Curvature and torsion of implicit hypersurfaces and the origin of charge-currents

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ABSTRACT. A formal correspondence is established between the curvature theory of generalized implicit hypersurfaces, the classical theory of electromagnetism as expressed in terms of exterior differential systems, and thermodynamics. A covariant normal field to a generalized implicit hypersurface, when made homogeneous of degree zero, can be used to produce a Jacobian matrix whose similarity invariants are related to the curvatures of the hypersurface. The Jacobian adjoint matrix can be used to produce an N-1 form, or current, which is closed globally. When the closed vector density is assigned the role of an intrinsic charge current density, and the components of the normal field are assigned the roles of the electromagnetic potentials, the theory is formally equivalent to an exterior differential system that generates the PDE's of both the Maxwell Faraday equations and the Maxwell Ampere equations. The interaction energy density between the potentials and the induced closed charge current density is exactly the similarity curvature invariant of highest degree (N-1) for the implicit surface.

1 Introduction

The origin of charge has long been a mystery to physical theory, perhaps even more elusive than the concept of inertial mass. A major objective of this article is to examine the conjecture that the charge-current density of electromagnetism may have part, if not all, of its origins in the differential geometry and topology of curvature and torsion. The concept is in a sense similar to the idea that mass density and gravity have their origins in the concept of metric curvature. The curvatures of interest herein are not, however, those generated by a symmetric metric, but instead are those curvatures related to the similarity invariants of a generalized implicit hypersurface. The generalized implicit hypersurfaces considered herein may not admit a global foliation as their normal fields need not satisfy the Frobenius integrability conditions. Hence such generalized hypersurfaces can support topological torsion as well as curvature. Although other (for example, hydrodynamic) interpretations of the results to be presented are possible, the electromagnetic nomenclature will be used for purposes of more rapid comprehension.

2 IMPLICIT SURFACE THEORY

2.1 The implicit surface generated by a 1-form if Action, A

An arbitrary 1-form of Action, A_0, whose coefficient functions may be considered as a set of electromagnetic potentials, can also play the role of the normal direction field to a generalized implicit hypersurface. The closure of the exterior differential system, $F_0 - dA_0 = 0$, always generates a system of PDE's which contain the Maxwell-Faraday equations, thereby establishing the first half of Maxwell theory [1]. The 1-form of Action can be rescaled by use of a Holder norm, λ, such that the resulting 1-form $A = A_0/\lambda$, is homogeneous of degree zero in its coefficient functions. The curvature features of the implicit hypersurface are completely specified in terms of the similarity invariants of the Jacobian matrix, $[\mathbb{J}_{mn}] = [\partial A_m/\partial x^n]$.

It is important to realize that the method to be discussed involves curvatures, torsion and energy densities, but does not depend explicitly upon a metric, gauge constraints, or the Einstein field equations.

2.2 The induced charge-current density J_s

The remaining half of electromagnetic theory is the Maxwell-Ampere equations, which depend upon the existence of a globally closed charge-current density. Although the Jacobian matrix described above is globally singular, $\det[\mathbb{J}] = 0$, it is always possible to construct algebraically the matrix of cofactors transposed, or $[\mathbb{J}]^{adjoint}$. If the components of the 1-form, A, do not form a null eigen vector, it is remarkable that multiplication of these components by $[\mathbb{J}]^{adjoint}$ yields an N-1 form density, or current, J_s, which is globally closed, and therefor can be utilized to play the role of a charge-current density (see Appendix). The global closure implies that there exists an N-2 form G_s such that $J_s - dG_s = 0$. The PDE's associated with this exterior differential system are known to contain the Maxwell-Ampere equations [1]. Note that the two exterior differential systems establish topological constraints on the variety. For example, the domain of support for the 2-form F is not compact without boundary, while the domain of support for G_s can be compact without boundary. The N-2 form, G_s , (like the 1-form of Action) is not uniquely determined by the exterior differential system, as it may contain closed, or closed and exact, components that do not contribute to the charge current density, $J_s = dG_s$.

