy-momenta P_c) contributions of the objects of physical phenomena in Nature and for the mathematical representation of this partitioning among carriers and among energy-momenta contributions.

f) We use the name **theoretical structure** for a set of defining considerations about the distribution of action among carriers and the mathematical form in which the results are presented. In this respect we will examine in this paper the approaches of Newton, Maxwell, Einstein, Schrödinger, Hohenberg-Kohn and Rumer as basic structures integrated through START.

1.3 Action Density

In this section we shall present our fundamental concepts as an Action **Density** Functional Theory and discuss the origin and consequences of the formulation of an Action **Amplitude** Function Theory.

For the study of the distribution of action we consider that:

a) In the space–time–action picture, where the basic mathematical properties of space-time are assumed to correspond to the physical space-time, the action density $a(\mathbf{x}, t)$ is inhomogeneously distributed, corresponding to the different material objects to which this action corresponds, in a possible relative motion in the spatial directions with speeds $0 \leq v \leq c$. The sets of distributions which move with relative velocities $0 \leq v < c$ with respect to a given observer are called matter-like.

b) The matter-like energy distributions are to be considered as sources of (infinite extension, in principle) decaying deformations of action distribution of several types: first, a part A_0 uniformly decaying with distance, which observers will interpret as gravitation; second, of a collection $(A, B, C, ...)$ of vortices fields, superimposed on the A_0 part, which can be felt selectively by responses of given internal vortices of the other matter-like distributions. This second property is not given *a priori* but it is a consequence of the description of the objects, as developed in the previous section, shown below.

c) We introduce now a third fundamental concept: energy–momentum carriers. At a macroscopic level the energy carrier is defined by a density distribution and by the integral properties of the distribution. We shall keep this concept without considering that the distribution could be reduced to a point singularity in space (line in space–time). We shall, anyhow, use the (not very fortunate) name **carrier density** for this quantity, the main reason being that its integral will be taken to be, as mentioned above, an integer. An extra reason is the definition of identical carriers as a density in a space volume V_s such that at time t_0

$$\int_{V_s} \rho_b d\mathbf{x} = n_b, \quad \int_{V_s} \partial_t a_b \Delta x_0 d\mathbf{x} = \int_{V_s} \rho_b \varepsilon_b d\mathbf{x} = n_b \varepsilon_b = E_b, \qquad (8)$$

and $E = \sum_b E_b$ for a collection $\{b\}$ of (by practical construction) independent types of carriers. In agreement with our freedom of description we could also allow the n_b not to be integers, provided E_{t_0} is not changed.

1.3.1 Classification of action (energy) carriers

The definition of energy carrier in (8) is useful for all possible cases. In the development of the different theories the action and the energy–momentum will be partitioned into the carrier fields. There will be a limitation when studying the interactions: some of these carriers (type $\mathfrak{a}$) will be called restricted interaction carriers because there will be no possibility of the energy–momentum attributed to the carrier fields to present rotational distributions as far as $curl(div(a(X))) = 0$.

A second type of carriers will be those for which the energy–momentum itself is also partitioned into the carrier fields as in the last terms of (8), this will allow that the individual energy–momentum fields posses rotational distributions $curl(\mathbf{p}(X)) \neq 0$, and by necessity $div(curl(\mathbf{p}(X))) = 0$. This second type of carriers, whose definition allows a larger type of interactions, will be called carriers type $\mathfrak{b}$.

A third type $\mathfrak{c}$ of carriers will be present in the systematic development of the theory (see [9, 10]), which are those collectively of type $\mathfrak{b}$, for which a further splitting of the field intensities is made which violates the basic symmetries of space–time for individual carriers c_i but not for the collection $\sum_i c_i$.

The carriers of type $\mathfrak{a}$ are those considered in standard (Newtonian) mechanics and in the ordinary formulation of General Relativity where

only the total energy-momentum content is considered. The carriers of type b are those considered in standard electrodynamics where the splitting of the energy-momentum content is considered. The carriers of type C are considered in the standard model of elementary particles where symmetry breaking interactions are considered (this will not be considered here, see [9, 10]).

1.3.2 The Density

The conditions to be obeyed by the analytical function carrier density $\rho_c(\mathbf{x}, t)$ are:

D1.- $\rho_c(\mathbf{x}, t)$ is a real quantity $\rho_c(\mathbf{x}, t) \subset \mathbb{R}$.

D2.- The density $0 \leq \rho_c(\mathbf{x}, t) < \infty$ in order to represent a finite amount of action.

D3.- The derivatives of the density $-\infty < \partial_\mu \rho_c(\mathbf{x}, t) < +\infty$ in order to represent an *assigned* finite amount of energy–momentum.

Theorem 1 *If $\Psi(\mathbf{x}, t)$ is an analytical complex or multivector quantity, finite and with first and second finite derivatives everywhere, conditions D1, D2 and D3 are fulfilled identically if $\rho_c(\mathbf{x}, t) = |\Psi_c(\mathbf{x}, t)|^2$. Here $|f|^2$ means the real quadratic form of any more general function f, even if f itself is not necessarily a real function and we define $|f|^2 = f^+ f$ and therefore $\partial_\mu |f|^2 = (\partial_\mu f^+) f + f^+ (\partial_\mu f)$.*

Proof. Condition D1 is fulfilled from the definition $\rho_c(\mathbf{x}, t) = |\Psi_c(\mathbf{x}, t)|^2$. Condition D2 is fulfilled from the definition of $\Psi(\mathbf{x}, t)$ and of $|f|^2$. Condition D3 is fulfilled from the definition $\partial_\mu |\Psi(\mathbf{x}, t)|^2 = (\partial_\mu^+ \Psi(\mathbf{x}, t))\Psi(\mathbf{x}, t) + \Psi(\mathbf{x}, t)^+ (\partial_\mu \Psi(\mathbf{x}, t))$ because a phase factor $e^{i(\Delta\mathcal{E}t - \Delta\mathbf{P}\cdot\mathbf{x})/\hbar}$ can freely be included in $\Psi(\mathbf{x}, t)$ in order to have the desired value of the **assigned** energy-momentum contributions for that carriers system, using the operator $\widehat{p_\mu} = -i\hbar\partial_\mu$. ■

It is seen that the conditions D1, D2, D3 and $\int_V \rho_c(\mathbf{x}, t) d\mathbf{x} = N_c$ also define the $\Psi(\mathbf{x}, t)$ to be quadratic integrable Hilbert functions. In any formulation this last definition of ρ can be used as a condition introduced via a Lagrange multiplier. Of course from the definition of $\Psi(\mathbf{x}, t)$ and the ideas of this paragraph this function should be called "the Schrödinger function" because, as discussed below, it was introduced by him in 1926 [16].

Gauge freedom Carrier density and density of action should then, to allow freedom of description, be gauge invariant physical quantities, thus we need to develop a procedure which can allow gauge freedom, that is, a procedure which allows for arbitrary, but correct and useful, descriptions. In any of the theoretical structures to be discussed below the gauge fields represent the additional energy–momentum terms assigned to the carrier in our description process. They introduce both the gauge freedom to split the energy–momentum per carrier and the gauge freedom to define the sharing of energy attributed to the interactions.

Because the definition of carrier density depends on the attributed energy per carrier we can not separate the definition of the gauge from the definition of the carriers themselves.

This is possible with the introduction above of the density amplitude Ψ, required to contain the necessary information in a form compatible to the basic concept that the energy–momentum components are obtained by using the operator $-i\hbar\partial_\mu$ applied to the function which describes the splitting of the action density into a carrier density ρ and the action per carrier. The definition required by D1, D2 and D3 above

$$\rho_c = |\Psi_c|^2 , \tag{9}$$

allows then gauge independence. A set of Lagrange conditions and multipliers can be used to define the carriers and their desired properties. This procedure can be carried at any level of description, hence the universality of mathematical descriptions presented here, which in fact give a self existing status to density functional theory. There should be no confusion from the fact that (9), here proposed for density functional theory, is formally equivalent to the use of Wave Equations in Quantum Mechanics.

1.4 Density Functional Theory and Interactions

1.4.1 Density Functional Theory from START

We proceed now to establish the basic aspects of density functional theory.

- The total energy of the system is a functional of the density, which can be defined in two steps. The first is to establish that there is

a ground, minimum energy, state of the system which defines the carriers themselves:

$$E_0^{(N)} = \int E_0^{(N)}(\mathbf{x})d\mathbf{x} = \int \rho_0^{(N)}(\mathbf{x})\varepsilon d\mathbf{x} = N\varepsilon, \tag{10}$$

$$\int \rho_0^{(N)}(\mathbf{x})d\mathbf{x} = N \quad (N = \text{ number of carriers }), \tag{11}$$

where the density of energy $E_0^{(N)}(x)$ at a given space point x has been factorized as the product of the average energy ε per carrier and the carrier density $\rho_0^{(N)}$. This by itself is the definition of carriers of a given type: they are all equivalent and the energy of the carrier is a constant in space, for all points of the distribution and, in a given system, the same for all carriers of the given type.

- The constant defining the average energy per carrier is a functional itself of the carrier density

$$\varepsilon = \varepsilon\left[\rho_0^{(N)}(\mathbf{x})\right], \tag{12}$$

being a functional it is a real number. Because the reference energy has to be freely defined this constant may be either positive, negative, or null. The functional may be, in some cases, constructed from the density and its space derivatives at the given point x, then it becomes a **local density functional** (LDF).

- The energy density, assuming **indistinguishable (independent or interacting) carriers** of a given type (at a given time t' in relation to the observer) is now subjected to the description needs or desires of the observer describing the system, in a local in space partitioning which in general follows a traditional scheme of using the concepts of constitutional energy, kinetic energy, and potential energy

$$\varepsilon = \mathbf{m}_c c^2 + \mathbf{kin}[\rho_0^{(N)}(\mathbf{x})] + \mathbf{V}(\mathbf{x}) + \mathbf{W}_{int,xc}[\rho_0^{(N)}(\mathbf{x})] + \varepsilon_0[\rho_0^{(N)}(\mathbf{x})], \tag{13}$$

where we have defined the average energy per carrier as, first, the (by the definition of carriers) constitutional energy of the carriers $\mathbf{m}_c c^2$, actual, arbitrary in definition, local kinetic energy per carrier $\mathbf{kin}[\rho_0^{(N)}(\mathbf{x})]$ and the external potential energy per carrier (in

its simplest form) $\mathbf{V}(\mathbf{x})$. Second the correction $\mathbf{W}_{int,xc}[\rho_0^{(N)}(\mathbf{x})]$ to the average kinetic and external potential energy per carrier arising from the assumed interaction among carriers and in due case: the correction arising from the statistics the carriers should follow and the assumed correlations of the type of carriers under consideration; this term is needed to define independent carriers from interacting carriers, as far as this energy appears as a property of the **carrier in the system**, it corrects for the fact that the constitutional energy and the external potential refer to the isolated carrier or more correctly stated to the one carrier system. Third and finally we have to include a local energy term $\varepsilon_0[\rho_0^{(N)}(\mathbf{x})]$; a basic term required to compensate for any difference in the sum of the previous ones with respect to the (constant) average energy per carrier ε, this term defines that the carriers are indistinguishable among themselves, condition included in the introduction of the concept of average energy per carrier. The last two terms define in practice an actual carrier in the system (a pseudo-carrier in condensed matter physics language) as different from an isolated carrier.

