A New Look on the Electromagnetic Duality Suggestions and Developments

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ABSTRACT. In this paper a new look on the electro-magnetic duality is presented and appropriately exploited. The duality analysis in the nonrelativistic and relativistic formulations is shown to lead to the idea the mathematical model field to be a differential form valued in the 2-dimensional vector space $\mathcal{R}^2$. A full $\mathcal{R}^2$ covariance is achieved through introducing explicitly the canonical complex structure $\mathcal{I}$ of $\mathcal{R}^2$ in the nonrelativistic equations. The connection of the relativistic Hodge $*$ with $\mathcal{I}$ is shown and a complete coordinate free relativistic form of the equations and the conservative quantities is obtained. The duality symmetry is interpreted as invariance of the conservative quantities and conservation equations.

1 Introduction

The recent developments in superstring and brane theories show that the concept of duality plays and probably will play more and more fundamental role in theoretical and mathematical physics. But the roots of duality in field theory, as well as the roots of the whole field theory, can be found in classical electrodynamics, i.e. in Maxwell equations. Therefore, a detailed and thorough understanding of the duality nature of classical electrodynamics seems to be a very important first step in getting well acquainted with the duality notion.

At first sight the electromagnetic duality seems quite simple and not much promising theoretical tool, but a closer look reveals the opposite. In fact, as we shall see later, a careful study of the duality brings us to the conclusion that the adequate mathematical model object of the electromagnetic field should have a more complicated structure than just the couple $(\mathbf{E}, \mathbf{B})$, or $\mathbf{F}_{\mu\nu}$. As it is well known [1], one of the most frequently used ways of introducing duality makes use of complex electro-magnetic vectors $(\mathbf{E}+i\mathbf{B})$ and transformation of the kind $(\mathbf{E}+i\mathbf{B}) \rightarrow e^{i\varphi}(\mathbf{E}+i\mathbf{B})$, but such a formal step seems closer to a peculiarity notice than to reveal an important feature of the theory, while the real presence of the $\mathcal{R}^2$-complex structure inside the electromagnetic theory seems to stay not fully understood as an important structure property of this theory. Following the above mentioned complex vector approach of consideration many questions stay unanswered, e.g. since on real amnifolds there are only real vector fields where the *complex* vector fields come from; why $i\mathbf{B}$ and not $i\mathbf{E}$; why in the relativistic formulation $i=\sqrt{-1}$ disappears and, instead, a complex structure (through the Hodge $*$) in the 6-dimensional module of 2-forms on the Minkowski space appears, etc.

As we shall see, an adequate (nonrelativistic or relativistic) formulation of classical electrodynamics really needs complex structure, but *not* complex vectors on real manifolds. The usual duality transformations turn to be those linear isomorphisms of the standard complex structure $\mathcal{I}$ in $\mathcal{R}^2$, which are also isometries of the standard euclidean metric in $\mathcal{R}^2$. In our approach the duality transformations appear as closer related to the invariance of the conserved quantities of the theory than to the invariance properties of the equations. And this should be expected since we can treat them as natural extension of the usual space-time isometries, which, as we know, are basic tools in formulating and computing the conservative characteristics of any physical field.

The first man who noticed the duality properties of electromagnetic equations was Heaviside [2]. A further development of Heaviside's notice was given later by Larmor [3]. A more detailed study of electromagnetic duality was made by Rainich [4] in the frame of General Relativity. A comparatively complete presentation may be found in the extensive paper of Misner and Wheeler [5] also in the frame of General Relativity in connection with their attempt to geometrize classical physics and to give topological interpretation of charges. Electromagnetic duality has always been in sight of those trying to introduce magnetic charges and currents in the theory, [6],[7]. In [8] one may find duality considerations in the frame of nonlinear electrodynamics of continuous media. A formal generalization for p-forms is given in [9]. A modern consideration of electromagnetic duality, directed to superstring and brane theories may be found in [1]; the possible nonabelian generalizations are considered in [10].

In this paper we pursue three main purposes. First, we are going to give a brief nonrelativistic and relativistic reviews of what is usually called electromagnetic duality in vacuum and in presence of electric and magnetic charges, without referring to other physical theories. Second, we shall present a new understanding of this duality, leading to a new and, in our view, more adequate mathematical nature of the (nonrelativistic and relativistic) model objects plus complex structure. Finally, we shall represent classical electrodynamics entirely in terms of the new nonrelativistic and relativistic mathematical model objects.

2 Electromagnetic Duality

2.1 Nonrelativistic consideration. We consider first the pure field Maxwell equations

$$\mathrm{rot}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \quad \mathrm{div}\mathbf{B} = 0, \tag{1}$$

$$\mathrm{rot}\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 0, \quad \mathrm{div}\mathbf{E} = 0. \tag{2}$$

First we note, that because of the linearity of these equations if $(\mathbf{E}_i, \mathbf{B}_i), i = 1, 2, ...$ are a collection of solutions, then every couple of linear combinations of the form

$$\mathbf{E} = a_i\mathbf{E}_i, \ \ \mathbf{B} = b_i\mathbf{B}_i \tag{3}$$

(sum over the repeated $i = 1, 2, ...$) with arbitrary constants (a_i, b_i) gives a new solution.

We also note, that in the static case the pure field Maxwell equations reduce to:

$$*\mathbf{dE} = 0, *\mathbf{dB} = 0, *\mathbf{d} * \mathbf{B} = 0, *\mathbf{d} * \mathbf{E} = 0, \tag{4}$$

where $*$ is the euclidean Hodge $*$-operator (see the Note at the end of the paper), $\boldsymbol{d}$ is the exterior derivative and the vector fields $\mathbf{E}$ and $\mathbf{B}$ are identified with the corresponding 1-forms through the euclidean metric g. The substitution $\mathbf{E} \rightarrow *\mathbf{E}, \ \mathbf{B} \rightarrow *\mathbf{B}$, because of the relation $*^2 = \pm id$, turns the first equation of (4) into the fourth one and vice versa, the fourth - into the first one; also, the second equation is turned into the third one, and vice versa, the third - into the second one. Hence, in this special case we can talk about $*$-symmetry of the equations.