These two Maxwell exterior differential systems, $F - dA = 0$, and $J_s - dG_s = 0$ can be used to deduce two additional differential systems that augment, but do not change, the PDE's of the classical Maxwell theory. These augmentations depend upon the existence of two 3-forms, previously defined as Topological Torsion, $A\hat{}F$, and Topological Spin, $A\hat{}G_s$ [2]. Constraints of equilibrium and uniqueness (which are not invoked herein) will cause these 3-forms to be null. Exterior differentiation of $A\hat{}G_s$ leads to the equation:

$$d(A\hat{}G_s) = F\hat{}G_s - A\hat{}J_s, \tag{1}$$

and demonstrates that twice the difference between the magnetic and electric energy densities of the field, $F\hat{}G_s$, is cohomologous with the interaction energy density, $A\hat{}J_s$. A major feature of this article is to present the idea that the interaction energy density, $A\hat{}J_s$, is proportional to the hypersurface curvature similarity invariant of degree (N-1). This similarity invariant is defined as the Adjoint curvature and is equal to the trace of the Jacobian Adjoint matrix, $[\mathbb{J}]^{adjoint}$. Note that closed but not exact gauge contributions to A and to G can influence the value of the competing terms, $F\hat{}G_s$ and $A\hat{}J_s$.

2.3 The interaction energy density, $A\hat{}J_s$, and Curvature

The interaction energy density, $A\hat{}J_s$, satisfies the cohomological constraint 1 and can be evaluated, given A and G_s. The Lagrangian field energy density term, $F\hat{}G$, in electromagnetic format is equal to twice the difference of the magnetic and electric energy density of the electromagnetic field. The term has different signs depending on whether the system is dominated by a plasma or electrostatic state. In regions where $A\hat{}G$ is closed (has zero divergence), the closed 3 dimensional integrals of $A\hat{}G$ have values whose ratios are rational and are therefore countable (quantized in units of h).

It can be shown that this interaction density is precisely equal to the Adjoint curvature of the hypersurface whose normal direction field is generated by the 1-form, A. On a variety of four dimensions, this result implies that interaction energy, $A\hat{}J_s$ is a cubic function of the hypersurface curvatures, while the Gaussian sectional curvature (and therefor mass energy density) is quadratic in the surface curvatures. When the Jacobian matrix is of maximal rank N-2, the interaction energy vanishes. Note that if the interaction energy density is zero, the charge current density need not be zero. A special situation exists when J_s is proportional to the Topological Torsion 3 form, $A\hat{}dA$, for then the interaction energy density vanishes due to orthogonality of its two factors. An example of this special case is given below, where the Hopf map is used to formulate an implicit surface 1-form. Maple programs are available for computing the features of generalized implicit hypersurfaces, demonstrating the claim that an intrinsic charge-current exists, and proving that the intrinsic charge-current interaction with the potentials is equal to the Adjoint curvature of the implicit hypersurface [3].

2.4 Topological evolution, internal energy density, dissipation

Given a 1-form of Action A and a closed charge current density J_s, it is possible to use Cartan's magic formula of topological evolution to demonstrate a correspondence between the implicit surface theory and the first law of thermodynamics. Cartan's Magic formula invokes the Lie derivative, with respect to a direction field, acting on exterior differential forms [4] as the fundamental generator of equations of evolution. The method does not depend upon metric, nor connection, and has a direct relationship to the Calculus of Variations. The name "Lie derivative" is said to be due to Slebodzinski, [5], but appears in archaic format in E. Cartan's book, "Lecons sur les Invariant Integraux". The name "Cartan's Magic formula" is due to Marsden [6]. All of these authors overlooked the concept that Cartan's Magic formula is the cohomological expression linking topological evolution and the first law of thermodynamics in a non-statistical manner [7].