- Physics studies both the existence of a system and, mainly, its response to external excitations. In the simplest approximation the necessary description is that of the possible stationary states of the system. In each of these system's states the carriers should continue to be indistinguishable and then for each stationary state s of the system the energy $E_s(\mathbf{x})$ should obey a condition equivalent to (10) for the N carrier system

$$E_s = \int E_s(\mathbf{x})d\mathbf{x} = \int \rho_s(\mathbf{x})\varepsilon_s d\mathbf{x} = N\varepsilon_s = E_0 + h\nu_s, \tag{14}$$

$$\int \rho_s(\mathbf{x})d\mathbf{x} = N \quad (N = \text{ number of carriers }), \quad \varepsilon_s = \varepsilon_s\left[\rho_s\left(x\right)\right], \tag{15}$$

$$\varepsilon_s = \mathbf{m}_c c^2 + \mathbf{kin}[\rho_s(\mathbf{x})] + \mathbf{V}(\mathbf{x}) + \mathbf{W}_{int,xc}[\rho_s(\mathbf{x})] + \varepsilon_s[\rho_s(\mathbf{x})]. \tag{16}$$

The study of the different excitation energies of the system $h\nu_s$ is now equivalent to the Heisenberg approach to study a physical system through its excitations spectra, which was properly termed as

quantum mechanics because of the direct use of Planck's constant h. Note that the functionals are given the same symbol as far as they are thought to depend only on the type of carrier.

- Among the stationary states of the system there is a simple series which corresponds in experimental practice where one of the carriers is being removed from the system with increasing excitation energy, in the sense that the remaining part of the system no longer needs to consider the carrier which has been removed. In that case we follow (14,15,16) with the conditions adjusted to $N \longrightarrow N-1$.

Density functional theory describes the self-organization of the carriers system with density $\rho(\mathbf{x})$ in the presence of some external potential.

In (10,11) we are, in fact, defining the carriers. First, considering the energy density to be given by the product $\rho(\mathbf{x})\varepsilon$ of a density of carriers and an average energy per carrier (the same for all, in a form which makes them indistinguishable). Second because the domain of integration defines the system of equivalent carriers (within this domain). In (13) the kinetic energy is considered to have the value which corresponds to independent carriers. The last function in (13) is then required to compensate both for the definition of energy contributions of the equivalent carriers and for considering them as independent carriers.

1.4.2 The Density as the Basic Variable

It is convenient to define the action in a form which distinguishes the part corresponding to the self-organization of the distribution and the part which corresponds to the 'external' influences on the distribution.

The volume (in space) of integration is considered large enough for the 'kinetic' energy to be internal, there should be no need to change the integration domain as a function of time. If the external influence is represented by the external potential $\mathbf{V}(X)$ we can write for the total (invariant) action

$$A = \int dt \left[\mathcal{E}_I[\rho(X)] + \int d\mathbf{x}\, \mathbf{V}(X)\rho(X) \right], \tag{17}$$

where the functional $\mathcal{E}_I[\rho(X)]$ corresponds to the energy of the distribution of carriers $\rho(X)$. This functional $\mathcal{E}_I$ has the interesting property

that at a given time

$$\frac{\delta \mathcal{E}_I}{\delta \rho(X)} = -\mathbf{V}(X). \tag{18}$$

This is a basic relation in Action–DFT as far as there is an intrinsic definition of the external potential. This shows the tautological nature of the concept of carriers, once they are defined the external potential is defined through the definition of the carriers themselves by $\mathcal{E}_I[\rho(X)]$. The tautological cycle is closed when given $\mathbf{V}(X)$ and $\rho(X)$ the kinetic energy and the interaction terms define $\mathcal{E}_I[\rho(X)]$. Reminder: in practice more general forms of $\mathbf{V}(X)$ should also be acceptable.

From the definitions above we can extend the description to consider a set $\{b\}$ of types of carriers, each carrier type with density ρ_b. In this case for each b the 'the external potential' depends in all types $b' \neq b$.

1.4.3 Introducing Gauge Freedom for the Description of the Action

We have emphasized that the fact that we are arbitrarily defining the terms above requires the possibility of changing the description of the action (energy) partitioning without changing the description of the density. The density $\rho(X)$ at space–time point X is required to be gauge invariant, whereas the description of the energy (action) is gauge dependent. This is achieved by constructing the energy density as the product of an average energy per carrier ε with two conjugated quantities $\Psi(X)$and $\Psi^\dagger(X)$ such that $\rho(X) = \Psi^\dagger(X)\Psi(X)$ is gauge invariant. Here we have defined an auxiliary quantity which can be essentially written in terms of a basic action $a_0(X)$ and the action introduced through a gauge phase $\phi(X)$ both in units of $\hbar$, as proposed by Klein and by Fock as early as 1926 [11, 6]:

$$\Psi(X) = \sqrt{\rho(X)}\; e^{-ia_0(X)+i\phi(X)}, \tag{19}$$

where we are restricted (even if $\phi(X)$ can be very general [9] and can represent electroweak, color and gravitational interactions), by definition, to

$$\hbar \frac{\partial \left(a_0(X) - \phi(X)\right)}{\partial t} = \varepsilon, \tag{20}$$

showing the gauge freedom of the description of the energy (action) associated with the carrier. By definition, at all space position points $\mathbf{x}$ we have the same energy per carrier ε which only in the simplest cases would be the sum of a constitutional, a kinetic and a potential energy part in the sense above. We have then recovered the equivalent to the Hohenberg–Kohn Theorems [7] and formally the Kohn–Sham minimization procedure [12] as the definition of these two terms

$$\delta\left(\mathcal{E}\left[\rho\right]-\varepsilon\left\{\int\rho(\mathbf{x})\,d\mathbf{x}-N\right\}\right)=0, \tag{21}$$

allows the direct self-consistent determination of $\rho(\mathbf{x})$ and ε (see [10]).

1.4.4 Schrödinger Amplitude Functions in START

The freedom of description of the energy-momentum partitioning is a fundamental issue in the construction of physical theories. Since at least the XIX^{th} century an action function was introduced which was useful for this purpose, in the XX^{th} century the de Broglie phase factor $\exp(-i\triangle a/\hbar)$ [2] allowed the freedom of energy-momentum description and the use of the gauge fields. Later the concept of non-commuting gauge fields [19] was successfully introduced to describe a larger set of energy-momentum partitioning among carriers (fundamental interactions). We follow now the Schrödinger procedure.

1. Let the Schrödinger (1926) [16] definition $S(\mathbf{x},t)$ of action in terms of an auxiliary function $\Psi(\mathbf{x},t)$ be

$$S(\mathbf{x},t)=\mathsf{K}\ln\Psi(\mathbf{x},t)=\mathsf{K}^{\dagger}\ln\Psi^{\dagger}(\mathbf{x},t),$$

that is: action is considered a sum of terms. The $\Psi(\mathbf{x},t)$ has one factor for each contribution to the energy–momentum The action $S(\mathbf{x},t)$ is required to correspond to the stationary states of the system to be described, this is ensured through a variational optimization procedure. The choice $\mathsf{K}=i\hbar=-\mathsf{K}^{\dagger}$ agrees with the previous paragraphs.

2. Let the carrier density ρ be the real quantity

$$\rho(\mathbf{x},t)=\Psi^{\dagger}(\mathbf{x},t)\Psi(\mathbf{x},t),$$

where $\rho(\mathbf{x},t)$, Ψ and $\Psi^{\dagger}$ are: unique-valued, continuous and twice-differentiable, with the condition $\rho(\mathbf{x},t)|_{\text{ space boundary }}=0$.

3. Let the canonically conjugated variables be: $(\mathbf{x}, t)$ and $\Box S = \mathrm{K}\Box \ln \Psi = \mathrm{K}^\dagger \Box \ln \Psi^\dagger$, with, again, $\Box = e^\mu \partial_\mu$ the space-time gradient operator.

4. The local energy description be ($\mathcal{E}, \mathcal{P}, \mathcal{E}_0$ are not densities, they can be gauged as below)

$$-|\mathsf{K}|^2 \frac{(\Box\Psi^\dagger)\cdot(\Box\Psi)}{\Psi^\dagger\Psi} c^2 = \mathcal{E}^2 - (\mathcal{P}c)^2 = (\mathcal{E}_0)^2 = \left(m_0 c^2\right)^2,$$

with the Lagrangian (density of energy) function

$$J = -|\mathsf{K}|^2 (\Box\Psi^\dagger)\cdot(\Box\Psi)c^2 - \left(m_0 c^2\right)^2 \Psi^\dagger\Psi,$$

and variationally search for the extremum energy $\mathcal{E}$ (minimum of action for a stationary state system) $\delta J = 0$ to obtain from the standard variational approach

$$-|\mathsf{K}|^2 \left[\Psi^\dagger \left(\Box^2\Psi\right) + \left(\Box^2\Psi^\dagger\right)\Psi\right] = m_0 c^2 \Psi^\dagger\Psi,$$

and the equation for the auxiliary function Ψ

$$-|\mathsf{K}|^2 \Box^2\Psi = \left(m_0 c^2\right)^2 \Psi.$$

(In the case where an interaction is assumed to exist $(E-V)^2 - (Pc - eA)^2 = \left(m_0 c^2\right)^2$).

From the Schrödinger variational search for the minimum of action for a stationary state system $\delta J = 0$ we obtain the Schrödinger-Klein-Gordon Equation (SKG), (consider first $V = A = 0$) it is

$$\left[-|\mathsf{K}|^2 \left(\frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) c^2\right) - \left(m_0 c^2\right)^2\right] \Psi = 0. \qquad (22)$$

We must emphasize that in the relativistic (and in the non relativistic) case we obtain, through the Schrödinger optimization procedure, the Ψ (or $\Psi^\dagger$) function which minimizes the action of the system. The geometric factorization of the operator in the SKG equation transform it into the Dirac equation. In the next section we follow an alternative procedure which illustrates directly the meaning of the components of the Schrödinger Amplitude Function.

1.4.5 Linear form of the Schrödinger-Klein-Gordon Equation

We want to express the Schrödinger-Klein-Gordon equation (SKG (22)) in the linear form

$$\hat{H}_{linear}\psi = m_0 c^2 \psi. \tag{23}$$

Consider the simple case of the free carrier, that is, of the equation (here $k^2 = (m_0 c/\hbar)^2$, $i^2 = -1$)

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi - \nabla^2\psi + k^2\psi = 0, \tag{24}$$

and write (following a procedure analog to that used by Charlier, Bérard, Charlier and Fristot [1] to obtain a Schrödinger-like equation from the SKG)

$$\psi = \sum_{a=1}^{m} \phi_a, \tag{25}$$

$$\frac{\partial\psi}{\partial x_\mu} = \sum_{a=1}^{m} c_\mu^a \phi_a,$$

$m = 8$ in order to have a faithful representation of both definitions and the coefficients are given by the matrix

$$8c_\mu^a = ik \begin{pmatrix} +1+1+1+1-1-1-1-1 \\ +1-1+1-1-1+1-1+1 \\ +1+1-1-1+1+1-1-1 \\ +1-1-1+1+1-1-1+1 \end{pmatrix},$$

with these definitions we write the 4 equations

$$\frac{\partial}{\partial x_\mu}\sum_{a=1}^{m}\phi_a = -ik\sum_{a=1}^{m} c_\mu^a \phi_a,$$

and use them in equation (22,24) to obtain:

$$\sum_\mu \frac{\partial}{\partial x_\mu}\sum_{a=1}^{8} c_\mu^a \phi_a = -ik\sum_{a=1}^{m}\phi_a,$$

which is a linear form of the energy-momentum conservation equation (22).