The important observation made by Heaviside [1], and later considered by Larmor [2], is that the substitution

$$\mathbf{E} \to -\mathbf{B}, \quad \mathbf{B} \to \mathbf{E} \tag{5}$$

transforms the first couple (1) of the pure field Maxwell equations into the second couple (2), and, vice versa, the second couple (2) is transformed into the first one (1). This symmetry transformation (5) of the pure field Maxwell equations is called *special duality transformation*, or SD-transformation. It clearly shows that the electric and magnetic components of the pure electromagnetic field are interchangeable and the interchange (5) transforms solution into solution. In the transformed solution the magnetic component is the former electric component and vice versa, i.e. the electric component may be considered as magnetic if needed, and then the magnetic component should be considered as electric. This feature of the pure electromagnetic field reveals its *dual* nature.

It is important to note that the SD-transformation (5) does not change the energy density $8\pi\mathbf{w} = \mathbf{E}^2 + \mathbf{B}^2$, the Poynting vector $4\pi\boldsymbol{S} = c(\mathbf{E} \times \mathbf{B})$, and the (nonlinear) Poynting relation

$$\frac{\partial}{\partial t}\frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} = -\mathrm{div}\boldsymbol{S}.$$

Hence, from energy-momentum point of view two dual, in the sense of (5), solutions are indistinguishable.

Note that the substitution (5) may be considered as a transformation of the following kind:

$$(\mathbf{E}, \mathbf{B}) \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} = (-\mathbf{B}, \mathbf{E}). \tag{6}$$

The following question now arises naturally: do there exist constants (a, b, m, n), such that the linear combinations

$$\mathbf{E}' = a\mathbf{E} + m\mathbf{B}, \;\; \mathbf{B}' = b\mathbf{E} + n\mathbf{B}, \tag{7}$$

or in a matrix form

$$(\mathbf{E}', \mathbf{B}') = (\mathbf{E}, \mathbf{B}) \begin{Vmatrix} a & b \\ m & n \end{Vmatrix} = (a\mathbf{E} + m\mathbf{B}, b\mathbf{E} + n\mathbf{B}), \tag{8}$$

form again a vacuum solution? Substituting $\mathbf{E}'$ and $\boldsymbol{B'}$ into Maxwell's vacuum equations we see that the answer to this question is affirmative iff $m = -b, n = a$, i.e. iff the corresponding matrix S is of the form

$$S = \begin{Vmatrix} a & b \\ -b & a \end{Vmatrix}. \tag{9}$$

The new solution will have now energy density $\mathbf{w}'$ and momentum density $\boldsymbol{S}'$ as follows:

$$\mathbf{w}' = \frac{1}{8\pi}\left(\mathbf{E'}^2 + \mathbf{B'}^2\right) = \frac{1}{8\pi}(a^2+b^2)\left(\mathbf{E}^2 + \mathbf{B}^2\right),$$

$$\boldsymbol{S}' = (a^2+b^2)\frac{c}{4\pi}\mathbf{E}\times\mathbf{B}.$$

Obviously, the new and the old solutions will have the same energy and momentum if $a^2+b^2 = 1$, i.e. if the matrix S is *unimodular*. In this case we may put $a = \cos\alpha$ and $b = \sin\alpha$, where $\alpha = const$, so transformation (8) becomes

$$\tilde{\mathbf{E}} = \mathbf{E}\cos\alpha - \mathbf{B}\sin\alpha,\ \ \tilde{\mathbf{B}} = \mathbf{E}\sin\alpha + \mathbf{B}\cos\alpha. \tag{10}$$

Transformation (10) is known as *electromagnetic duality transformation*, or D-transformation. It has been a subject of many detailed studies in various aspects and contexts [1],[3]-[9]. It also has greatly influenced some modern developments in non-Abelian Gauge theories, as well as some recent general views on duality in field theory, esp. in superstring and brane theories (classical and quantum). In the next section we shall study this transformation from a new point of view, following the idea that $(\mathbf{E}, \mathbf{B})$ are two vector components of one mathematical object having some more complicated nature.

From physical point of view a basic feature of the D-transformation (10) is, that the difference between the electric and magnetic fields becomes non-essential: we may superpose the electric and the magnetic vectors, i.e. vector-components, of a general electromagnetic field to obtain new solutions. From mathematical point of view we see that Maxwell's equations in vacuum, besides the usual linearity (3) mentioned above, admit also "cross"-linearity, i.e. linear combinations of $\mathbf{E}$ and $\mathbf{B}$ of a definite kind define new solutions: with every solution

$(\mathbf{E}, \mathbf{B})$ of Maxwell's vacuum equations a 2-parameter family of solutions can be associated by means of linear transformations given by matrices of the kind (9). If these matrices are unimodular, i.e. when $a^2 + b^2 = 1$, then all solutions of the family have the same energy and momentum. In other words, the space of all solutions to the pure field Maxwell equations factors over the action (8) of the group of linear maps $S : \mathcal{R}^2 \rightarrow \mathcal{R}^2$ represented by matrices of the kind (9).

It is well known that matrices S of the kind (9) do not change the canonical complex structure $\mathcal{I}$ in $\mathcal{R}^2$: we recall that if the canonical basis of $\mathcal{R}^2$ is denoted by $(\varepsilon^1, \varepsilon^2)$ then $\mathcal{I}$ is defined by $\mathcal{I}(\varepsilon^1) = \varepsilon^2$, $\mathcal{I}(\varepsilon^2) = -\varepsilon^1$, so if S is given by (9) we have: $S.\mathcal{I}.S^{-1} = \mathcal{I}$. Hence, **the electromagnetic D-transformations** (10) **coincide with the unimodular symmetries of the canonical complex structure $\mathcal{I}$ in $\mathcal{R}^2$**. This important in our view remark clearly points out that the canonical complex structure $\mathcal{I}$ in $\mathcal{R}^2$ should be an **essential element** of classical electromagnetic theory, so we should not forget about it and in no way neglect it. Moreover, in my opinion, we **must find an appropriate way to introduce $\mathcal{I}$ explicitly** in the theory.