For evolutionary processes in the direction of the charge current density, Cartan's magic formula becomes

$$L_{(J_s)}A = i(J_s)dA + d(i(J_s)A) = W + dU = Q \qquad (2)$$

Using electromagnetic notation, on a variety $\{x, y, z, t\}$ the (virtual) work 1-form becomes

$$W = i(J_s)dA = (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})_k dx^k + (\mathbf{J} \cdot \mathbf{E})dt \tag{3}$$

which is recognized as the product of the Lorentz force density times the differential displacement plus the dissipative power density times the increment dt.

In certain cases the induced charge current density, J_s will have a component proportional to the Topological Torsion field, $A\hat{}dA = i(T)dx\hat{}dy\hat{}dz\hat{}dt$. An example of this case is presented below. If J_s is proportional to the Topological Torsion current, T, it follows that the evolution of the implicit surface is given by the expression,

$$L_{(J_s)}A = L_{(T)}A = i(T)dA + 0 = (\Gamma)\ A = (\mathbf{E} \cdot \mathbf{B})A = Q. \tag{4}$$

It follows that the heat 3 form, $Q\hat{}dQ$, and the Topological Torsion 3 form, $A\hat{}F$, are proportional:

$$Q\hat{}dQ = (\mathbf{E} \cdot \mathbf{B})^2 A\hat{}dA. \tag{5}$$

From classical thermodynamics, when a process produces a heat 1-form Q which does not admit an integrating factor, then such a process is thermodynamically irreversible. From Frobenius theory, an integrating factor does not exist if $Q\hat{}dQ \neq 0$. If the implicit surface 1-form is of Pfaff dimension 4, then $A\hat{}dA \neq 0$, and the topological parity 4 form, $dA\hat{}dA = 2(\mathbf{E}{\cdot}\mathbf{B})dx\hat{}dy\hat{}dz\hat{}dt \neq 0$. So if the induced charge-current density has a component in the direction of the Topological Torsion field, then the associated process is thermodynamically irreversible. Such irreversible processes are artifacts of 4 dimensions.

Similarly, evaluation of the internal energy density for a process defined by the dynamics of the charge-current density becomes $U = (i(J_s)A) = \mathbf{A} \cdot \mathbf{J}_s - \rho\phi$, which is identical to the coefficient of the interaction energy density. The dissipative irreversible component of the evolutionary process, which is proportional to the Topological Torsion current, does not contribute to the internal energy, as $i(T)A = 0$. Hence a correspondence has been established between the curvature theory of implicit surfaces, the charge-current density interaction, and the internal energy of a thermodynamic system. In the reversible situations, where

$(\mathbf{E}\cdot\mathbf{B}) = 0$, the implicit hypersurface method thereby seems to offer an alternative, non-quantum mechanical, understanding of what otherwise would be called superconducting currents. It is possible to have charge currents without dissipation. In one case, (the Meisner effect) the **B** field is excluded from the superconducting region, and in another case (the Hall effect) a large **B** field is present along with a non-dissipative but transverse current.

2.5 Four Dimensional Hypersurfaces

In this example, the Hopf map is used to deduce a 1-form of Pfaff dimension 4:

$$A_0 = b(ydx - xdy) + a(tdz - zdt). \tag{6}$$

This 1-form of Potentials depends on the coefficients a and b which are presumed to take on values ± 1. There are two cases corresponding to left and right handed "polarizations": $a = b$ or $a = -b$. (There actually are 6 cases to consider, by cyclically permuting the variables, and these can be combined to represent spinor solutions.[8]) The details of the calculation are presented elsewhere [3], but the results of the similarity curvature invariants are summarized below. Both the Mean curvature and the Adjoint cubinc (interaction) curvature of the implicit Hopf hypersurface in 4D vanish, for any choice of a or b. The Gauss curvature is non-zero, positive, real and is equal to the square of the radius of a 4D euclidean sphere.

$$Mean\ Curvature\ (linear\ sum\ of\ \ curvatures) = 0, \tag{7}$$
$$Gauss\ Curvature\ (quadratic\ in\ curvatures) > 0 \tag{8}$$
$$A\hat{\ }J_s\ \ Adjoint\ Curvature\ (cubic\ in\ curvatures) = 0 \tag{9}$$