If a representation of the same relation is given through the use in (25) of a set of real 8×8 we obtain the Dirac equation: $\Box\psi = -ik\psi$. The Dirac equation is thus a faithful representation of the linearized form of the Schrödinger–Klein–Gordon equation. The components ϕ_a of the Schrödinger auxiliary function are, according to the original definition, the contributions to the action **function** needed for a description of the energy-momentum distribution. We could otherwise maintain the representations (25) and write the equivalent to the Dirac equation without the use of a matrix representation.

Observe that the fact that we can now define a d'Alembertian operator

$$\Box = \gamma^\mu \partial\mu,$$

allows the generation of a Clifford algebra representation of the full geometry of space–time through the definition

$$\gamma\,(e^\mu) = \gamma^\mu,$$

of the matrix representation of the geometric algebra of space–time. The γ^μ faithfully obey the same relations as the basis vectors e^μ. The set of 8×8 real matrices $[2\mathbb{R}(4)]$ representing the e^μ is:

$$\gamma(e^0) = \begin{pmatrix} +1 & & & & & & & \\ & +1 & & & & & & \\ & & +1 & & & & & \\ & & & +1 & & & & \\ & & & & -1 & & & \\ & & & & & -1 & & \\ & & & & & & -1 & \\ & & & & & & & -1 \end{pmatrix}$$

$$\gamma(e^1) = \begin{pmatrix} & & & & & & -1 & \\ & & & & & & & +1 \\ & & & & -1 & & & \\ & & & & & +1 & & \\ & & +1 & & & & & \\ & & & -1 & & & & \\ +1 & & & & & & & \\ & -1 & & & & & & \end{pmatrix}$$

$$\gamma(e^2) = \begin{pmatrix} & & & & & & & -1 \\ & & & & & & -1 & \\ & & & & & +1 & & \\ & & & & +1 & & & \\ & & & -1 & & & & \\ & & -1 & & & & & \\ & +1 & & & & & & \\ +1 & & & & & & & \end{pmatrix}$$

$$\gamma(e^3) = \begin{pmatrix} & & & & & & -1 & \\ & & & & & & & +1 \\ & & & & +1 & & & \\ & & & & & -1 & & \\ & & & -1 & & & & \\ & & & & +1 & & & \\ +1 & & & & & & & \\ & -1 & & & & & & \end{pmatrix}$$

The Dirac matrices γ_D^μ and wave function Ψ_D are obtained trough a complex structure similitude transformation $S\left(\gamma\left(e^\mu\right)\right)S^{-1} = \gamma_D^\mu$ and $\Psi_D = S\psi$ with, for example:

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & & & \\ & 1 & i & & \\ & & 1 & i & \\ & & & 1 & i \end{pmatrix},\ S^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ -i & & & & & & & \\ & & -i & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & -i & & \\ & & & & & & -i & \end{pmatrix}.$$

Representations of algebras are not unique, but they are related by similitude transformations.

2 Space–Time–Action Relativity Theory

Our basic and more fundamental idea is that the physical world can be described as a distribution of action density in space–time. The properties of matter fields and their interaction are represented by the mathematical properties of this distribution. Action is considered as a fundamental variable, not as a quantity resulting from some calculation.

In [9, 10] we analyzed a classical theory of fields in (complex) space–time geometry and arrived to the conclusion that this geometry corresponds to a unified space–time–action geometry.

2.1 Space–Time to Space–Time–Action

We present a deductive type of theory. The basic description of physical nature will be derived from the, here assumed primitive, formal concepts of space and time as a frame of reference and of a distribution of action, action density in space-time, which obey the principles presented below. We assume that the theory here developed has as fundamental purpose to provide a useful and coherent formal representation of our perception of the physical world; therefore that when we mention the objects of the physical world: matter, light, position, elapsed time, etc. this objects correspond to our anthropological concepts and observations.

2.1.1 Formal Presentation

The ideas developed in START (Space–Time–Action Relativity Theory) are derived from the systematic use of the following principles and postulates [Keller 2000a,b; 2001].

First Principle: *Principle of Relativity:* The *Principle of Relativity* in full requires that the laws of Physics should have the same form for all observers.(Poincaré–Einstein Relativity {Poincaré 1904 [14], Einstein 1905 [3]}). As a basic requirement to connect to experiment it assumes: Constancy of the **value of the observed velocity of light** in vacuum. Independently of the state of movement of the source or of the observer .

First Postulate: There is a geometry, corresponding to space–time, where the First Principle is satisfied (Minkowski space–time with local pseudo-Euclidean structure). Here it is clear that the velocity of light is a fundamental geometrical parameter and the First Principle could be rephrased to assign a unit value to it.

Second Principle: *Principle of Existence:* Each observer considers the physical entity as an amount of action A contained in a given space-time volume V_{ST}, A is a relativistic invariant (in the sense of Minkowski's discussion). Independently of the state of movement of the phenomenon or of the observer.

Second Postulate: There is a geometry corresponding to the union of space–time and action where the First and Second Principles are satisfied (pseudo-Euclidean structure).

Main Theorem KT: Complex Structure Theorem**.** The geometry where the Second Postulate is satisfied is a five-dimensional basis geometry, mathematically corresponding to a particular complexification of space–time. The relation between a 5-dimensional geometry and the complexification of the basis set has been briefly presented in the introduction and will be discussed below.

Third Principle: *Principle of Quantization:* The exchanges in action (among carriers) always occur as integer multiples of h. This makes Planck's principle a universal principle which requires the definition of the entities we have called **action carriers**, because if there are not differentiated action carriers there is not a proper definition of the exchanges of action. This is also a guide and a limitation in the definition of the action carriers and of the V_{ST} associated to the system with total action A.

Fourth Principle: *Principle of Choice:* The distribution of action in space–time corresponding to a physical system is unique and nevertheless any faithful description of this distribution should be equally acceptable. The acceptability of a description, is interpreted here as also implying an optimization of the action distribution and a mechanism to allow the system of carriers to arrive to this optimized state.

Third Postulate: The equivalent acceptable descriptions, for a physical system of carriers, are related by isometries and gauge transformations in the space–time–action geometric space corresponding to the Second Postulate.

Proof of the KT Theorem. : We have the kinematical concept of space–time interval ds^2 with a quadratic form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \ (\mu, \nu = 0, 1, 2, 3) \tag{26}$$

generated by the dx^μ and we want to include as a fifth coordinate the dynamical concept of action and its distribution at each space–time point $X = x^\mu e_\mu$, use the real scalar function $\mathcal{A}(X)$

$$d\mathcal{A}(X) = p_\mu(X) dx^\mu \quad \text{which defines} \quad p_\mu(X) = \partial \mathcal{A}(X)/\partial x^\mu, \tag{27}$$

here $p_\mu(X)$ is a distribution itself, the geometrical meaning is introduced if we consider $p_\mu(X) = \tan \Theta(X, \mu)$. Join formally, using $i^2 = -1$ and

the Clifford numbers $\mathbf{j^2} = +\mathbf{1}$, $\mathbf{j}e_\mu = -e_\mu\mathbf{j}$, $e_\mu \cdot e_\nu = e_\nu \cdot e_\mu = g_{\mu\nu}$ into

$$dS^\mu = dx^\mu(\mathbf{1} + i\mathbf{j}\kappa_0 \tan\Theta(X,\mu)), \tag{28}$$

to obtain from the real quadratic form (in units of distance square)

$$dS^2 = \tfrac{1}{2}\left\{[dx^\mu(\mathbf{1} + i\mathbf{j}\kappa_0 \tan\Theta(X,\mu))e_\mu] \cdot [dx^\nu(\mathbf{1} - i\mathbf{j}\kappa_0 \tan\Theta(X,\nu))e_\nu] + \mathbf{h.c.}\right\}, \tag{29}$$

$$dS^2 = g_{\mu\nu}dx^\mu dx^\nu \mathbf{1}\left(1 - \kappa_0^2 \tan\Theta(X,\mu)\tan\Theta(X,\nu)\right), \tag{30}$$

or, in five-dimensional-like formulation

$$dS^2 = g_{uv}dx^u dx^v = ds^2 - \kappa_0^2 \left|dK(X)\right|^2; \quad u,v = 0,1,2,3,4, \tag{31}$$

where $\kappa_0^2 \left|dK(X)\right|^2$ corresponds to the sum of the squares of action contributions. Note that in order to obtain the desired metric the combination $(i\mathbf{j})^2 = -\mathbf{1}$ had to be used. This has introduced a new vector function (remark: $\mathbf{e}_\mu$ and $\mathbf{j}\mathbf{e}_\mu$ are linearly independent vectors)

$$dK(X) = dK(X)^\mu \mathbf{j}\mathbf{e}_\mu = \sum_\mu \tan\Theta(X,\mu)dx^\mu \mathbf{j}\mathbf{e}_\mu.$$

This vector K mathematically corresponds (in Brillouin's terminology) to the contraction of a tensor density, while in geometric analysis to a vector function $dK(dX,P)$ of two space-time vectors: $dX = dx^\mu e_\mu$ and $P = p_\mu e^\mu$. It is important to notice that it is not the actual value of the action density which is dynamically important but its space-time variations. Even more important for dynamics is that, when the action density is considered a sum $a = \sum_c a_c$ over carriers, the contributions to $dK = \sum_c (dK)_c$ per carrier could be non-zero even is the sum could itself be null. That is the dynamics could be purely **relative dynamics**. The basic dynamical equation is proposed to be

$$\delta \int dS^2 = 0, \tag{32}$$

in a joint minimization of trajectory and action. A particular theory is derived if a Lagrangian is proposed (containing description constrains in addition to this principle). Gravitation will require extremum (shortest) trajectories for a test carrier besides, through the common procedure of Lagrangian minimization, the minimization of action. The universal

constant κ_0, the ratio of a fundamental distance to the fundamental unit of action, expresses the action as an equivalent distance in such a form that $(dx^4)^2 = |(\kappa_0 dK)|^2$, with $g_{mn} = \mathrm{diag}(+1,-1,-1,-1,-1)$ defines the metric of the equivalent five dimensional geometry basis vectors, using $m,n = 0,1,2,3,4$. Here the diagonal time-like term $g_{00} = +1$ and the 'action intervals' term $g_{44} = -1$. ■

For the units to be used in the unified geometry consider the definition (m_0 electron rest mass, c speed of light, $h = 2\pi\hbar$ Planck's constant, e electron charge, r_0 classical electron radius and $\alpha = e^2/\hbar c$)

$$r_{\mathrm{Compton}} = \frac{\hbar}{2m_0 c} = \frac{r_0}{2\alpha}, \quad \kappa_0 = \frac{d_0}{h} = 4\pi r_{\mathrm{Compton}}/h = \frac{1}{m_0 c}. \tag{33}$$

The classical limit of the unification of action to space–time is obtained when $\kappa_0 (\triangle a)_I \to \infty$ for a unit action interval in a form similar to the classical limit of the unification of space and time being obtained when $c(\triangle t)_I \to \infty$ for a unit time interval. Note that in the MKS system of units $|\kappa_0 (\triangle a)_I| \gg |c(\triangle t)_I|$. (also natural but not used here are Planck units in terms of the Planck's constant, the velocity of light and the Newton's gravitational constant $length = l_P = \left(\frac{\hbar G_N}{c^3}\right)^{\frac{1}{2}}$, $time = t_P = \left(\frac{\hbar G_N}{c^5}\right)^{\frac{1}{2}}$ and $mass = m_P = \left(\frac{\hbar c}{G_N}\right)^{\frac{1}{2}}$, [13]).