In presence of electric current $\boldsymbol{j}_e$ and electric charges ρ_e Maxwell equations (1)-(2) lose this D-symmetry. In order to retain it *magnetic charges* with density ρ_m and magnetic currents $\boldsymbol{j}_m = \rho_m \boldsymbol{v}$ are usually introduced, and of course, the Lorentz force is correspondingly modified. The new system of equations looks as follows:

$$\mathrm{rot}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\frac{4\pi}{c}\boldsymbol{j}_m, \quad \mathrm{div}\mathbf{B} = 4\pi\rho_m, \tag{11}$$

$$\mathrm{rot}\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\boldsymbol{j}_e, \quad \mathrm{div}\mathbf{E} = 4\pi\rho_e, \tag{12}$$

$$\mu\nabla_{\boldsymbol{v}}\boldsymbol{v} = \rho_e\mathbf{E} + \frac{1}{c}(\boldsymbol{j}_e \times \mathbf{B}) + \rho_m\mathbf{B} - \frac{1}{c}(\boldsymbol{j}_m \times \mathbf{E}), \tag{13}$$

where μ is the mass density of the particles and it is assumed that particles do not vanish and do not arise. Now the whole, or extended, D-transformation looks in the following way:

$$\begin{aligned} &\tilde{\mathbf{E}} = \mathbf{E}\cos\alpha - \mathbf{B}\sin\alpha, \\ &\tilde{\boldsymbol{j}}_e = \tilde{q}_e\boldsymbol{v} = \boldsymbol{j}_e\cos\alpha - \boldsymbol{j}_m\sin\alpha = (\rho_e\cos\alpha - \rho_m\sin\alpha)\boldsymbol{v} \end{aligned} \tag{14}$$

$$\begin{aligned}\tilde{\mathbf{B}} &= \mathbf{E}\sin\alpha + \mathbf{B}\cos\alpha,\\ \tilde{\boldsymbol{j}}_m &= \tilde{q}_m\boldsymbol{v} = \boldsymbol{j}_e\sin\alpha + \boldsymbol{j}_m\cos\alpha = (\rho_e\sin\alpha + \rho_m\cos\alpha)\boldsymbol{v}.\end{aligned} \tag{15}$$

Hence, the 1-parameter family of transformations (14-15) is a symmetry of the system (11-13).

The corresponding considerations concerning one particle carrying electric and magnetic charges may be found in [6]. We shall omit this simple case.

Finally we note that D-transformations change the two well known invariants: $I_1 = (\mathbf{B}^2 - \mathbf{E}^2)$ and $I_2 = 2\mathbf{E}.\mathbf{B}$ in the following way:

$$\tilde{I}_1 = \tilde{\mathbf{B}}^2 - \tilde{\mathbf{E}}^2 = (\mathbf{B}^2 - \mathbf{E}^2)\cos 2\alpha + 2\mathbf{E}.\mathbf{B}\sin 2\alpha = I_1\cos 2\alpha + I_2\sin 2\alpha, \tag{16}$$

$$\tilde{I}_2 = 2\tilde{\mathbf{E}}.\tilde{\mathbf{B}} = (\mathbf{E}^2 - \mathbf{B}^2)\sin 2\alpha + 2\mathbf{E}.\mathbf{B}\cos 2\alpha = -I_1\sin 2\alpha + I_2\cos 2\alpha. \tag{17}$$

It follows immediately that

$$\tilde{I}_1^{\,2} + \tilde{I}_2^{\,2} = I_1^2 + I_2^2,$$

i.e. the sum of the squared invariants is a D-invariant.

2.2 Relativistic consideration. Recall Maxwell's pure field equations in relativistic form

$$\mathbf{dF} = 0,\ \ \mathbf{d} * \mathbf{F} = 0. \tag{18}$$

The Hodge $*$-operator is defined by the relation

$$\alpha \wedge *\beta = -\eta(\alpha,\beta)\sqrt{|det(\eta_{\mu\nu})|}dx \wedge dy \wedge dz \wedge d\xi,$$

where α and β are a p-forms on the Minkowski spacetime M, $\xi = ct$ is the time coordinate, and the Minkowski metric η has signiture $(-,-,-,+)$.

We note first 2 simple symmetries of (18).

1^o. The transformation $\mathbf{F} \rightarrow *\mathbf{F}$ keeps the system (18) the same. This follows from the property $*(*\mathbf{F}) = -\mathbf{F}$ of the $*$-operator when restricted on 2-forms.

2^o. Any *conformal* change of the Minkowski metric $\eta \rightarrow f^2\eta$, where f is everywhere different from zero function on the Minkowski space M, keeps the restriction of the Hodge $*$ to 2-forms on M the same, so (18) is conformally invariant.

Recalling the explicit form of $\mathbf{F}$ and $*\mathbf{F}$ in canonical coordinates

$$\begin{aligned}\mathbf{F} =& \mathbf{B}_3 dx \wedge dy - \mathbf{B}_2 dx \wedge dz + \mathbf{B}_1 dy \wedge dz \\ &+ \mathbf{E}_1 dx \wedge d\xi + \mathbf{E}_2 dy \wedge d\xi + \mathbf{E}_3 dz \wedge d\xi\end{aligned} \tag{19}$$

$$\begin{aligned}*\mathbf{F} =& \mathbf{E}_3 dx \wedge dy - \mathbf{E}_2 dx \wedge dz + \mathbf{E}_1 dy \wedge dz \\ &- \mathbf{B}_1 dy \wedge dz - \mathbf{B}_2 dy \wedge d\xi - \mathbf{B}_3 dz \wedge d\xi\end{aligned} \tag{20}$$

we see that $*$ replaces $\mathbf{E}$ with $-\mathbf{B}$ and $\mathbf{B}$ with $\mathbf{E}$, i.e. the action of $*$ gives the SD-transformation. On the other hand an extended SD-transformation may be introduced by

$$(\mathbf{F}, *\mathbf{F}) \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix} = (*\mathbf{F}, -\mathbf{F}).$$

We note the difference: the $*$-operator transforms a p-form β into a $(4-p)$-form $*\beta$, while the above SD-transformation transforms a *couple* of forms to another *couple* of forms.

As in the nonrelativistic case, this SD-transformation is readily extended to the full D-transformation

$$(\mathbf{F}, *\mathbf{F}) \rightarrow (\mathcal{F}, \tilde{\mathcal{F}}) = (\mathbf{F}, *\mathbf{F}) \begin{Vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{Vmatrix},$$

i.e.

$$\mathcal{F} = \mathbf{F}\cos\alpha + *\mathbf{F}\sin\alpha, \ \tilde{\mathcal{F}} = -\mathbf{F}\sin\alpha + *\mathbf{F}\cos\alpha. \tag{21}$$

The above transformations transform solutions to solutions. Moreover, *in contrast* to the nonrelativistic case, where linear combinations of $\mathbf{E}$ and $\mathbf{B}$ of *special* kind transform solutions to solutions, here every linear combination of $\mathbf{F}$ and $*\mathbf{F}$, i.e. a transformation of the kind

$$\mathcal{F}_g = a\mathbf{F} + b * \mathbf{F}, \ \tilde{\mathcal{F}}_g = m\mathbf{F} + n * \mathbf{F}, \tag{22}$$

where the subscribe g means "general", with arbitrary constants (a, b, m, n) gives again a solution. As we shall see this is because some of the special properties of S are now hidden in the $*$-operator through the pseudo-metric η, and the components $\mathbf{F}$ and $*\mathbf{F}$ are interrelated.