The computations indicate that the Hopf implicit surface has three curvature eigen values, $\{0, +i\omega, -i\omega)$. Hence the Hopf surface is a 3D imaginary *minimal* hyper surface in 4D, has two non-zero imaginary curvatures, and is of positive Gauss curvature! Real minimal surfaces in 3D have negative Gauss curvature. Strangely enough, the charge-current density is not zero, but is proportional to the non-zero topological Torsion vector that generates the 3 form $A\hat{\ }F$. The Topological Torsion vector has a direction field proportional to the radius of a 4D sphere, representing an expansion (or contraction) of space-time. The Topological Parity 4 form is not zero, and its coefficient $(4ba)$ depends on the sign of the coefficients a and b. In other words the 'handedness' of the different components of the 1-form of Action determines the orientation of the normal field with respect to the implicit surface. From section 1.4 it is known that a process described by a vector proportional to the Topological Torsion vector, in a domain where the topological parity $(4ba)$ is non-zero, is thermodynamically irreversible.

2.6 Possible applications to gravitational collapse

Every Pfaffian 1-form whose coefficients are functionally homogeneous of degree zero can be used to describe the normal field to an implicit surface. The equation 1 can be put into correspondence with the principle of equivalence, where $F\hat{}G$ plays the role of the gravitational field and where $A\hat{}J$ plays the role of inertial energy density. When the Topological Spin is closed (has zero divergence) then the gravitational energy density is equivalent to the inertial energy density. The curvature similarity invariants can be computed from the Jacobian matrix of the homogeneous 1 form. For those p-branes which are 3 dimensional implicit surfaces in 4 dimensions, the interaction (inertial) energy density of is exactly the cubic curvature similarity invariant of the implicit hypersurface. As the curvature radii get smaller and smaller, the electromagnetic interaction energy - being proportional to the cube of the curvatures - could conceivably prevent, if not impede, gravitational collapse. It seems intuitive that a collapsing mass system generates high temperatures, which intuitively would ionize the matter to produce an electromagnetic plasma. Certainly such terms involving electromagnetic interaction should be included in the dynamics of collapsing mass systems. Note that this effect, like the Bohm-Aharanov effect, does not depend explicitly upon the field strengths, **E** and **B**. Such considerations appear to have been neglected in metric based curvature theories that claim to generate black holes.

3 Summary

The usual dogma of electromagnetic teaching is that given a charge current density, one tries to deduce $\mathbf{E}, \mathbf{B}, \mathbf{D}$, and $\mathbf{H}$ fields by some system of constraints and constitutive relations. In practice, this route to solutions involves the creation of a set of potentials, $\mathbf{A}$, ϕ. The approach employed in this article starts with the potentials and deduces the charge currents, by utilizing a new result of implicit surface theory to construct a global divergence free vector field from any given covariant vector field. It is most remarkable that the globally closed current so deduced does NOT require that $F = dA \neq 0$. From the Electromagnetic point of view such a result would imply the extraordinary existence of **D** and **H** fields without **E** and **B** fields, and no electromagnetic forces of the Lorentz type. Even in such cases there still exist interaction energies between the potentials and the charge current densities, which are cubic, not quadratic, in the curvatures of the induced hypersurface.

For many years, the physical properties of the potentials were ignored because the potentials could admit gauge contributions that did not effect the **E** and **B** fields. However, quantum mechanics, and the Bohm Aharonov effect have changed all that, such there now are admissions that there are regions that contain potentials but the field intensities are excluded. These regions are often superconducting regions, and according to the arguments presented herein are regions where interaction energies are cubic in curvatures of the implicit hypersurface.