2.2 The Basic Operators

From the considerations above the fifth coordinate corresponds to the use of x^4 to represent **action density** in space–time, not the accumulated action. Otherwise the action for a given amount of **energy distribution in space** would be an ever increasing function of time. This will be a distribution of energy in space "moving" in time. The given definitions, and a dimensionless imaginary circular variable action density $a(X) = -2\pi i \mathcal{A}(X)/h$, allow the introduction of a series of operators for computing the dynamical quantities defined so far. First, there is an energy–momentum operator $\widehat{\mathbf{p}}$, which would then be, with the **representation** $\gamma^\mu = \gamma(e^\mu)$ and with the action units included in its definition,

$$\widehat{\mathbf{p}} = i\hbar\gamma^\mu \partial_\mu \,. \tag{34}$$

This operator is fundamental to define and study the auxiliary action amplitude function Ψ and the possibility of changing the description by gauging.

Some concepts related to our third principle above. Because action has to be supplied or emitted in multiples of h, when a given energy $\mathcal{E}_0$ or momentum p_0 is considered, a characteristic associated time τ or distance λ are automatically defined as either a period or a wavelength:

$$\mathcal{E}_0 = \frac{h}{\tau} \quad \text{or} \quad p_0 = \frac{h}{\lambda}, \tag{35}$$

also from (34) $\mathcal{E}_0$ and p_0 are geometrically in the tangent space of ST and the associated action in the $e_4 e_\mu$ planes which, from (35), have symplectic structure. This introduces the concept of cyclic variables for (35) in the form $\tau = 2\pi\tau_0$ or $\lambda = 2\pi\lambda_0$.

Then defining, as usual, $\hbar = h/2\pi$ we can write (35) in the form

$$\mathcal{E}_0 = \hbar/\tau_0 = \hbar\omega \quad \text{or} \quad p_0 = \hbar/\lambda_0 = \hbar\varpi. \tag{36}$$

With the definitions above the energy–momentum operator $\widehat{\mathbf{p}} = i\hbar\gamma^\mu\partial_\mu = i\hbar\Box$ has become the usual in Quantum Mechanics. (Here also $\Box = \gamma^\mu\partial_\mu$ is the space–time d'Alembertian operator ($\mu = 0, 1, 2, 3$)). We should remind the reader that our definitions here are direct consequence of the principles above, not additional postulates.

2.3 The Kinetic Energy in START

From our definitions we are considering two quantities: energy $i\hbar \int dV\, \partial_t\, a(X)$ and the corresponding momenta $i\hbar \int dV\, \partial_{x_i}\, a(X)$. One of the basic relations in relativistic dynamics is the transformation of the above quantities with respect to observers in relative motion with a relative velocity v_{12}.

For an observer 1 the integrated energy–momentum $P_\mu = (\mathcal{E}/c, 0, 0, 0)$ can be written

$$\mathcal{E} = mc^2, \tag{37}$$

as by definition for this observer that object is at rest and then the energy corresponds to a mass m and no momenta are involved. From the definition we see that there is a geometrical constraint that any space variation of the action distribution, in the volume of integration, should be symmetric for the observer in relation to which the distribution is at rest (in order that the integral to vanish). Notice that it is not a point-like object, even if it can be considered small in relation to some characteristic distance.

For an, in relative motion with respect to the first, observer 2 the same relations hold. The energy for this observer will be $\mathcal{E}'=m'c^2 = i\hbar \int dV'\partial_{t'}\, a'(X')$, larger than $\mathcal{E}$, given that in his frame of reference the derivatives have to be computed with respect of its own, shorter time intervals, dt', where

$$\frac{dt'}{\sqrt{1-\frac{v_{12}^2}{c^2}}} = dt \quad \text{from} \quad (cdt)^2 - (v_{12}dt)^2 = (cdt')^2. \tag{38}$$

For the observer 2 the measurement corresponding to the prime system

$$\mathcal{E}' = \frac{\partial}{\partial t'}\mathcal{A} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{\partial \mathcal{A}}{\partial t} = \frac{\mathcal{E}}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{39}$$

The action A related to that physical entity is invariant

$$\mathcal{A} = i\hbar \int_{V_{ST}} dV_{ST} a(X) = i\hbar \int_{V'_{ST}} dV'_{ST} a'(X). \tag{40}$$

Then for the observer 1 in relative motion the action $\mathcal{A}$ corresponds to a smaller time $\Delta t'$ and consequently the associated energy is $\mathcal{E}' = m'c^2 = \mathcal{E}/\sqrt{1-v^2/c_2}$. Observer 1 concludes that in the moving system

$$\mathcal{E}' = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \rightarrow \quad \text{with} \quad \lim(v \ll c) \rightarrow mc^2 + \tfrac{1}{2}mv^2 + ..., \tag{41}$$

and also, because the prime system is considered in motion, he can call the energy $\mathcal{E}'$ the sum of the rest (mass) energy $\mathcal{E}$ and the kinetic energy $\mathcal{E}_k$ which for the slow motion is approximated by the typical $\mathcal{E}_k = \frac{1}{2}mv^2$ term. Otherwise for the system in motion a characteristic inverse length of the energy distribution has been changed from p_0/h to p'_0/h, together with a characteristic inverse time $\mathcal{E}/h \rightarrow \mathcal{E}'/h$, these quantities should be part of the gauge-free description used in the theory.

Considering (41) again, we can write

$$\mathcal{E} = h\nu_0, \quad \mathcal{E}' = h\nu = h(\nu_0 + \Delta\nu), \tag{42}$$

and then, mainly in the low velocity approximation, we can write

$$\mathcal{E}_k = h\nu_k \quad \text{where} \quad \nu_k = \Delta\nu \geq 0. \tag{43}$$

Here ν corresponds to the characteristic frequency appearing in the Dirac equation, where the total energy is considered, whereas $\lambda = p/h$ and ν_k would be the characteristic wavelengths and frequencies as, for example, in the Schrödinger equation, as derived from START.

2.4 Dynamical Principles

In space–time–action geometry the main dynamical principle should be that all elementary trajectories be minimal. That is, from our definition of carriers above where $d\mathcal{A} = \{\epsilon_c \int \rho_c(X)dx\}dt$, a minimization of a total action A (in the case when we admit that the $\kappa_0 >> 1$ predominates) or a minimization of a START trajectory. Defining the (square of the) differential $(dS)^2 = (ds)^2 - (d\mathbf{a})^2$, where $(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu$ is the space–time differential and $(d\mathbf{a})^2$ the equivalent length of the action differential, the minimal principle

$$\delta(dS)^2 = 0, \tag{44}$$

could be separated into a kinematical principle similar to the one of (general) relativity and an additional principle of minimum action

$$\delta(ds')^2 = 0, \quad \delta(d\mathbf{a}')^2 = 0. \tag{45}$$

Where we have defined

$$(ds')^2 = (ds)^2 - \left[(d\mathbf{a})^2 - (d\mathbf{a}')^2\right], \tag{46}$$

as a modified space–time interval square which, in fact, corresponds to considering a curved effective space–time as will be shown below. The action interval square $(d\mathbf{a}')^2$ corresponds to some 'inactive' part of the action in relation to a given geometrical description.

For some phenomena, light as the main example, (45) is separately obeyed given that $(cdt)^2 - (dx)^2 = 0$ and $g_{\mu\nu}p^\mu x^\nu = 0$ because $\varepsilon = pc = h\nu = hc/\lambda$. Otherwise our principle of minimal action corresponds to the one universally accepted in the formulation of physical principles.

2.4.1 Point-like STA Trajectories

To make connection with classical relativistic approaches a point–like carrier can be defined using a Dirac's delta function distribution

$a_0(X, x_p) = \kappa_0 a_p(t)\delta(x - x_p)$, following a point–like trajectory x_p reduced to a line in space. For the observer there would be two types of trajectories to study fields:

1) massive (m_0 parameter), that when at rest have STA interval

$$dS^2 = g_{AB}dx^A dx^B = 0 = (cdt)^2 - \left(\kappa_0 m_0 c^2 dt\right)^2, \tag{47}$$

which in fact defines $\kappa_0 = 1/m_0 c$ in terms of m_0 and c (in our case the electron mass and the velocity of light); and

2) massless ($e = h\nu$, $p = h/\lambda$, $c = \lambda\nu$ parameters) for which $x^2 + y^2 + z^2 = (ct)^2$ and $a^2 = 0$, where as a consequence

$$dS^2 = g_{AB}dx^A dx^B = 0 = (cdt)^2 - \left(dx^2 + dy^2 + dz^2\right) - (\kappa_0 d\mathcal{A})^2. \tag{48}$$

The ordinary density of action can be considered as a density of point–like trajectories $a(X) = \rho(X) \int a_0(X', x)dx'$.

3 Maxwell Equations from START

Let us formally show that the Maxwell equations in their standard textbook form are analytical properties of the third derivatives of the action density attributed to a test carrier (with 'electric' charge) as induced by a collection of interacting carriers (carriers type b, those for which the energy–momentum is partitioned in an interdependent form). Then the energy per carrier can be considered the derivative of a scalar field but the momentum for carriers of type b can not be solely considered the gradient of a scalar field. In this particular case assume that we describe a set of carriers as interacting by partitioning an amount of energy (the interaction energy $\mathfrak{E}_e(X)$) among them, allowing the partitioning of the momentum, which is a quantity with a sign related to directions in space, to be the sum of the momenta of the different carriers, that is the sum could be any number including zero whereas the individual momenta will be described as the result of the overall momentum $\left(\partial a_e(X)/\partial x^i\right)\mathbf{e}^i$ plus the momentum $\Delta_R \quad p_{e,i}\mathbf{e}^i$ induced by the interactions among the carriers. These interaction momenta fields might then have a non null rotational part.