It is important to note that transformation (21) keeps the energy-momentum tensor

$$\mathbf{Q}_\mu^\nu = \frac{1}{4\pi}\left[\frac{1}{4}\mathbf{F}_{\alpha\beta}\mathbf{F}^{\alpha\beta}\delta_\mu^\nu - \mathbf{F}_{\mu\sigma}\mathbf{F}^{\nu\sigma}\right] = \frac{1}{8\pi}\left[-\mathbf{F}_{\mu\sigma}\mathbf{F}^{\nu\sigma} - (*\mathbf{F})_{\mu\sigma}(*\mathbf{F})^{\nu\sigma}\right], \tag{23}$$

and its divergence

$$\nabla_\nu \mathbf{Q}_\mu^\nu = -\frac{1}{4\pi}\left[\mathbf{F}_{\mu\nu}(\nabla_\sigma \mathbf{F}^{\sigma\nu}) + (*\mathbf{F})_{\mu\nu}(\nabla_\sigma(*\mathbf{F})^{\sigma\nu})\right] \tag{24}$$

the same: $\mathbf{Q}_\mu^\nu(\mathbf{F}) = Q_\mu^\nu(\mathcal{F})$, $\nabla_\nu \mathbf{Q}_\mu^\nu(\mathbf{F}) = \nabla_\nu \mathbf{Q}_\mu^\nu(\mathcal{F})$.

We see that the relativistic formulation of Maxwell theory naturally admits general $\mathcal{R}^2$-covariance as far as transformations (22) are implied to act on two 2-forms of the kind $(\mathbf{F}, *\mathbf{F})$. As for the D-transformations, they are closely related to the symmetries of the energy-momentum quantities and relations.

The two quantities

$$(4\pi)^2 Q_{\mu\nu}Q^{\mu\nu} = I_1^2 + I_2^2,\ (4\pi)^2 Q_{\mu\sigma}Q^{\nu\sigma} = \frac{1}{4}(I_1^2 + I_2^2)\delta_\mu^\nu$$

also enjoy the D-invariance. We note also that the eigen values of $\mathbf{Q}_\mu^\nu$ are D-invariant, as it should be, while the eigen values of $\mathbf{F}$ and $*\mathbf{F}$, given respectively by

$$\lambda_{1,2} = \pm\sqrt{-\frac{1}{2}I_1 + \frac{1}{2}\sqrt{I_1^2 + I_2^2}}, \quad \lambda_{3,4} = \pm\sqrt{-\frac{1}{2}I_1 - \frac{1}{2}\sqrt{I_1^2 + I_2^2}},$$

$$\lambda_{1,2}^* = \pm\sqrt{\frac{1}{2}I_1 + \frac{1}{2}\sqrt{I_1^2 + I_2^2}}, \quad \lambda_{3,4}^* = \pm\sqrt{\frac{1}{2}I_1 - \frac{1}{2}\sqrt{I_1^2 + I_2^2}}$$

are not D-invariant. Only when $I_1 = I_2 = 0$, the so called *null field case*, the eigen values of $\mathbf{F}$ and $*\mathbf{F}$ are D-invariant since in this case they are zero.

The relativistic generalization of equations (11)-(13) is

$$\nabla_\nu \mathbf{F}^{\nu\mu} = -4\pi j_e^\mu, \tag{25}$$

$$\nabla_\nu(*\mathbf{F})^{\nu\mu} = -4\pi J^\mu_m, \tag{26}$$

$$\mu c^2 u^\nu \nabla_\nu u_\mu = -\mathbf{F}_{\mu\nu} j^\nu_e - (*\mathbf{F})_{\mu\nu} J^\nu_m, \tag{27}$$

where μ is the invariant mass density, $j^\mu_e = \rho_e u^\mu$, $J^\mu_m = (\boldsymbol{J}_m, J^4) = (-\boldsymbol{j}_m, -\rho_m)$, and u^μ is the 4-velocity vector field.

The above system (25)-(27) enjoys the following symmetry transformation:

$$\mathbf{F} \to *\mathbf{F};\ j_e \to J_m;\ J_m \to -j_e.$$

This invariance is a particular case of the more general invariance transformation given by relations (21) plus

$$j'_e = j_e \cos\alpha - J_m \sin\alpha,\ J'_m = j_e \sin\alpha + J_m \cos\alpha,$$

which is readily checked. It should be noted that these invariances make use of the property $*^2 = -id$ of the restricted to 2-forms Hodge $*$-operator, and of the *constancy* of the phase angle α in the D-transformation.

Finally we'd like to note the different physical sense of equation (27) compare to equations (25)-(26): equation (27) equalizes changes of energy-momentum densities, it is a direct differential form of the energy-momentum balance relation between the field and the particles, carrying mass and electric and magnetic charge; equations (25)-(26) are relativistic forms of the differential equivalents of the time changes of the flows through 2-surfaces of **E** and **B**, moreover, these two equations identify quantities of *different physical nature*: it is hard to believe that differentiating field functions (the left hand side) we could obtain currents of charged mass particles (the right hand side).

3 The Suggestion and Developments

3.1 Nonrelativistic formulation. We summurize some of the D-features of the field description through Maxwell equations.

1. The D-invariance of the field equations is a mathematical representation of the dual *electro-magnetic* $(\mathbf{E}, \mathbf{B})$-nature of the field. This dual nature is explicitly seen in the nonrelativistic form of Maxwell's equations: **E** and **B** depend on each other but they can be always distinguished from each other.

2. All energy-momentum quantities and relations are D-invariant.

3. The D-transformation is represented by a rotation in a 2-dimensional vector space and acts through superposing the electric and magnetic components of the field and the corresponding currents.