4 Appendix: Generalized Implicit hypersurfaces.

The classic implicit surface is generated by assigning a constant value to a function, $\phi(x, y, z..)$. It is important to recall that an implicit surface, in contrast to a parametric surface, can consist of more than one disconnected components. The gradient field to the given function represents a normal field to the surface, and tangent vectors which reside on the surface are orthogonal to the normal field at all points. As the normal field for the classic implicit surface is a gradient field, its associated 1-form is exact. If this normal gradient field is rescaled by a factor such that it is homogeneous of degree zero in its functional arguments, then the Jacobian matrix of the rescaled normal field can be used to generate the curvatures of the implicit surface.

This procedure can be extended to the study of generalized implicit surfaces whose normal field is not representable by an exact 1-form. The 1-form representing the normal field can have arbitrary Pfaff dimension. If the Pfaff dimension (class) of the 1-form is greater than 2, then the implicit surface can support topological torsion, $A\hat{}dA \neq 0$. It is necessary that the Pfaff dimension be greater than 2 if the implicit surface admits an envelope.

4.1 The Holder norm and similarity curvature invariants.

After division by a suitable function of the coefficient potentials, λ, an original 1-form of Action, $A_0 = (U(x, y, z, ...)dx + V(x, y, z, ...)dy + W(x, y, z, ...)dz...)$,can be made homogeneous of degree zero in terms of those coefficient functions that define the potentials. It is to be emphasized that the homogeneity condition is not on the arguments of the coefficients, but on the coefficient functions themselves. The scaling function of choice, λ, is a Holder norm and is defined in terms of the covariant coefficients of the 1-form, $\lambda = (aU^p + bV^p + cW^p + ...)^{n/p}$. The index n will be defined as the homogeneity index; the index p will be described herein as the isotropic index, and the constants $(a, b, c...)$ are constant scale factors whose signs determine the signature. By choosing the index n to be unity, $n = 1$, the 1-form, A, defined as

$$A = A_0/\lambda = (Udx + V dy + W dz...)/\lambda = A_k dx^k \tag{10}$$

becomes homogeneous of degree zero in its coefficients. That is, if every coefficient function is increased by a factor β then the coefficient functions A_k does not change. This 1-form, A_0/λ, which is homogeneous of degree zero in its coefficients, is used to define the normal field to an implicit hypersurface in the variety. The geometrical properties of this hypersurface can be expressed classically in terms of the similarity invariants of the associated *singular* Jacobian dyadic (or matrix), $[\mathbb{J}_{mn}] = \left[\partial A_m(x^k)/\partial x^n\right]$. Classically these similarity invariants are "symmetric" functions of the surface curvatures. Examples may be found at [3].

The fundamental conjecture of extended implicit surfaces utilized herein has two parts: (1) The determinant of the Jacobian matrix $[\mathbb{J}]$, defined above as a vector valued set of homogeneous gradient functions, has value zero for n=1, any isotropic index p, and any signature (a,b,c....). (2) The components of a Current $|\mathbf{J}_s\rangle$ constructed from the matrix product $|\mathbf{J}_s\rangle = [\mathbb{J}]^{adjoint} \cdot |\mathbf{A}\rangle$ has zero divergence globally. That is, the n-1 form $i(\mathbf{J}_s)dx\hat{\ }dy\hat{\ }dz...$, composed from the components of $\mathbf{J}_s$, is closed in an exterior derivative sense:

$$dJ_s = d(i(\mathbf{J}_s)dx\hat{\ }dy\hat{\ }dz...) = 0. \tag{11}$$

The conjecture has been proved abstractly to dimension N=8, but is presumed to work for any N.

The curvature similarity invariants of the Jacobian matrix $[\mathbb{J}]$ can be computed algebraically by forming the Cayley-Hamilton characteristic polynomial. When the Gauss subset of the Holder norm is used ($p = 2,\ n = a = b = c... = 1$), then the trace of the adjoint matrix is exactly equal to the coefficient of the interaction energy density, $A\hat{}J_s$. For implicit surfaces in 4 dimensional space, which is all that is needed in this article, this similarity invariant is cubic in the curvatures of the hypersurface. For explicit constructive proof of the fundamental theorem in 4 dimensions using Maple, see [9].

5 Acknowledgments

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