In the reference frame of a given observer the induced action density (arising from the interaction), denoted by $a_e(X)$, per unit charge ($\Rightarrow$puch) of a test carrier at space–time point $X = x^\mu e_\mu$ (here the greek indices $\mu = 0, 1, 2, 3$ and $x^0 = ct$ whereas the space vectors $\mathbf{q} = q^i\mathbf{e}_i = q_i\mathbf{e}^i$

, $\mathbf{e}_i = e_0 e_i$, $i = 1, 2, 3$ are written in bold face letters and we use the standard definitions of "dot" and "cross" products), the related energy density $\mathfrak{E}_e(X)$ and the total (external plus induced) momentum density $\mathbf{p}_e$, **per unit charge of the test carrier**, would be

$$\mathfrak{E}_e(X) = \frac{\partial a_e(X)}{\partial t}, \qquad \mathbf{p}_e = p_{e,i}\mathbf{e}^i = (\frac{\partial a_e(X)}{\partial x^i} + \Delta_R \quad p_{e,i})\mathbf{e}^i, \qquad (49)$$

and the, by definition, electric field intensity $\mathbf{E}$ is the force (puch) corresponding to this terms

$$\mathbf{E} = (\frac{\partial \mathfrak{E}_e(X)}{\partial x^i} + \frac{\partial p_{e,i}}{\partial t})\mathbf{e}^i = \nabla \mathfrak{E}_e(X) + \frac{\partial \mathbf{p}_e}{\partial t},$$

with time dependence

$$\frac{\partial \mathbf{E}}{\partial t} = (\frac{\partial^2 \mathfrak{E}_e(X)}{\partial t \partial x^i} + \frac{\partial^2 p_{e,i}}{\partial t \partial t})\mathbf{e}^i = 2\frac{\partial^3 a_e(X)}{\partial t \partial x^i \partial t}\mathbf{e}^i + \frac{\partial^2 (\Delta_R p_{e,i})}{(\partial t)^2}\mathbf{e}^i.$$

By definition of interacting carriers, we have added in (49) the term $\Delta_R \quad p_{e,i}\mathbf{e}^i$ as the effect of the conservation of **interaction transverse momenta** between the field representing the rest of the carriers with that sort of charges. This is by definition the origin, in START, of a magnetic field intensity $\mathbf{B} = B_k \mathbf{e}^k$ that will appear as the curl of the momentum (puch) of an interaction field acting on a carrier of type b. The axial vector

$$\mathbf{B} = \left(\frac{\partial p_{e,i}(X)}{\partial x^j}\right)\mathbf{e}^j \times \mathbf{e}^i = \nabla \times \mathbf{p}_e,$$

with time dependence

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\partial^2 p_{e,i}(X)}{\partial t \partial x^j}\mathbf{e}^j \times \mathbf{e}^i.$$

.Otherwise the space variation of $\mathbf{E}$, including the **interaction transverse momenta**,

$$\nabla \quad \mathbf{E} = \nabla \cdot \mathbf{E} + \nabla \times \mathbf{E}, \qquad (50)$$

will then also include a transversal (rotational) term

$$\nabla \times \mathbf{E} = \frac{\partial^2 p_{e,j}(X)}{\partial x^i \partial t}\mathbf{e}^i \times \mathbf{e}^j = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{(2nd Maxwell Equation)}$$

relation which is the direct derivation in START of this well known Maxwell equation. The scalar term $\nabla \cdot \mathbf{E}$ being a divergency of a vector field should be defined to be proportional to a source density

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho = \sum_i \left(\frac{\partial^3 a_e(X)}{\partial x^i \partial x^i \partial t} \right) = \frac{\partial}{\partial t}\nabla^2 a_e(X),$$

(1st Maxwell Equation)

and will be given full physical meaning below.

For the space variation of $\mathbf{B}$ we have

$$\nabla \quad \mathbf{B} = \nabla \cdot \mathbf{B} + \nabla \times \mathbf{B}.$$

The first term vanishes identically in our theory because it corresponds to the divergence of the curl of a vector field

$$\nabla \cdot \mathbf{B} = 0, \qquad \text{(3rd Maxwell Equation)}$$

while the last term, using $U \times V \times W = V(U \cdot W) - (U \cdot V)W$

$$\nabla \times \mathbf{B} = \nabla \left(\nabla^2 a_e(X) \right) - \nabla^2 \mathbf{p}_e = \mu_0 (\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}),$$

(4th Maxwell Equation)

where the additional dimensional constant μ_0 is needed to transform from time units (used in the conceptual definition of a current $\mathbf{J} = \nabla \left(\nabla^2 a_e(X) \right) / \mu_0$) into distance units and $\epsilon_0 \mu_0$ will have then units of T^2/D^2 or inverse velocity squared, in fact (see below) $\epsilon_0 \mu_0 = c^{-2}$corresponding to have used twice the derivative with respect to t and not to $x^0 = ct$. The 4th Maxwell Equation, defining $\mathbf{J}$, is related to the analogous of the 1st Maxwell Equation and to the analogous of the 2nd Maxwell Equation, also to a Lorentz transformation of the 1st Maxwell Equation.

As we have mentioned for carriers of type a the magnetic-like field will vanish $\mathbf{B} = 0$ and then the interaction will be simpler, obeying only the source first Maxwell equation, this is for example the standard approach to the case of the gravitational interaction. Because all carriers will have the action partitioned as in the case of carriers of type a a gravitational-like interaction is universal in START. Otherwise the full Maxwell-like interaction will be related to the concept of charges for carriers of type b. A more comprehensive theory of gravitation should use

the formal structure of the Maxwell equations and effects depending in the currents of the masses should be included. This, because gravitation is 38 orders of magnitude weaker than electromagnetic interactions, require extremely large speeds and densities, situation not foreseeable in our observable physical world, can not be expected to be directly inspected experimentally.

The derived Maxwell equations are formally equivalent to the original Maxwell equations, then they are: first local equations and second linear in the sources (ρ and $\mathbf{J}$).

Both the 4th Maxwell Equation, defining $\mathbf{J}$, related to a Lorentz transformation of the 1st Maxwell Equation, defining ρ, can immediately be integrated using geometric analysis techniques, the standard approach being of fundamental conceptual consequences in START. The space divergence of a non-solenoidal vector field like $\mathbf{E}$ is immediately interpreted as its 'source' given that

$$\Delta\mathbf{E} = (\nabla \cdot \mathbf{E})\, S\Delta\mathbf{x},$$

and this equation is integrated using the standard geometric theorem that the volume integral of a divergency $\nabla \cdot \mathbf{E}$ equals the surface integral of the normal (to the surface) component of the corresponding vector field $\mathbf{n} \cdot \mathbf{E}$. Where $\mathbf{n}$ is a unit vector perpendicular to the surface S (in the text-book formula below $S = 4\pi r^2$ corresponding to an integration sphere of radius r containing a spherically symmetric source density $\rho(r)$ generating a force field per unit charge $\mathbf{E} = E(r)\frac{\mathbf{r}}{r}$, note we are using here the notation $\mathbf{x} \to \mathbf{r}$ for symmetry purposes) of the integration volume $V = 4\pi r^3/3$:

$$\int_V (\nabla \cdot \mathbf{E})\, dV = \int_V \frac{4\pi}{\epsilon_0}\rho(r')r'^2 dr' = \frac{1}{\epsilon_0}Q = \int_S E(r)\frac{(\mathbf{r}\cdot\mathbf{n})}{r}dS = 4\pi r^2 E(r),$$

$$\mathbf{E} = E(r)\frac{\mathbf{r}}{r} = \frac{Q}{4\pi\epsilon_0 r^2}\frac{\mathbf{r}}{r}, \qquad \mathbf{r}\cdot\mathbf{n} = r.$$

That is: *the inverse square law of the Newtonian and Coulombic forces are geometrical consequences of the definition of interaction among charged carriers.* But this is not a derivation of the values of the *Newtonian and Coulombic* constants G and ϵ_0.

In the case of the, generated by a current, magnetic force field intensity $\mathbf{B}$, being a space bivector, it is also a direct geometrical consequence that its source can (must) be a current vector density $\mathbf{J}$. For a

small ($l \ll r$) current source at the origin of coordinates: (in the sphere $\mathbf{r}^t(\theta,\phi) \bullet \mathbf{r}^{ct} = 0$, $\left(\mathbf{r}^t\right)^2 = \left(\mathbf{r}^{ct}\right)^2 = 1$)

$$\int_V (\nabla \times \mathbf{B})\, dV = \int_S B(r) \left(\mathbf{r}^t(\theta,\phi) \times \mathbf{n}\right) dS = 4\pi r^2 f B(r) \mathbf{r}^{ct},$$

$$\int_V 4\pi\mu_0 \mathbf{J}\delta\left(r'\right) r'^2 dr' = \mu_0 M \mathbf{r}^{ct} = 4\pi r^2 f B(r) \mathbf{r}^{ct} \Rightarrow \mathbf{B} = B(r)\mathbf{r}^t = \frac{\mu_0 M}{4\pi r^2 f}\mathbf{r}^t,$$

and its *Amperian inverse square law is also a geometrical consequence of the definition of transverse interaction among charged carriers.*

3.1 Tautology between Fields and Charges

Now, coming back to our considerations in the introduction about carriers and action partitioning

Consider the particular case of an initial situation without electromagnetic phenomena being present $\mathbf{E} = 0$, $\mathbf{J} = 0$, $\mathbf{B} = 0$ and $\rho = 0$, and that in the process of creating a pair of 'interacting carriers' with charges Q, an initial pulse of current $\mathbf{J}(\mathbf{r},t) = Q\mathbf{v}(\mathbf{r})\delta(t_0)$ is assumed to have been generated, this induces an electric field for $t > t_0$ from the Maxwell Equations derived above:

$$\partial_t \mathbf{E} = -\mathbf{J}(\mathbf{r},t)/\epsilon_0 + \nabla \times \mathbf{B}/\epsilon_0\mu_0 = Q\mathbf{v}(\mathbf{r})\delta(t_0)/\epsilon_0,$$

$$\mathbf{E}(\mathbf{r},t;t > t_c) = -Q\mathbf{v}(\mathbf{r})/\epsilon_0 \quad \text{then} \quad \rho(\mathbf{r},t) = -Q\nabla \cdot \mathbf{v}(\mathbf{r}), \quad (51)$$

$$\text{and} \quad \partial_t \mathbf{B}(\mathbf{r},t;t > t_c) = -\nabla \times \mathbf{E}(\mathbf{r},t;t > t_c) = Q\nabla \times \mathbf{v}(\mathbf{r})/\epsilon_0, \quad (52)$$

showing that this virtual mechanism (in our process to establish a partitioning of energy and momentum among charged carriers) requires the actual allocation of physical properties, to the collection of created carriers, as far as the divergence of the current pulse creates a charge and the rotational of the current pulse of velocity field $\mathbf{v}(\mathbf{r})$ a magnetic multipole. We can see that the definitions are a circuitous procedure and that it can equally be said that we have sources that create fields or that the fields generate the concept of sources. Notice that a source (of a pair) with a circular current generates an electric and a magnetic field, as in the case of the electron, and the carrier also show the presence of the spin associated to the solenoidal current. Within the postulates above the action of circular currents will have to be quantized in terms of Planck's

$\hbar$, a pair of currents in terms of $\hbar/2$ each. The excess energy-momentum-angular momentum (electromagnetic wave) is itself (from the Maxwell Equations) an action carrier traveling at the speed $c = (\epsilon_0\mu_0)^{-\frac{1}{2}}$.We see that the concept of electromagnetic (light) wave is basic in the study of action density and its distribution in space–time, the quantization condition is also fundamental in this case.

In relation to (**??**) in the introduction, consider the process of creating a carriers-pair where the current pulse is r^{-2} where a decaying pulse,

$$\mathbf{v}(\mathbf{r}) = \frac{-1}{4\pi r^2}\frac{\mathbf{r}}{r}, \qquad 4\pi r^2\mathbf{v}(\mathbf{r}) = \frac{\mathbf{r}}{r},$$

then the generated **permanent** electric field is

$$\mathbf{E}(\mathbf{r}) = \frac{\partial^2 a_e(\mathbf{r})}{\partial r \partial t} = \frac{Q}{\epsilon_0 4\pi r^2}\frac{\mathbf{r}}{r},$$

representing the creation of the electric effect of a permanent charge Q .The linearity of the equations allows the consideration of a charge distribution $\rho = Q\rho_c$, and not necessarily of a, not compatible with the principles of START, point charge. We clearly see the inseparability of the concept of charge and of the field of that source charge. The concept of action density in space–time is fundamental in our discussion.