4. The rotation angle α does not depend on the space-time coordinates.

5. The electro-magnetic duality transformation keeps and emphasises the field's united nature.

The following suggestion comes from these notices: the electromagnetic field, considered as *one physical object*, has *two physically distinguishable interrelated vector components*, $(\mathbf{E}, \mathbf{B})$, so the adequate mathematical model-object must have two vector algebraically distinguishable and differentially interelated (through the equations) components and must admit 2-dimensional linear transformations of its components. In particular, the 2-dimensional rotations should be closely related to the invariance properties of the energy-momentum characteristics of the field. But every 2-dimensional linear transformation requires a "room where to act", i.e. a 2-dimensional real vector space has to be *explicitly pointed out* and incorporated in the theory. This 2-dimensional space has always been implicitly present inside the electromagnetic field theory, but has never been introduced explicitly. We shall introduce it through the following assumption.

The electromagnetic field is mathematically represented (nonrelativisticaly) by an $\mathcal{R}^2$-valued differential 1-form ω, such that in the canonical basis $(\varepsilon^1, \varepsilon^2)$ in $\mathcal{R}^2$ the 1-form ω looks as follows

$$\omega = \mathbf{E} \otimes \varepsilon^1 + \mathbf{B} \otimes \varepsilon^2. \tag{28}$$

Remark. In (28), as well as later on, we identify the vector fields and 1-forms on $\mathcal{R}^3$ through the euclidean metric and we write, e.g. $*(\mathbf{E} \wedge \mathbf{B}) = \mathbf{E} \times \mathbf{B}$. Also, we identify $(\mathcal{R}^2)^*$ with $\mathcal{R}^2$ through the euclidean metric.

Now we have to present equations (11)-(13) correspondingly, i.e. in terms of $\mathcal{R}^2$-valued objects.

We begin with the electric ρ_e and magnetic ρ_m densities, considering them as components of an $\mathcal{R}^2$ valued function $\mathcal{Q}$, i.e.

$$\mathcal{Q} = \rho_e \otimes \varepsilon^1 + \rho_m \otimes \varepsilon^2. \tag{29}$$

The two currents $\boldsymbol{j}_e$ and $\boldsymbol{j}_m$, considered as 1-forms, become components of an $\mathcal{R}^2$-valued 1-form $\mathcal{J}$ as follows

$$\mathcal{J} = \boldsymbol{j}_e \otimes \varepsilon^1 + \boldsymbol{j}_m \otimes \varepsilon^2. \tag{30}$$

As we mentioned earlier, the above assumption (28) requires a general covariance with respect to transformations in $\mathcal{R}^2$, so, the complex structure $\mathcal{I}$ has to be introduced explicitly in the equations. In order to do this we recall that the linear map $\mathcal{I} : \mathcal{R}^2 \rightarrow \mathcal{R}^2$ induces a map $\mathcal{I}_* : \omega \rightarrow \mathcal{I}_*(\omega) = \mathbf{E} \otimes \mathcal{I}(\varepsilon^1) + \mathbf{B} \otimes \mathcal{I}(\varepsilon^2) = -\mathbf{B} \otimes \varepsilon^1 + \mathbf{E} \otimes \varepsilon^2$. We recall also that every operator $\mathcal{D}$ in the set of differential forms is naturally extended to vector-valued differential forms according to the rule $\mathcal{D} \rightarrow \mathcal{D} \times id$, and id is usually omitted. Having in mind the identification of vector fields and 1-forms through the euclidean metric we introduce now $\mathcal{I}$ in Maxwell's equations (11)-(13) through ω in the following way:

$$*\mathbf{d}\omega - \frac{1}{c}\frac{\partial}{\partial t}\mathcal{I}_*(\omega) = \frac{4\pi}{c}\mathcal{I}_*(\mathcal{J}), \quad \delta\omega = -4\pi\mathcal{Q}, \tag{31}$$

where $\delta = *\mathbf{d}*$ is the codifferential. Two other equivalent forms of (31) are given as follows:

$$\mathbf{d}\omega - *\frac{1}{c}\frac{\partial}{\partial t}\mathcal{I}_*(\omega) = \frac{4\pi}{c} * \mathcal{I}_*(\mathcal{J}), \quad \delta\omega = -4\pi\mathcal{Q},$$

$$*\mathbf{d}\mathcal{I}_*(\omega) + \frac{1}{c}\frac{\partial}{\partial t}\omega = -\frac{4\pi}{c}\mathcal{J}, \quad \delta\omega = -4\pi\mathcal{Q}.$$

In order to verify the equivalence of (31) to Maxwell equations (11)-(13) we compute the marked operations. For the left-hand side of the first (31) equation we obtain

$$*\mathbf{d}\omega - \frac{1}{c}\frac{\partial}{\partial t}\mathcal{I}_*(\omega) = \left(\mathrm{rot}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\right) \otimes \varepsilon^1 + \left(\mathrm{rot}\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) \otimes \varepsilon^2,$$

and the right-hand side is

$$\frac{4\pi}{c}\mathcal{I}_*(\mathcal{J}) = -\frac{4\pi}{c}\boldsymbol{j}_m \otimes \varepsilon^1 + \frac{4\pi}{c}\boldsymbol{j}_e \otimes \varepsilon^2$$

The second equation $\delta\omega = -4\pi\mathcal{Q}$ is, obviously, equivalent to

$$\mathrm{div}\mathbf{E}\otimes\varepsilon^1 + \mathrm{div}\mathbf{B}\otimes\varepsilon^2 = 4\pi\rho_e\otimes\varepsilon^1 + 4\pi\rho_m\otimes\varepsilon^2$$

since $\delta = -\mathrm{div}$. Hence, (31) coincides with (11)-(13).

We shall emphasize once again that according to our general assumption (28) the field ω will have different representations in the different bases of $\mathcal{R}^2$. Changing the basis $(\varepsilon^1, \varepsilon^2)$ to any other basis $\varepsilon^{1'} = \varphi(\varepsilon^1), \varepsilon^{2'} = \varphi(\varepsilon^2)$, means, of course, that in equations (31) the field ω changes to $\varphi_*\omega$ and the complex structure $\mathcal{I}$ changes to $\varphi\mathcal{I}\varphi^{-1}$. In some sense this means that we have two fields now: ω and $\mathcal{I}$, but $\mathcal{I}$ is given beforehand and it is not determined by equations (31). So, in the new basis the $\mathcal{I}$-dependent equations of (31) will look like

$$*\mathbf{d}\varphi_*\omega - \frac{1}{c}\frac{\partial}{\partial t}(\varphi\mathcal{I}\varphi^{-1})_*(\varphi_*\omega) = \frac{4\pi}{c}(\varphi\mathcal{I}\varphi^{-1})_*(\varphi_*\mathcal{J}).$$

If φ is a symmetry of $\mathcal{I} : \varphi\mathcal{I}\varphi^{-1} = \mathcal{I}$, then we transform just ω to $\varphi_*\omega$, so if ω is a solution then $\varphi_*\omega$ is also a solution.

