On a paper by J. Smoller and B. Temple

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ABSTRACT. In a paper dealing with a new formulation of the Oppenheimer-Volkoff (O-V) equations, J. Smoller and B. Temple prove, under mild assumptions on the equation of state, that black holes never form in solutions of the O-V equations. No attempt is made to extend this conclusion to other situations. In the present paper we prove that the concept of black hole is universally inconsistent with the Einstein theory of gravitation.

Dans un article traitant d'une nouvelle formulation des équations de Oppenheimer-Volkoff, J. Smoller et B. Temple montrent, sous des conditions faiblement restrictives, que le concept de trou noir n'apparaît jamais dans les solutions de ces équations. Les auteurs ne cherchent pas à étendre cette conclusion à d'autres situations. Dans le présent article nous montrons que la notion de trou noir est universellement incompatible avec la théorie gravitationnele d'Einstein.

1 Introduction

According to O-V equations [1], black holes could form from gravitational collapse in massive stars. However this conclusion is based upon rather flimsy arguments regarding both the geometrical and the physical ideas involved in the formulation of the problem. In particular it is assumed that the pressure be identically zero. A recent paper by J. Smoller and B. Temple [2] introduces a new formulation of the O-V equations without making this simplified hypothesis and brings about an entirely different result, namely that black holes never form in solutions of these equations :

> ”When the pressure is not zero, black holes cannot form in static spherically symmetric solutions of the Einstein equations for a perfect fluid. This implies that the portion of the empty-space Schwarzschild solution inside the Schwarzschild radius is disconnected from the rest of the solution space of the O-V system in the sense that it cannot be obtained as a limit of the O-V solutions having non-zero density” [2].

We see that Smoller and Temple do not reject as unphysical the so-called Schwarzschild solution inside the Schwarzschild radius ; they only prove its inconsistency with their formulation of the O-V equation of state. In other words the paper by Smoller and Temple does not reject generally the concept of black hole. It only proves that the concept of black hole is inconsistent with the new formulation of the equation of state. So the question remains :

> Is the concept of black hole universally inconsistent with the Einstein theory of gravitation ?

Of course we cannot expect to answer this question in the setting of the Smoller-Temple computation. These authors take for granted several misleading classical ideas and do all of their work with the so-called standard form :

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2\left(\sin^2\theta\ d\phi^2 + d\theta^2\right) \qquad (1.1)$$

Folklore has it that this form contains all the characteristic features of the gravitational field generated by a statical spherical distribution of matter. However, according to a previous investigation [3],[5], basic features regarding the gravitational field in question are not included in the metric (1.1) :

a) Although the underlying manifold is the product $\mathbb{R}\times\mathbb{R}^3$, the metric (1.1) is referred to polar coordinates, namely to the manifold with boundary $\mathbb{R}\times\left[0,+\infty\right[\times S^2$. So the world-line $\mathbb{R}\times\{(0,0,0)\}$ of the origin disappears. It follows in particular that the isotropy of the metric is not conceivable with respect to (1.1).

b) To the form (1.1) there corresponds a metric on $\mathbb{R}\times\mathbb{R}^3$ which is in general discontinuous at the origin. Moreover the boundary conditions of the problem cannot be formulated with respect to (1.1).

c) The parameter r occurring in (1.1) is wrongly considered as radial coordinate. In fact the parameter r has nothing to do with coordinates. It only serves to define the length $2\pi r$ of a non-Euclidean circle (and the area $4\pi r^2$ of a non-Euclidean sphere) the radius of which is neither given nor definable by the solution related to (1.1), namely by the so-called Schwarzschild solution. In particular the spherical distribution of matter has neither centre nor radius, it is inexistent with respect to (1.1).

In view of the preceding elucidations, it follows that the derivation of the so-called Schwarzschild solution is inconsistent with fundamental mathematical principles. Now, since the notion of black hole results from an interpretation of this solution, it follows that the answer to the posed question does not depend essentially on the equation of state inside the matter, but on a reexamination of the problems related to the vacuum solutions.