In actual systems there will be a media which represents globally a collection of carrier fields and we can define a more general intensity bivector G which keeps the linear-like description but corrects for the influence of that media

$$G \equiv \mathbf{D} + e_5\mathbf{H},$$

$$\square \wedge G = 0, \quad \square \cdot G = \frac{J}{\epsilon_0}.$$

defining the influence of the media as a coefficient affecting the original intensities

$$\mathbf{D} \equiv \epsilon_r(X)\mathbf{E}, \qquad \mathbf{B} \equiv \mu_r(X)\mathbf{H},$$

in our approach the $\epsilon_r(X)$ and $\mu_r(X)$ fields correspond to a redefinition of $a_e(X)$ and of the transverse part of $\mathbf{p}_e$.

The basic properties being local relations are not changed in this substitution.

3.2 Newtonian Gravity

The analysis above depends only on the assumption of the decomposition of the action and of the energy momentum into contributions per carrier. We have mentioned that there could be at least two types of carriers: universal, or type a, and with particular **pair-wise relative** charges, or type b. For the first type a universal type of interaction is possible and, because there are no relative charges, all carriers of action would participate in this category. In the equations above, even if related to the Maxwell equations, all steps of the analysis apply to carriers of type a, the only difference is that there the concept of charge is a universal quantity. In START there is only one universal quantity in relation to a given observer as reference: the energy related to any carrier, there is no room for positive and negative charges in this case. This is different for the **relative charges** which otherwise could naturally be of opposite sign as far as they are related to the process of **creating two charged carriers from any previously existing carriers** and thus breaking the description symmetry of the previous stage.

The solution of the first Maxwell equation, when applied to gravitation considering the mass $M = \mathcal{E}/c^2$, implies (as shown above) the Newtonian gravitational potential equation per unit test mass m:

$$\mathfrak{E} = V = -G\frac{M}{r} \text{ , that is } \mathbf{E} = -G\frac{M}{r^2},$$

the usual relations in the textbook formulation of Newtonian gravity. The constant $G = 1/4\pi\epsilon_0^{(g)}$. If we define $c^2\mu_0^{(g)}\epsilon_0^{(g)} = 1$ then $\mu_0^{(g)} = 4\pi G/c^2$.

This straightforward result will be used below. In this approach to gravitation there is no quantization properly, there being no exchange of action, only a description of the sharing of energy between a source carrier and a test carrier.

When general relativity is analyzed we will immediately find the need for the consideration of the momentum of the test particle moving according to the geodesic of the curved space–time. As mentioned above we could also include the transverse momentum in the interaction between sources of the gravitational field, where it was mentioned that gravitational forces are 38 orders of magnitude weaker than electric forces. On top of that we should add that, as by definition the magnetic forces are expressed as rates of change of momenta, then a factor of the inverse

of the velocity of light c will appear explicitly in the relation $\mathbf{E}_g$ to $\mathbf{B}_g$ equations when $t \Rightarrow ct$

$$\nabla \times \mathbf{E}_g = \frac{\partial^2 p_j^{(g)}(X)}{\partial x^i \partial t} e^i \times e^j = -\frac{\partial \mathbf{B}_g}{\partial t},$$
$$\nabla \times \mathbf{B}_g = \nabla \left(\nabla^2 a_e^{(g)}(X) \right) - \nabla^2 \mathbf{p}_e^{(g)} = \mu_0^{(g)} (\mathbf{J}_g + \epsilon_0^{(g)} \frac{\partial \mathbf{E}_g}{\partial t})$$
$$= \frac{4\pi G}{c^2} \mathbf{J}_g + \frac{1}{c^2} \frac{\partial \mathbf{E}_g}{\partial t},$$

and then if the equivalent of magnetic interactions were to be added to gravitation the factor of $1/c^2$ is to be added.

3.2.1 Force Intensities.

From the previous sections and definitions an interaction field intensity space–time bivector F can be considered as generated from another space–time vector distribution $J(\mathbf{x}, t) = J^\mu e_\mu = \rho e_0 + \mathbf{J} e_0$, usually called the current (of the generating carriers fields):

$$\Box F = \frac{J}{\epsilon_0}, \; F = \Box^{-1} \frac{J}{\epsilon_0} \quad ;$$

or, analytically inverting the relation, we obtain F from the source current which usually is considered the more important physical quantity. This current is nothing else that the (observer dependent) description of the carrier distribution $\rho_c(\mathbf{x}, t)$ defined by the reference observer. In our approach the space–time current J is not an additional concept, it is formally defined from the derivatives of the action. The generating four dimensional vector current J is usually analyzed, for a given observer frame of reference with time-like vector e_0, in the time-like (scalar) part density $\rho = Q\rho_c$ proportional to the carrier density ρ_c and the space-like (vector) part 'current density' $\mathbf{J}$, writing $F = \mathbf{E} + e_5 \mathbf{B}/\mu_0$, $\Box \Rightarrow (\partial_t + \nabla)$ and $e_0 J = \rho - \mathbf{J}$

$$(\partial_t + \nabla) \left(\mathbf{E} + e_5 \frac{\mathbf{B}}{\mu_0} \right) = \frac{(\rho - \mathbf{J})}{\epsilon_0},$$

the intensity vectors $\mathbf{E}$ and $\mathbf{B}$ can be analyzed in curlless and solenoidal parts,

$$\nabla \mathbf{E} = \nabla \cdot \mathbf{E} + e_5 \nabla \times \mathbf{E}, \; \nabla \mathbf{B} = \nabla \cdot \mathbf{B} + e_5 \nabla \times \mathbf{B},$$

and the basic equations separated for a given observer

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho/\epsilon_0 && \text{(space scalar part)} \\ \partial_t \mathbf{E} &= -\mathbf{J}/\epsilon_0 + e_5 \nabla \times \mathbf{B}/\epsilon_0\mu_0 && \text{(space vector part)} \\ \partial_t \mathbf{B} &= -\nabla \times \mathbf{E} && \text{(space bivector part)} \\ e_5 \nabla \cdot \mathbf{B} &= 0 && \text{(space trivector part)} \end{aligned} \tag{53}$$

4 The electron as a general relativity test particle

In general relativity, meant to be a comprehensive theory, the best known solutions are developed for the so called matter-free space and a test particle. We use this concept to show that (46) corresponds to a description of the action distribution which agrees with the conceptual development of General Relativity.

4.1 The Schwarzschild solution

In our present theory there are two fundamental carrier structures: the massless fields and the massive electron field with basic relation

$$\mathcal{E}^2 = \left(\mathcal{E}_0 + \Delta\mathcal{E}\right)^2, \; \mathcal{E}^2 - \mathcal{E}_0^2 = (pc)^2, \tag{54}$$

where $\Delta\mathcal{E}$ is any gauge-free energy contribution and $\mathcal{E}_0 = m_0 c^2$ the energy, at rest relative to some observer, considered to be the mass of the carrier.

The concept of test particle in general relativity in the Schwarzschild solution is compatible with the Newtonian limit for the interaction gravitational energy

$$\Delta\mathcal{E}\,(r) = -m_0 \frac{GM}{r}, \tag{55}$$

where M is the total mass of 'the system' of radius r_s we are exploring with the test particle and, conceptually, with the START use of the action square difference, writing $\mathcal{E} = \mathcal{E}_0 + \Delta\mathcal{E}$ for large (classical limit) values of $r > r_s$

$$\begin{aligned} \mathcal{E}^2 - \mathcal{E}_0^2 &= \mathcal{E}_0^2 + 2\mathcal{E}_0\Delta\mathcal{E} + \left(\Delta\mathcal{E}\right)^2 - \mathcal{E}_0^2 = (pc)^2 \\ &= 2\mathcal{E}_0\Delta\mathcal{E} + \left(\Delta\mathcal{E}\right)^2 \to -2m_0c^2 m_0 \frac{GM}{r} + \left(m_0 \frac{GM}{r}\right)^2, \end{aligned} \tag{56}$$

this corresponds to the energy and radial momentum terms in $(d\mathcal{A})^2 - (d\mathcal{A}')^2$ if $(d\mathcal{A}')^2 = (m_0 c^2 dt)^2$, and substituting in (46) using $\kappa_0 = 1/m_0 c$ and space spherically symmetric coordinates t, r, θ, ϕ we obtain

$$(dS)^2 = \left(1 - 2\frac{GM}{c^2 r} + \left(\frac{GM}{c^2 r}\right)^2\right) c^2 (dt)^2 \qquad (57)$$
$$- \left(1 + \frac{2GM}{c^2 r} - \left(\frac{GM}{c^2 r}\right)^2\right) (dr)^2 - r^2 \left[(d\theta)^2 + \sin^2\theta\,(d\phi)^2\right],$$

which is the Schwarzschild [Schwarzschild 1916] metric in the limit of $r \gg GM/c^2$ (sometimes (57) is called the Eddington form [Eddington 1928], notice $(1-x)^{-1} \cong (1+x)$, $x \ll 1$, this relation is also used below.).

It is customary to write [Snygg 1997] the interval square using in our case $f(r) = 1 + b^2(r)$ and $h(r) = 1 - b^2(r)$

$$f(r) = \left(1 - 2\frac{GM}{c^2 r} + \left(\frac{GM}{c^2 r}\right)^2\right) \text{ and } \; h(r) = \left(1 + \frac{2GM}{c^2 r} - \left(\frac{GM}{c^2 r}\right)^2\right), \qquad (58)$$

for $c^2 r \gg GM$ we obtain the Schwarzschild relation $f \cong h^{-1}$, which can afterwards be used for all r in matter free space following the Einstein's definition which requires the curvature to be identically zero.

The result (57) shows that our approach provides a conceptual understanding of the role of sources carriers and test particles in general relativity. It also shows the possibility of extending the analysis to circumstances more difficult to consider within the traditional approaches.

Once we have obtained the Schwarzschild metric we can now find the **curved hypersurface in START corresponding to the curved space–time** where the test particles are assumed to move. Formally we need to define a set of vectors e_μ, $\mu = 0, 1, 2, 3$, $g_{\mu\nu} = \text{diag}\,(1, -1, -1, -1)$, and their reciprocal, in terms of a vierbein using the Minkowski space reference vectors

$$\left\{e_\mu^{(0)} = \widehat{e}_\mu,\, \mu = 0, 1, 2, 3;\; e_\mu^{(0)} e_\nu^{(0)} + e_\nu^{(0)} e_\mu^{(0)} = 2g_{\mu\nu}^{(0)}\right\}.$$

Use the induced metric

$$g_{\alpha\beta} = \text{diag}\left[c^2 f\left(r\right), -h\left(r\right), -r^2, -r^2 \sin^2\theta\right], \tag{59}$$

From this, it is clear (see [Snygg 1997]) how to construct an orthonormal system of vectors (in the Eddington's representation)

$$\begin{aligned} e_t &= c\left(f\left(r\right)\right)^{\frac{1}{2}} \widehat{e}_0, \quad e_r = \left(h\left(r\right)\right)^{\frac{1}{2}} \widehat{e}_1, \\ e_\theta &= r\widehat{e}_2, \qquad\qquad e_\phi = r\,\sin\theta\widehat{e}_3. \end{aligned} \tag{60}$$

The extension of this analysis to the space–time regions where the sources are considered to exist (as an action distribution) will be given in the next section.