In order to write down the Poynting energy-momentum balance relation we recall the product of vector-valued differential forms. Let $\Phi = \Phi^a\otimes e_a$ and $\Psi = \Psi^b\otimes k_b$ are two differential forms on some manifold with values in the vector spaces V_1 and V_2 with bases $\{e_a\}, a = 1, ..., n$ and $\{k_b\}, b = 1, ..., m$, respectively. Let $f : V_1 \times V_2 \to W$ is a bilinear map valued in a third vector space W. Then a new differential form, denoted by $f(\Phi, \Psi)$, on the same manifold and valued in W is defined by

$$f(\Phi, \Psi) = \Phi^a \wedge \Psi^b \otimes f(e_a, k_b).$$

Clearly, if the original forms are p and q respectively, then the product is a $(p+q)$-form.

Assume now that $V_1 = V_2 = \mathcal{R}^2$ and the bilinear map is the exterior product:
$\wedge : \mathcal{R}^2 \times \mathcal{R}^2 \to \Lambda^2(\mathcal{R}^2)$. Let's compute the expression $\wedge(\omega, \mathbf{d}\omega)$.

$$\begin{aligned}\wedge(\omega,\mathbf{d}\omega) &= \wedge(\mathbf{E}\otimes\varepsilon^1 + \mathbf{B}\otimes\varepsilon^2, \mathbf{dE}\otimes\varepsilon^1 + \mathbf{dB}\otimes\varepsilon^2)\\ &= (\mathbf{E}\wedge\mathbf{dB} - \mathbf{B}\wedge\mathbf{dE})\otimes\varepsilon^1\wedge\varepsilon^2 = -\mathbf{d}(\mathbf{E}\wedge\mathbf{B})\otimes\varepsilon^1\wedge\varepsilon^2\\ &= -\mathbf{d}(**(\mathbf{E}\wedge\mathbf{B}))\otimes\varepsilon^1\wedge\varepsilon^2 = *\delta(\mathbf{E}\times\mathbf{B})\otimes\varepsilon^1\wedge\varepsilon^2 =\\ &= -*\mathrm{div}(\mathbf{E}\times\mathbf{B})\otimes\varepsilon^1\wedge\varepsilon^2 = -\mathrm{div}(\mathbf{E}\times\mathbf{B})dx\wedge dy\wedge dz\otimes\varepsilon^1\wedge\varepsilon^2.\end{aligned}$$

Following the same rules we obtain

$$\wedge\left(\omega, *\frac{1}{c}\frac{\partial}{\partial t}\mathcal{I}_*\omega\right) = \frac{1}{c}\frac{\partial}{\partial t}\frac{\mathbf{E}^2+\mathbf{B}^2}{2}dx\wedge dy\wedge dz\otimes\varepsilon^1\wedge\varepsilon^2,$$

and

$$\frac{4\pi}{c}\wedge(\omega, *\mathcal{I}_*\mathcal{J}) = \frac{4\pi}{c}\left(\mathbf{E}.\boldsymbol{j}_e - \mathbf{B}.\boldsymbol{j}_m\right)dx\wedge dy\wedge dz\otimes\varepsilon^1\wedge\varepsilon^2.$$

Hence, the generalized Poynting energy-momentum balance relation is given by

$$\wedge\left(\omega, \mathbf{d}\omega - *\frac{1}{c}\frac{\partial}{\partial t}\mathcal{I}_*\omega\right) = \frac{4\pi}{c}\wedge(\omega, \mathcal{I}_*\mathcal{J})\,. \tag{32}$$

As for the generalized Lorentz force $\vec{\mathcal{F}}$, staying on the right-hand side of eq.(13), it is presented by

$$\begin{aligned}\vec{\mathcal{F}}\otimes\varepsilon^1\wedge\varepsilon^2 &= \left[\frac{1}{c}\left(\boldsymbol{j}_e\times\mathbf{B} - \boldsymbol{j}_m\times\mathbf{E}\right) + \rho_e\mathbf{E} + \rho_m\mathbf{B}\right]\otimes\varepsilon^1\wedge\varepsilon^2\\ &= \frac{1}{c}*\wedge(\omega, \mathcal{J}) + *\wedge(\omega, \mathcal{I}_*\mathcal{Q})\,.\end{aligned}\tag{33}$$

Since the orthonormal 2-form $\varepsilon^1\wedge\varepsilon^2$ is invariant with respect to rotations (and even with respect to unimodular transformations in $\mathcal{R}^2$) we have the duality invariance of the above energy-momentum quantities and relations.

Hence, in our approach we have achieved a full covariance of the equations, given in the form (31). Indeed, the covariance with respect to arbitrary transformations in $\mathcal{R}^3$ is obvious, so we show now the covariance of (31) with respect to nonsingular linear transformations $\varphi:\mathcal{R}^2\rightarrow\mathcal{R}^2$. Let ω satisfies (31), so we have to show that $\varphi_*(\omega) = \mathbf{E}\otimes\varphi(\varepsilon^1)+\mathbf{B}\otimes\varphi(\varepsilon^2)$ also satisfies (31). We apply φ from the left on (31) and obtain

$$\begin{aligned}\varphi\left(*\,\mathbf{d}\omega - \frac{1}{c}\frac{\partial}{\partial t}\mathcal{I}_*\omega\right) &= \left(*\mathbf{d}\varphi_*\omega - \frac{1}{c}\frac{\partial}{\partial t}\varphi_*\mathcal{I}_*\omega\right)\\ &= \left(*\mathbf{d}\varphi_*\omega - \frac{1}{c}\frac{\partial}{\partial t}\varphi_*\mathcal{I}_*\varphi_*^{-1}\varphi_*\omega\right) =\\ &= \left(*\mathbf{d}\varphi_*\omega - \frac{1}{c}\frac{\partial}{\partial t}(\varphi\mathcal{I}\varphi^{-1})_*\varphi_*\omega\right) = \left(*\mathbf{d} - \frac{1}{c}\frac{\partial}{\partial t}\mathcal{I}'_*\right)\varphi_*\omega,\end{aligned}$$

where $\mathcal{I}' = \varphi\mathcal{I}\varphi^{-1}$. For the right-hand side of (31) we obtain

$$(\varphi\mathcal{I})_*\mathcal{J} = (\varphi\mathcal{I}\varphi^{-1})_*\varphi_*\mathcal{J} = \mathcal{I}'_*\varphi_*\mathcal{J},$$

so, our assertion is proved.