2 Space-Time metric and Equations of Gravitation

Isotropic space-time metric on $\mathbb{R} \times \mathbb{R}^3$ means : Space-time metric on $\mathbb{R} \times \mathbb{R}^3$ invariant by the action of the group, denoted by $S\Theta(4)$, consisting of the matrices

$$\begin{pmatrix} 1 & O_H \\ O_V & A \end{pmatrix}$$

with $O_H = (0,0,0)$, $O_V = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $A \in SO(3)$. It is shown rigorously [5] that such a metric can be written as

$$ds^2 = a_{00}\,(t, \|x\|)\,dt^2 + 2a_{01}\,(t, \|x\|)\,(xdx)\,dt + a_{11}\,(t, \|x\|)\,dx^2 + a_{22}\,(t, \|x\|)\,(xdx)^2$$

The functions occurring in it are assumed C^∞ on $\mathbb{R} \times \mathbb{R}^3$, i.e. C^∞ with respect to the coordinates t, x_1, x_2, x_3 at every point of $\mathbb{R} \times \mathbb{R}^3$, even at $(t, 0, 0, 0)$. In order to be so, it is necessary and sufficient that the functions $a_{00}(t,u)$, $a_{01}(t,u)$, $a_{11}(t,u)$, $a_{22}(t,u)$ be C^∞ on $\mathbb{R} \times \mathbb{R}$ and moreover even with respect to $u \in \mathbb{R}$.
Since $a_{00} = a_{00}(t, \|x\|) > 0$, we can introduce the C^∞ functions

$$f = \sqrt{a_{00}}, \qquad\qquad f_1 = \frac{a_{01}}{\sqrt{a_{00}}}$$

which allow to write

$$ds^2 = (fdt + f_1 (xdx))^2 + a_{11}dx^2 + \left(a_{22} - f_1^2\right)(xdx)^2$$

and thus to make explicit the corresponding spatial (positive definite) metric:

$$-a_{11}dx^2 - \left(a_{22} - f_1^2\right)(xdx)^2$$

Next we introduce the positive C^∞ functions ℓ_1 and ℓ by setting

$$\ell_1^2 = -a_{11}, \qquad \ell^2 = -a_{11} - \left(a_{22} - f_1^2\right)\|x\|^2$$

Then, with $\|x\| = \rho$, it follows in particular that the C^∞ function $a_{22} - f_1^2$ can be written as

$$\frac{\ell_1^2 - \ell^2}{\rho^2}$$

(The differentiability of the last expression for $\rho = 0$ can also be checked directly by taking into account the condition $\ell_1(t, 0) = \ell(t, 0)$ and the special properties of the functions ℓ_1 and ℓ). Thus we obtain the general isotropic metric in its geometrical form:

$$ds^2 = f^2dt^2 + 2ff_1 (xdx)\, dt - \ell_1^2dx^2 + \left(\frac{\ell_1^2 - \ell^2}{\rho^2} + f_1^2\right)(xdx)^2 \qquad (2.1)$$

where f, f_1, ℓ, ℓ_1 are functions of (t, ρ).

It is shown that the Ricci tensor $\{R_{\alpha\beta}\}$ resulting from (2.1) is invariant by the action of the group $S\Theta(4)$ on $\mathbb{R} \times \mathbb{R}^3$. Then, according to the theory of $S\Theta(4)$-invariant tensor fields, its components are defined by means of four functions of (t, ρ) as follows :

$$R_{00} = Q_{00}, R_{0i} = x_iQ_{01}, R_{ii} = Q_{11} + x_i^2Q_{22}, R_{ij} = x_ix_jQ_{22},$$
$$(i, j = 1, 2, 3; i \neq j)$$

The curvature scalar $R = Q$ is also a function of (t, ρ).

It is easily seen that, if an energy-momentum tensor $\{W_{\alpha\beta}\}$ satisfies the equations of gravitation related to (2.1), then it is $S\Theta(4)$-invariant, so that its components are also defined by four functions of (t, ρ) in the following way :

$$W_{00} = E_{00}, W_{0i} = x_iE_{01}, W_{ii} = E_{11} + x_i^2E_{22}, W_{ij} = x_ix_jE_{22},$$
$$(i, j = 1, 2, 3; i \neq j)$$

By using the preceding notations, we can write down from the outset the system of the equations of gravitation relative to (2.1) as a system of four equations. There is no need to introduce polar coordinates in the computations.