Notice that in (57), because of the use of κ_0, the test particle parameters have cancelled. This is the reason for this approach to be universal. As a result the solutions for the case of gravitation do not depend at all in the test particle considered, provided that its definition is the Einstein definition: a sufficiently small mass to be introduced into the system without noticeably disturbing it. The action related to the intrinsic energy–momentum of the test particle is 'inactive' in the analysis. Otherwise we could have introduced in (55), (56) and (57) other contributions to the action, but this can not be done in a universal form if they are of electromagnetic type (a factor e/m_0 can not be removed). Anyhow, it is suggestive that it can be done, new terms appear for **that** test particle, including cross terms like $\left(QM/r^2\right)\left(e/m_0\right)$ and others. Also in (57) there is a quadratic, repulsive, term which can have physical significance, for distances much smaller than the ones where general relativity has been successfully tested.

One of the possible symmetries in START is the transformation of position vectors $\mathbf{y}$ in START to a new set $\{\mathbf{y} = x^u e_u;\ u = 0, 1, 2, 3, 4\}$

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}) = x'^u e'_u. \tag{61}$$

which describes the curvature of the space–time part necessary for representing physical interactions, at the expense of defining 'test' carriers which are now considered as non-interacting amongst themselves, but defining the new representation (61) of position vectors and coordinates in START.

4.1.1 General relativity in START

From our previous analysis, the structure equivalent to general relativity [Einstein 1915a, 1915b, 1915c, 1916] in START is the following:

- In the flat space–time–action geometry a distribution of action is given and analyzed as corresponding to the total matter and interaction fields (radiation) content. For a given observer at a given time this will appear as an energy, momentum, and stress distribution. The usual name for this is the energy–momentum–stress density tensor.

- Basically one obtains the structure corresponding to general relativity by the process of transforming this $1 + 3 + 1$ geometrical description into an equivalent $1 + 3$ description given by a curved space–time.

- Even if the projection of the surface in five dimensions as a four-dimensional space corresponds to the curved space–time of general relativity, the physical meaning of this curved space–time is given by defining the trajectories of 'test' particles as the geodesics in this 4-D space. The definition of the test particle is that of a sufficiently small and sufficiently small energy–momentum physical entity such that its trajectory can be considered as that generated by a moving point which will not noticeably change the assumed energy–momentum–stress distribution.

- General relativity was constructed with gravitation as the basic interaction; this is very important because gravitation has proved this far to be universal, that is, any two amounts of energy, within experimental conditions, attract each other through a gravitational interaction.

- Other interactions than gravitation could also be included in general relativity either as the amount of energy–momentum they represent or as part of the description of the energy–momentum which is attributed as belonging to a given 'test' particle. See below.

The analysis we have presented here corresponds to changing the status of general relativity from a physical model to a part of a deductive theory where a specific form of description has been selected: that of

not using action as a fifth variable but as a guide for constructing the curvature of space–time to obtain the characteristic equations of general relativity.

4.1.2 A charged carrier as a test particle in general relativity

A charged particle at rest which is acted on by gravitational and electromagnetic interactions will have for the (attributed) total energy (at distances large enough such that the collection of masses with which the test carrier interacts are collectively represented by the volume integral of a mass density $\mathcal{M}(\mathbf{r})$) in the presence of the mass $M = \int_D \mathcal{M}(\mathbf{r})dv$, the following description:

$$\varepsilon = m_0 c^2 - m_0 \frac{GM}{r} + e\frac{Q}{r}.$$

Substituting this in (54)–(58) will change the functions $f\,(r)$ and $h\,(r)$ into

$$f(r) = 1 - 2\frac{GM}{c^2 r} + \left(\frac{GM}{c^2 r}\right)^2 - \frac{e}{m_0} Q \frac{GM}{c^4 r^2} + 2\frac{eQ}{m_0 c^2 r} + \left(\frac{eQ}{m_0 c^2 r}\right)^2,$$

$$h(r) = 1 + 2\frac{GM}{c^2 r} - \left(\frac{GM}{c^2 r}\right)^2 + \frac{e}{m_0} Q \frac{GM}{c^4 r^2} - 2\frac{eQ}{m_0 c^2 r} - \left(\frac{eQ}{m_0 c^2 r}\right)^2.$$

The analysis of these functions would lead to the following conclusions:

1. Besides the attractive gravitational term there is a (quadratic) repulsive term which will dominate at intermediate distances. Time coordinates do not become imaginary or discontinuous.

2. The electric part of the interaction depends explicitly in the e/m_0 ratio of the test particle, and it can then not be a universal behavior of a test particle.

Otherwise, when the relations corresponding to general relativity are derived from START, those entering into the experimental proofs of the validity of general relativity (considered this far) are not changed and retain their validation status.

4.2 The Mathematical Structure of General Relativity from START

Once we have seen that an electron used as a test particle in the START geometry allows us to obtain the Schwarzschild metric we can now proceed to a systematic derivation of the structure of general relativity from START.

The main considerations are the following.

a) General relativity is a geometric theory describing the trajectories of test particles as the natural trajectories, geodesics, in curved space–time geometry.

b) The curved space–time is obtained by incorporating, within STA, equivalent distances from the action part into the ST part. At the level of the test particle the action that was incorporated into the initially flat space–time was the **additional** interaction action, gravitational in our example, for the test particle. This corresponds to the amount of action which would be described as belonging to the particle in classical mechanics when the particle is considered as being 'acted' upon by the gravitational field. The mathematical description of the fact that the particle is now a non-disturbing test object is introduced by subtracting from the square of the total action the square of the action corresponding to the direct description of the particle. That is, general relativity is a theory where the geometry describes everything that is to be described, through the curved space–time, and the test particle is only an auxiliary in this description.

c) The quadratic form obtained was afterwards analyzed using intrinsic geometrical techniques to have, in accordance to the basic geometric postulate of Einstein, a purely geometrical theory. The basic equations, everywhere in space, are the transfer of the intervals corresponding to the relevant action (squared) to the flat quadratic form of space–time.

d)We can directly consider that the quadratic form defines the metric tensor of the new geometry, and then use the definition of the curvature from the metric tensor in the generated curved space–time, to obtain a relation between the curvature and the energy–momentum–stress tensor. The result is a derivation of the starting

equations of Einstein, which in his formulation are a basic postulate, where the presence of matter induces the curvature of space–time.

We write the equations related to the considerations above.

4.2.1 The metric in General Relativity

Once we have created the equivalent curved space–time the metric in GR is given through the use of the line element (here $g_{\mu\nu} = g_{\mu\nu}^{\mathrm{GR}}$ from the choice of action allocation to geometry and $g_{\mu\nu}^{0}$ corresponds to flat space–time)

$$dS^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = g_{\mu\nu}^{0} \left(1 + \Delta g_{\mu\nu}\right) dx^{\mu} dx^{\nu}, \tag{62}$$

which in turn defines local vector frames (up to a gauge transformation)

$$e_{\mu}^{GR} = \mathbf{h}\left(e_{\mu}\right), \quad \text{such that} \quad g_{\mu\nu} = e_{\mu}^{\mathrm{GR}} \cdot e_{\nu}^{\mathrm{GR}},$$

with $\mathbf{h}\left(x\right)$ a vector-valued function of vectors usually represented through a vierbein h_{μ}^{ν}. In general, from our complex type of structure,

$$\mathbf{h}\left(\mathbf{v}\right) = \mathbf{v} + f^{\mu\nu}\left(x\right) \mathbf{v} \cdot \left(e_{\mu} e_{\nu}\right). \tag{63}$$

In (58) above

$$\mathbf{h}\left(\mathbf{v}\right) = \mathbf{v} + ib\left(r\right) \mathbf{v} \cdot \left(e_0 e_r\right), \quad b^2 = \kappa_0^2 \left(\Delta a\right)^2,$$

with Δa the action being allocated to be represented as space–time curvature.

Anyhow there are additional Lorentz boosts and rotations which can be applied, without changing dS^2 to a set of reference functions $\mathbf{h}^{(0)}$. They are generated by the bivector functions $\Omega\left(x\right)$ to obtain $\mathbf{h} = \Omega \mathbf{h}^{(0)}$. The set of fields $\{\mathbf{h}^{(0)}\left(x\right), \Omega\left(x\right)\}$ define the local geometry of the 4-D space–time ST (curved when action has been described as part of the carrier space ST). In this space the invariant volume is given by $d^4x\sqrt{-g}$, where $g = \det\left(g_{\mu\nu}\right)$.

In terms of these fields the components of the covariant derivative for multi-vectors M are, using $2\left(A \times B\right) \equiv AB - BA$

$$D_{\mu} M \equiv e_{\mu} \cdot \nabla M + \Omega\left(e_{\mu}\right) \times M, \quad DM \equiv \mathbf{h}\left(e^{\mu}\right) D_{\mu} M. \tag{64}$$

For a spinor ψ

$$D\psi = h\left(e^{\mu}\right)\left(e_{\mu}\cdot\Box\psi + \tfrac{1}{2}\Omega\left(e_{\mu}\right)\psi\right), \tag{65}$$

and then the $\Omega\left(e_{\mu}\right)$ appear as the Fock–Ivanenko gauge fields. We shall consider two types of gauge fields. The type $\Omega\left(e_{\mu}\right)$ which is a vector-valued function of a vector and the type

$$\mathcal{A}\left(x\right) = \left(A_R + iA_I\right) + ie_0e_1e_2e_3\left(B_R + iB_I\right), \tag{66}$$

(complex) vector and (complex) axial vector fields, which can also be considered as resulting from proper and improper Lorentz transformations R (the inclusion of action has enlarged the set of isometries to that corresponding to complex Lorentz transformations), they all will obey

$$\Omega'\left(e_{\mu}\right) = \widetilde{R}\Omega\left(e_{\mu}\right)R - 2\widetilde{R}e_{\mu}\cdot\nabla R, \tag{67}$$

using $\widetilde{R}R = 1$ and $M \rightarrow M' = \widetilde{R}MR$, as the Lorentz transformation of the multi-vectors. In our theory $\widetilde{R}$ can be $\widetilde{R} \in \{\widetilde{K}\}$ a more general transformation belonging to the group of complex Lorentz transformations $\widetilde{K}$.