Note the following simple forms of the energy density

$$\frac{1}{8\pi} * \wedge(\omega, *\mathcal{I}_*\omega) = \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi}\varepsilon^1 \wedge \varepsilon^2,$$

and of the Poynting vector,

$$\frac{c}{8\pi} * \wedge(\omega, \omega) = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} \otimes \varepsilon^1 \wedge \varepsilon^2,$$

the D-invariance is obvious. As for the general $\mathcal{R}^2$ covariance of the second equation of (31) it is obvious.

Resuming, we may say that pursuing the correspondence: *one physical object - one mathematical model-object*, we came to the idea to introduce the $\mathcal{R}^2$-valued 1-form ω as the mathematical model-field. This, in turn, set the problem for general $\mathcal{R}^2$ covariance of the equations and this problem was solved through introducing explicitly the canonical complex structure $\mathcal{I}$ in the dynamical equations (31) of the theory. The duality transformation appears now as an invariance property of the energy-momentum quantities and relations.

3.2 Relativistic formulation. As it was mentioned in the previous section in the relativistic formulation of Maxwell equations we have got already a general $\mathcal{R}^2$-covariance, but the two components, subject to the general $\mathcal{R}^2$-linear transformation, are *not* algebraically independent, in fact they are $(\mathbf{F}, *\mathbf{F})$. We look now at the situation from another point of view.

First we note that we shall follow the main idea of the nonrelativistic formulation, namely, to consider as a mathematical-model field some $\mathcal{R}^2$-valued differential form, being closely connected to the canonical complex structure $\mathcal{I}$ in $\mathcal{R}^2$. But in contrast to the nonrelativistic case here we consider an $\mathcal{R}^2$-valued 2-form on the Minkowski space-time $M = (\mathcal{R}^4, \eta)$. In general such a 2-form Ω looks as $\Omega = \mathbf{F}_1 \otimes \varepsilon^1 + \mathbf{F}_2 \otimes \varepsilon^2 = \mathbf{F}_a \otimes \varepsilon^a$. Let now we are given two linear maps:

$$\Phi : \Lambda^2(M) \rightarrow \Lambda^2(M),$$

$$\varphi : \mathcal{R}^2 \to \mathcal{R}^2.$$

These maps induce a map $(\Phi, \varphi) : \Lambda^2(M, \mathcal{R}^2) \to \Lambda^2(M, \mathcal{R}^2)$ by the rule:

$$(\Phi, \varphi)(\Omega) = (\Phi, \varphi)(\mathbf{F}_a \otimes \varepsilon^a) = \Phi(\mathbf{F}_a) \otimes \varphi(\varepsilon^a).$$

It is natural to ask now is it possible the joint action of these two maps to keep Ω unchanged, i.e. to have

$$(\Phi, \varphi)(\Omega) = \Omega.$$

In such a case the form Ω is called (Φ, φ)-*equivariant*. If φ is a linear isomorphism and we identify Φ with (Φ, id) and φ with (id, φ), we can equivalently write

$$\Phi(\Omega) = \varphi^{-1}(\Omega).$$

If we specialize now: $\varphi = \mathcal{I}$ we readily find that the $(\Phi, \mathcal{I})$-equivariant forms Ω must satisfy

$$(\Phi, \mathcal{I})(\Omega) = -\Phi(\mathbf{F}_2) \otimes \varepsilon^1 + \Phi(\mathbf{F}_1) \otimes \varepsilon^2 = \mathbf{F}_1 \otimes \varepsilon^1 + \mathbf{F}_2 \otimes \varepsilon^2,$$

Hence, we must have $\Phi(\mathbf{F}_1) = \mathbf{F}_2$ and $\Phi(\mathbf{F}_2) = -\mathbf{F}_1$, i.e. $\Phi^2 = -id$. In other words, the defining property of the complex structure $\mathcal{I}$: $\mathcal{I}^2 = -id$, is carried over to a linear map Φ in $\Lambda^2(M)$: $\Phi^2 = -id$. Since the Hodge $*$, restricted to 2-forms in Minkowski space, satisfies this condition, and according to expressions (19)-(20) in standard coordinates its action coincides with the special duality transformation, it is a natural candidate for Φ. Hence, working with $(*, \mathcal{I})$-equivariant 2-forms on Minkowski space, we can replace the action of $\mathcal{I}$ with the action of the Hodge $*$-operator. And that's why in relativistic electrodynamics we have general $\mathcal{R}^2$ covariance if we work with forms Ω of the kind $\Omega = \mathbf{F} \otimes \varepsilon^1 + *\mathbf{F} \otimes \varepsilon^2$. In the nonrelativistic formulation this is not possible to be done since we work there with 1-forms on $\mathcal{R}^3$ and no map $\Phi : \Lambda^1(\mathcal{R}^3) \to \Lambda^1(\mathcal{R}^3)$ with the property $\Phi^2 = -id$ exists, and we have to introduce the complex structure through $\mathcal{R}^2$ only.