$$\begin{aligned} Q_{00} - \frac{Q}{2} f^2 + \frac{8\pi k}{c^4} E_{00} &= 0 \\ Q_{01} - \frac{Q}{2} f f_1 + \frac{8\pi k}{c^4} E_{01} &= 0 \\ Q_{11} + \frac{Q}{2} \ell_1^2 + \frac{8\pi k}{c^4} E_{11} &= 0 \\ Q_{22} - \frac{Q}{2} \left(\frac{\ell_1^2 - \ell^2}{\rho^2} + f_1^2 \right) + \frac{8\pi k}{c^4} E_{22} &= 0 \end{aligned}$$

Usually it is convenient to replace the last equation by the equation :

$$Q_{11} + \rho^2 Q_{22} + \frac{Q}{2} \left(\ell^2 - \rho^2 f_1^2 \right) + \frac{8\pi k}{c^4} \left(E_{11} + \rho^2 E_{22} \right) = 0$$

On the other hand the computations are greatly simplified if, instead of f_1 and ℓ_1, we introduce the functions $h = \rho f_1$ and $g = \rho \ell_1$, which are also significant geometrically and physically. The function h satisfies the condition $|h| \leq \ell$ which serves to characterize the nature of the coordinate t as time coordinate, whereas the function g is the curvature radius of the spheres centered at the origin. Of course h and g are C^∞ with respect to $(t, \rho) \in \mathbb{R} \times [0, +\infty[$, but, since $\rho = \|x\|$ is not differentiable at the origin, they are not differentiable on the subspace $\mathbb{R} \times \{(0,0,0)\}$ of $\mathbb{R} \times \mathbb{R}^3$. Consequently, whenever we need to check the differentiability of the metric tensor on the subspace $\mathbb{R} \times \{(0,0,0)\}$, we must return to the functions $f_1 = \frac{h}{\rho}$ and $\ell_1 = \frac{g}{\rho}$ which appear in (2.1).

3 Stationary Fields. Vacuum Solutions comparable with Newton's Theory.

If the metric tensor is independent of t, the functions f, $h = \rho f_1$, ℓ, $g = \rho \ell_1$ depend only on ρ, and an easy computation gives

$$\begin{aligned} Q_{00} &= f \left(-\frac{f''}{\ell^2} + \frac{f'\ell'}{\ell^3} - \frac{2f'g'}{\ell^2 g} \right) \\ Q_{01} &= \frac{h}{\rho f} Q_{00} \end{aligned}$$

$$Q_{11}=\frac{1}{\rho^2}\left(-1+\frac{g'^2}{\ell^2}+\frac{gg''}{\ell^2}-\frac{\ell' gg'}{\ell^3}+\frac{f'gg'}{f\ell^2}\right)$$

$$Q_{11}+\rho^2 Q_{22}=\frac{f''}{f}+2\frac{g''}{g}-\frac{f'\ell'}{f\ell}-\frac{2\ell' g'}{\ell g}+\frac{h^2}{f^2}Q_{00}$$

The equations of gravitation (without cosmological constant) outside the matter imply $Q = R = 0$, so that they reduce to the system :

$$Q_{00}=0, Q_{01}=0, Q_{11}=0, Q_{11}+\rho^2 Q_{22}=0$$

On the other hand, since $Q_{00} = 0$ implies $Q_{01} = 0$, we obtain finally a system of three equations :

$$-f''+\frac{f'\ell'}{\ell}-\frac{2f'g'}{g}=0 \tag{3.1}$$

$$-1+\frac{g'^2}{\ell^2}+\frac{gg''}{\ell^2}-\frac{\ell' gg'}{\ell^3}+\frac{f'gg'}{f\ell^2}=0 \tag{3.2}$$

$$f''+2\frac{fg''}{g}-\frac{f'\ell'}{\ell}-\frac{2f\ell' g'}{\ell g}=0 \tag{3.3}$$