In practice the metric appears as an independent field in START which is defined according to the Principle of Choice of Acceptable Descriptions, then once it is chosen the condition of flat STA is that the total curvature vanishes. Otherwise (from the integral of the selected contributions to action) with $g \equiv -\det\left(g_{\mu\nu}\right)$,

$$A = \int a\left(x\right)\sqrt{-g}\,dx_0dx_1dx_2dx_3, \tag{68}$$

we can derive the effective energy–momentum-stress density tensor, corresponding to the selected contributions to action, as

$$\kappa_0\frac{\delta A}{\delta g^{\mu\nu}\left(x\right)} \equiv \frac{\kappa_0}{2}\mathcal{T}_{\mu\nu}\left(x\right), \tag{69}$$

(the factor $\frac{1}{2}$ is needed for convention reasons); also, from the Ricci scalar curvature $\mathcal{R}$ which results from the chosen line elements

$$\frac{\delta\mathcal{R}}{\delta g^{\mu\nu}\left(x\right)} = \mathcal{R}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathcal{R}, \text{ with } \mathcal{R}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathcal{R} + \frac{\kappa_0}{2}\mathcal{T}_{\mu\nu} = 0. \tag{70}$$

to obtain the equivalent to the GR basic equation. Here we should stress that our formulation is not, neither for the gravitational part nor for the electromagnetic part, a Kaluza–Klein theory. Nevertheless, a change of variables in (70) would allow the formal writing of $\mathcal{R}_{ab} = 0, \quad a, b = 0, 1, 2, 3, 4$. We only require a set of coordinates x^μ covering the (curved or not) region of space–time around x, the existence of tangent vectors $e_\mu = \partial_\mu x$ allowing the definition of vectors in that space

$$\mathbf{V} = V^\mu e_\mu \to \mathbf{V}' = V'^\mu e'_\mu, \quad V'^\mu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right) V^\nu,$$

and $e'_\mu = \partial_{x'^\mu} x = \left(\partial x^\nu / \partial x'^\mu\right) e_\nu$ and of its derivatives in terms of the coefficients of affine connection $\Gamma^\rho_{\nu\mu}$

$$\partial_\mu e_\nu \equiv \Gamma^\rho_{\nu\mu} e_\rho, \tag{71}$$

such that

$$\partial_\mu V = \left(\partial_\mu V^\mu + \Gamma^\nu_{\rho\mu} V^\rho\right) e_\nu \equiv \left(\nabla_\mu V^\nu\right) e_\nu, \tag{72}$$

where we define the covariant derivative of the (contravariant) component of a vector.

The set of definitions above are usually related to the Riemann curvature tensor, to the Ricci contractions and to the concept of parallel transport along a curve $x^\mu(\tau)$,

$$\frac{dx^\rho}{d\tau}\left(\nabla_\rho V^\mu\right) = 0, \tag{73}$$

using the arbitrary parameters τ, defining the curve along which the vector $\mathbf{V}$ remains unchanged. This is when

$$\frac{dV^\mu}{d\tau} = -\Gamma^\mu_{\nu\rho}\left(\frac{dx^\rho}{d\tau}\right) V^\nu; \quad \text{or} \quad \frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\nu\rho}\left(\frac{dx^\rho}{d\tau}\right)\left(\frac{dx^\nu}{d\tau}\right). \tag{74}$$

To find the curve itself the vector $\mathbf{V}$ is considered the tangent vector $V^\mu \to dx^\mu/d\tau$ to the curve $x^\mu(\tau)$ to obtain from (74) the well known condition for an extremum trajectory. The metric itself can be considered to be the symmetric functional $g(,)$ of the basis vector pairs $g^{GR}_{\mu\nu} = g\left(\mathbf{h}(e^{(0)}_\mu), \mathbf{h}(e^{(0)}_\nu)\right)$. Our theory is otherwise a purely geometric theory where the description of space–time surfaces in STA is changed to that corresponding to that of a particular action density distribution following our principle of acceptable choice of description.

5 Rumer (Kaluza–Klein) Theory deduced from START

The Rumer form of the Kaluza–Klein–Einstein–Bergmann [8, 11, 5] theory is deduced from START when besides deriving the metric tensor from the square of the line element dS, as the symmetric part of dS^2, the antisymmetric, then imaginary, elements are kept and considered in turn as as real elements of an extended metric tensor in a 5-D geometry. That is consider again the complex line element

$$dS = e_\mu dS^\mu = dx^\mu(\mathbf{1} + \mathbf{j}\kappa_0 i\tan\Theta(X,\mu)) = e_\mu(dx^\mu + i\mathbf{j}\kappa_0 p(X,\mu)dx^\mu),$$

from which we have obtained the real quadratic form (in units of distance square)

$$dS^2 = g_{\mu\nu}dx^\mu dx^\nu \left(\mathbf{1} - \kappa_0^2 \tan\Theta(X,\mu)\tan\Theta(X,\nu)\right), \tag{75}$$

or, in the diagonal local system of coordinates

$$dS^2 = ds^2 - \kappa_0^2 \left|dK(X)\right|^2; \tag{76}$$

where $\kappa_0^2 \left|dK(X)\right|^2$ corresponds to the sum of the squares of action contributions.

Compute again the complex square, keeping now the scalar and the bivector parts

$$\begin{aligned}(dS)^2 &= \frac{1}{2}\left[\{e_\mu dx^\mu(\mathbf{1} + \mathbf{j}\kappa_0 i\tan\Theta(X,\mu))\} \quad \{e_\nu dx^\nu(\mathbf{1} - \mathbf{j}\kappa_0 i\tan\Theta(X,\nu))\} + (\mathbf{hc})\right] \\ &+ \frac{1}{2}\left[\{e_\mu dx^\mu(\mathbf{1} + \mathbf{j}\kappa_0 i\tan\Theta(X,\mu))\} \quad \{e_\nu dx^\nu(\mathbf{1} - \mathbf{j}\kappa_0 i\tan\Theta(X,\nu))\} - (\mathbf{hc})\right] \\ &= g_{\mu\nu}dx^\mu dx^\nu \left(\mathbf{1} - \kappa_0^2\tan\Theta(X,\mu)\tan\Theta(X,\nu)\right) + e_{\mu\nu}dx^\mu dx^\nu i\mathbf{j}\kappa_0(p(\mu) - p(\nu)) \\ &= dS^2 + e_{\mu\nu}dx^\mu dx^\nu i\mathbf{j}\kappa_0(p(X,\mu) - p(X,\nu)),\end{aligned}$$

where the antisymmetric product of two vectors, the bivectors $e_{\mu\nu}$ are also the generators of spin angular momentum.

From the principles of General Relativity of considering the changes in energy-momentum for the test particle, consider that in the case of an electromagnetic interaction the test particle of charge e receives and additional energy momentum $p(X,\mu) = eA_\mu(X)$

$$i\mathbf{j}\kappa_0 p(X,\mu) = i\mathbf{j}\frac{e}{m_0 c}A_\mu(X),$$

using the action equivalent distance $\kappa_0 = 1/m_0 c$. Here as in the case of general relativity $\mathbf{j}$ is a unit vector with property $\mathbf{j}^2 = -1$. The relation

of the electromagnetic field with the intrinsic spin of the electron as the test particle is explicitly found here and is recovered in the analysis of the Dirac equation both in its multivector formulation and its twistor formulation. The resulting scheme, where a spherically symmetric approximation to gravitation is included in the terms $f(r)$ and $h(r)$ as above, can be presented as a 5×5 matrix

$$g_{AB} = \begin{pmatrix} f(r) & & & & i\mathbf{j}\frac{eA_0}{m_0c} \\ & -h(r) & & & i\mathbf{j}\frac{eA_1}{m_0c} \\ & & -1 & & i\mathbf{j}\frac{eA_2}{m_0c} \\ & & & -1 & i\mathbf{j}\frac{eA_3}{m_0c} \\ -i\mathbf{j}\frac{eA_0}{m_0c} & -i\mathbf{j}\frac{eA_1}{m_0c} & -i\mathbf{j}\frac{eA_2}{m_0c} & -i\mathbf{j}\frac{eA_3}{m_0c} & 0 \end{pmatrix},$$

which is the usual 5-D presentation of the Kaluza–Klein idea, in the case of the electromagnetic field, where the fifth dimension is the additional action, for the test particle, expressed in the form of an equivalent distance. When the g_{AB} matrix is symmetrized as in general relativity the additional antisymmetric terms are all null and then they are not relevant for that type of study, only the diagonal term which includes anyhow the possibility of electromagnetic contributions.

Besides the many papers which have been written about the Kaluza–Klein proposition and their extension to the idea of hyper-space with one additional dimension (at least) for each additional interaction included, the direct inclusion of action as a fifth dimension was proposed as early as the 1949–1956 by the Russian physicist Y.B. Rumer[**?**] under the name of 'Action as a spatial coordinate. I-X'.

In the work of Rumer the main foreseen application is to the case of optics in what he called 5-optics. We should remember that in this case the action $dA = 0$ and then the fifth coordinate turns out to be identically null. Then there is no need for mathematical procedures to make this coordinate disappear which is the problem of higher dimensional procedures. Also this author proposes that a periodic boundary condition is applied to this coordinate in the sense that

$$x_5(A) = x_5(A + h),$$

the distance equivalent to action is a circular coordinate with period equal to the Planck's constant h. In our study the physical elementary trajectories can all be taken as densities of null trajectories, as defined

above, and then this condition is automatically obeyed. In START there is no need to eliminate the fifth-dimension because it is a null dimension, by definition.

6 Hypothesis and Principles in START

The set of hypothesis and principles which are explicitly included in our theory, we have called START, as presented in this paper, are:

Physics is the science which describes the basic phenomena of Nature within the procedures of the Scientific Method.

We consider that the mathematization of the anthropocentric primary concepts of space, time and the existence of the physical objects (action carriers), is a suitable point of departure for creating intellectual structures which describe Nature.

We introduce a set of principles: Relativity, Existence, Quantization and Choice as the operational procedure, and a set of 3 mathematical postulates to give this principles a formal, useful, structure.

We derive, in this paper some of, the fundamental structures of Physics: General Relativity, Density Functional Theory, Newtonian Gravitation and the Maxwell formulation of Electromagnetism. A fundamental common concept is the definition of energy (action) carriers. It is clear that most of the here presented relations are known relations as far as we are **deriving** the structures and theories from START.

6.1 Lagrangians and symmetries in START

We are analyzing the consequences of assuming an action density distribution to describe the physical world. A closely related function for the purpose of deriving the mathematical structures is the Lagrangian of the system obtained by adding to the total action $A = \sum_a \int a_a(X) dx^3 cdt$ obtained from a sum of action density contributions, a series of vanishing functions multiplied by the set of factors known as Lagrange multipliers $\mathcal{L}$. They have the form of a choice of description. For example, to state that we consider a set of n_a of non-interacting carriers of type a and that the particles densities are factorized

$$\mathcal{L}_a = \lambda_a \left[\int \psi_a^+ \psi_a dx^3 - n_a \right]. \tag{77}$$

The term $\sum_a a_a(X)$ is now called a contribution to the 'Lagrangian density'. Here $\int \psi_a^+ \psi_a dx^3 = n_a$ and the term in the square parenthesis vanishes, λ_a is the Lagrange multiplier which, for consistency, is the average energy per carrier of type a, and when $(dS)^2$ is being optimized we perform the variation (for each carrier $\mathcal{L}_a$ can contain as many conditions as needed)

$$\delta \left\{ A + \sum_a \int c dt \mathcal{L}_a \right\} = 0. \tag{78}$$

In START, because of its equivalent complex structure (28)–(29) and its quadratic forms (47)–(48), we have, besides the geometrical space–time Poincaré group $\mathcal{P}$ of transformations leaving the finite difference $(dx^0)^2 - (d\mathbf{x})^2$ invariant, an additional set of transformations related to the complex structure, which can also be considered those of a more general quadratic form $(dx^0)^2 - (d\mathbf{x})^2 - (dx^4)^2$. The additional operations are: a translation in the e^4 direction, three rotations in the $e^i e^4$, $i = 1, 2, 3$ planes and one 'boost' in the $e^0 e^4$ plane.

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