By adding (3.1) to (3.3) we obtain

$$\frac{f'g'}{f}=g''-\frac{\ell' g'}{\ell} \tag{3.4}$$

and substituting this expression of $\frac{f'g'}{f}$ into (3.2) we find

$$-1+\frac{g'^2}{\ell^2}+\frac{2gg''}{\ell^2}-\frac{2\ell' gg'}{\ell^3}=0$$

whence

$$\frac{d}{d\rho}\left(-g+\frac{gg'^2}{\ell^2}\right)=0$$

and

$$-g+\frac{gg'^2}{\ell^2}=-2A=const. \tag{3.5}$$

Moreover (3.4) can be written as

$$\frac{\ell'}{\ell}+\frac{f'}{f}=\frac{g''}{g'}$$

whence

$$f\ell = cg', (c = const.) \tag{3.6}$$

So the general stationary solution outside the matter is defined by the equations (3.5) and (3.6). The function h does not appear in them. It remains completely indeterminate. Of course this circumstance does not mean that h is empty of physical meaning, as is usually believed. In fact, h is involved in the propagation function of the light emitted radially from the spherical boundary of the matter, hence also in the definition of time along the radial geodesics. Specific choices of h give rise to significant physically definitions of time. This situation differs radically from that in special relativity where we have to do with a unique propagation function, namely $t - \frac{\rho}{c}$. The discussion of the relevant problems lies beyond the scope of the present paper.

Let us now consider the equation (3.5) which serves to define the curvature radius $g(\rho)$. If $A = 0$, we find $g' = \ell$, whence $f = c$, and the corresponding metric (2.1) is pseudo-Euclidean. We give up this trivial case and assume $A \neq 0$ in the sequel. Then the equation (3.5) gives a first significant information, namely that the obtained from it determination of $g(\rho) = \rho\ell_1(\rho)$ does not cover the whole half-line $[0, +\infty[$. In fact, for $\rho = 0$ we have $g(0) = 0$ and then the equation (3.5) implies $A = 0$, contrary to our assumption $A \neq 0$. So we are certain in advance that the solution $g(\rho)$ of (3.5) is defined on some half-line $[\alpha, +\infty[$ whith $\alpha > 0$. Since the function $\ell = \ell(\rho)$ is not given, it seems impossible to obtain explicitly the general solution of (3.5) relative to the radial coordinate ρ. However in the present case we have to do with a stationary field, so that the geodesic distance

$$\int_0^\rho \ell\,(u)\,du = \delta, (\rho = \|x\|)\,,$$

is well defined (For a non-stationary field, the geodesic distance is rather inconceivable). So, ρ appears as a strictly increasing function of δ, and we have

$$\frac{dg}{d\rho} = \frac{dg}{d\delta}\frac{d\delta}{d\rho} = \ell\frac{dg}{d\delta}, -g + \frac{g}{\ell^2}\left(\frac{dg}{d\rho}\right)^2 = -g + g\left(\frac{dg}{d\delta}\right)^2$$

$$-g + g\left(\frac{dg}{d\delta}\right)^2 = -2A = const., f = c\frac{dg}{d\delta},$$

Writing again ρ instead of δ, we obtain the system :

$$-g + gg'^2 = -2A = const. \tag{3.7}$$

$$f = cg' \tag{3.8}$$

This being said, we have now to bring out the relationship between the theories of Newton and Einstein.

Classically it is believed that, r being the parameter occurring in (1.1), $-\dfrac{km}{r}$ is identical with Newton's potential. This assertion is certainly erroneous. In fact, the parameter r in Newton's potential is the Euclidean distance between the centre of mass and the considered point, whereas the parameter r in (1.1) is the curvature radius of non-Euclidean spheres centered at the origin. The classical approach to the problem identifies erroneously the curvature radius $g(\rho)$ with a radial coordinate. Thus we see in particular that the commonly used term "Schwarzschild radius" is meaningless. The Schwarzschild radius is actually a curvature radius.

Now we pose the fundamental question : Among the solutions defined by (3.5) and (3.6), which are comparable with Newton's theory ?.

Since Newton's potential is defined by means of the Euclidean distance $\|x\|$, it is obvious that the required solutions are obtained by choosing as radial coordinate the geodesic distance between the origin and the point x. In other words the required solutions are those defined by the equations (3.7) and (3.8).

On the other hand $A \neq 0$ implies that $g = g(\rho) \geq \alpha > 0$, according to a preceding remark, and since

$$g - 2A = gg'^2 \geq 0 \tag{3.9}$$

we have finally the equation

$$\frac{dg}{d\rho} = \frac{\sqrt{g-2A}}{\sqrt{g}}$$

which defines $g = g(\rho)$ as a strictly increasing function of the distance ρ. The inverse function $\rho = F(g)$ is also strictly increasing and on account of the equation

$$\frac{d\rho}{dg} = \frac{\sqrt{g}}{\sqrt{g-2A}}$$

its explicit expression is easily obtained :

$$\rho = F(g) = B + \sqrt{g\,(g-2A)} + 2A\ell n\left(\sqrt{g-2A} + \sqrt{g}\right), B = const.$$

We see that $F(g) \to +\infty$ as $g \to +\infty$, hence also $g(\rho) \to +\infty$ as $\rho \to +\infty$. Moreover

$$\frac{\rho}{g\,(\rho)} = \frac{F\,(g)}{g} \to 1, \qquad as \qquad \rho \to +\infty,$$

so that

$$\frac{1}{g(\rho)} = \frac{1+\epsilon(\rho)}{\rho} \quad with \;\; \epsilon(\rho) \to 0 \quad as \quad \rho \to +\infty$$

and the equation (3.8) gives

$$f = c\sqrt{1 - \frac{2A}{g}} = c\sqrt{1 - \frac{2A}{\rho} - \frac{2A\epsilon(\rho)}{\rho}}$$

On account of the Newtonian approximation, we obtain now by a rigorous reasoning the value of the constant A :

$$A = \frac{km}{c^2} = \mu$$

Regarding the constant B, we write it as $B = \rho_0 - 2\mu\ell n\sqrt{2\mu}$ where ρ_0 is another constant, so that the function $g = g(\rho)$ is obtained lastly by the equation :

$$\rho = \rho_0 + \sqrt{g\,(g-2\mu)} + 2\mu\ell n\left(\sqrt{\frac{g}{2\mu} - 1} + \sqrt{\frac{g}{2\mu}}\right)$$

the validity of which requires $g \geq 2\mu$ in accordance with the condition (3.9). The solution does not allow to ascribe a definite value to the constant ρ_0, and this is why we must take into account all the possible determinations of $g(\rho)$ for the different values of ρ_0. However it is to be noticed that the allowable physically values of ρ_0 will be relatively small. In any case the function $g(\rho)$ is strictly increasing on the half-line $[\rho_0, +\infty[$ and its values describe the half-line $[2\mu, +\infty[$ with $g(\rho_0) = 2\mu$, $g'(\rho_0) = 0$.

Suppose first that $\rho_0 \leq 0$. If ρ_1 denotes the radius of the sphere bounding the matter, we have necessarily $\rho_1 > \rho_0$ and $\rho_1 > 0$ (the value $\rho_1 = 0$ is excluded because $g(\rho) = \rho\ell_1(\rho)$ vanishes for $\rho = 0$). The function $g(\rho)$ is physically valid for $\rho \geq \rho_1$, so that $g(\rho) > 2\mu$ for all $\rho \geq \rho_1$.

Suppose secondly that $\rho_0 > 0$. Since $f(\rho_0) = cg'(\rho_0) = 0$, the metric degenerates for $\rho = \rho_0$, so that it is physically meaningless for $\rho = \rho_0$. Consequently, ρ_1 being the radius of the spherical distribution of matter, we have $\rho_1 > \rho_0$ and $g(\rho) > 2\mu$ for all $\rho \geq \rho_1$.

4 Black holes never appear in solutions of the Einstein equations

We now return to the question : Is the concept of black hole universally inconsistent with the Einstein theory of gravitation ?

We have already noticed that we cannot answer it in the setting of the Smoller-Temple computations. In fact, these authors are restricted within the limits of the metric(1.1) which gives rise to misleading results. Moreover, since the theory of black holes is based upon the assumption that the so-called Schwarzschild solution inside the Schwarzschild sphere be physically valid, we have principally to examine the behaviour of the vacuum solutions.

This being said, the stationary solution, brought out in the previous section, points out a fundamental result : The positive constant 2μ is the greatest lower bound of the mathematical solution $g(\rho) = \rho\ell_1(\rho)$ outside the matter. Moreover, if ρ_1 is the radius of the spherical distribution of matter, we have $g(\rho) = \rho\ell_1(\rho) > 2\mu$ for every $\rho \geq \rho_1$, so that $\rho_1 > 0$.

Thus we can ascertain three fundamental results:

a) The given distribution of matter cannot be reduced to a point.

b) The so-called Schwarzschild vacuum solution for $r = g(\rho) \leq 2\mu$ is meaningless mathematically and physically.

c) Black holes never form in solutions of the Einstein equations.

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