Having in view these considerations our basic assumption for the algebraic nature of the mathematical-model object must read:

The electromagnetic field is (relativistically) represented by a $(*, \mathcal{I})$-*equivariant 2-form* Ω *on the Minkowski space-time*:

$$\Omega = \mathbf{F} \otimes \varepsilon^1 + *\mathbf{F} \otimes \varepsilon^2. \tag{34}$$

The relativistic pure field Maxwell equations (18), expressed through the
$(*, \mathcal{I})$-equivariant 2-form Ω have, obviously, general $\mathcal{R}^2$ covariance and are equivalent to

$$\mathbf{d}\Omega = 0. \tag{35}$$

In presence of electric and magnetic charges, making use of the definitions in the previous section, we introduce the generalized relativistic current $\mathcal{J}_r$ as follows

$$\mathcal{J}_r = j_e \otimes \varepsilon^1 + J_m \otimes \varepsilon^2.$$

Hence, equations (25)-(26) acquire the form

$$\mathbf{d}\Omega = -4\pi \mathcal{J}_r. \tag{36}$$

The generalized Lorentz force is given by

$$- * \wedge(\mathcal{J}_r, \Omega) = \Big(-\mathbf{F}_{\mu\sigma} j_e^\sigma - (*\mathbf{F})_{\mu\sigma} J_m^\sigma\Big) dx^\mu \otimes \varepsilon^1 \wedge \varepsilon^2. \tag{37}$$

The divergence of the energy-momentum tensor $\mathbf{Q}_\mu^\nu$ is given by

$$* \wedge \Big(\delta\Omega, \Omega\Big) = -\frac{1}{4\pi}\Big[\mathbf{F}_{\mu\nu}\nabla_\sigma \mathbf{F}^{\sigma\nu} + (*\mathbf{F})_{\mu\nu}\nabla_\sigma (*\mathbf{F})^{\sigma\nu}\Big] dx^\mu \otimes \varepsilon^1 \wedge \varepsilon^2. \tag{38}$$

The energy-momentum tensor $\mathbf{Q}$, considered as a symmetric 2-form $\mathbf{Q}_{\mu\nu} = \mathbf{Q}_{\nu\mu}$, is given by

$$\left(\mathbf{Q} \otimes \varepsilon^1 \wedge \varepsilon^2\right)(X, Y) = \frac{1}{8\pi} * \wedge\Big(i_X\Omega, *i_Y \mathcal{I}_* \Omega\Big), \tag{39}$$

where X and Y are 2 arbitrary vector fields, and i_X is the inner product by the vector field X. Indeed,

$$i_X\Omega = X^\mu \mathbf{F}_{\mu\nu} dx^\nu \otimes \varepsilon^1 + X^\mu (*\mathbf{F})_{\mu\nu} dx^\nu \otimes \varepsilon^2,$$

$$*i_Y \mathcal{I}_*\Omega = *\Big[Y^\mu \mathbf{F}_{\mu\nu} dx^\nu\Big] \otimes \varepsilon^2 - *\Big[Y^\mu (*\mathbf{F})_{\mu\nu} dx^\nu\Big] \otimes \varepsilon^1.$$

$$\wedge\Big(i_X\Omega, *i_Y\mathcal{I}_*\Omega\Big) = -X^\mu Y^\nu \Big[\mathbf{F}_{\mu\sigma}\mathbf{F}_\nu^\sigma + (*\mathbf{F})_{\mu\sigma}(*\mathbf{F})_\nu^\sigma\Big] dx \wedge dy \wedge dz \wedge d\xi \otimes \varepsilon^1 \wedge \varepsilon^2.$$

So, we obtain

$$\frac{1}{8\pi} * \wedge(i_X\Omega, *i_Y\mathcal{I}_*\Omega) = -\frac{1}{8\pi}X^\mu Y^\nu\Big[\mathbf{F}_{\mu\sigma}F^\sigma_\nu + (*\mathbf{F})_{\mu\sigma}(*\mathbf{F})^\sigma_\nu\Big]\varepsilon^1\wedge\varepsilon^2.$$

The presence of the 2-form $\varepsilon^1\wedge\varepsilon^2$ introduces invariance with respect to unimodular transformations in $\mathcal{R}^2$.

Another definition of the energy-momentum tensor, making no use of the complex structure $\mathcal{I}$, is through the canonical inner product g in $\mathcal{R}^2$ as a bilinear map instead of $\wedge$. Indeed, it is easy to see that the right-hand side of the relation

$$\mathbf{Q}_{\mu\nu}X^\mu Y^\nu = \frac{1}{2} * g\Big(i(X)\Omega, *i(Y)\Omega\Big)$$

is equal to

$$-\frac{1}{2}X^\mu Y^\nu\Big[\mathbf{F}_{\mu\sigma}\mathbf{F}^\sigma_\nu + (*\mathbf{F})_{\mu\sigma}(*\mathbf{F})^\sigma_\nu\Big].$$

Resuming, we note the main differences with respect to the nonrelativistic formulation. First, the mathematical model-object is a 2-form Ω on Minkowski space with values in $\mathcal{R}^2$, second, Ω is $(*,\mathcal{I})$-equivariant. As for the usual duality transformations, they appear as particular $\mathcal{R}^2$-invariance properties of the conserved quantities and of the corresponding conservation relations.

The general conclusion of this section is that the $\mathcal{R}^2$ valued nonrelativistic 1-form ω and the relativistic 2-form Ω seem to be natural and adequate mathematical model-objects of electromagnetic fields, while the duality transformations characterize the invariance properties of the conversed quantities and the corresponding conservation relations.

4 Conclusion

Here we are going to mention those points of the paper which from our point of view seem most important.

Classical electrodynamics works mainly with two concepts: *charge* and *field*. The charge carriers (called also field sources) are considered as point-like (or structureless) objects. The field is considered as generated by static or moving charges, and it is not defined at the points of its own source, hence, the point charges acquire topological sense. Passing to continuous charge distributions we write down currents on the right hand sides of Maxwell equations and forget about the topological nature of charges. Moreover, this identification of characteristics of the field represented by the left hand sides of the equations with characteristics of mass objects carrying electric (and possiply magnetic) charges, seems not well enough motivated from theoretical point of view. It would be more natural to write down equations which identify quantities of the *same* nuture, e.g. some energy-momentum balance relation between the field and the particles (recall our remark at the end of Sec.2).

The duality properties of the solutions reveal the internal structure of the field as having *two vector components*, which are

-differentially interrelated (through the equations), but

-algebraically distinguished.

Moreover, the adequate understanding of the duality properties requires explicitly introduced complex structure in the equations.

This resulted in making use of $\mathcal{R}^2$-valued differential forms, ω and Ω, as mathematical model objects, and corresponding complex structures $\mathcal{I}$ and the relativistic Hodge $*$ restricted to 2-forms. The equations admit a full $\mathcal{R}^2$ covariance, while the duality properties appear as invariance properties of the conservative quantities and conservation relations. In fact, the action of the D-transformations in the linear space of vacuum solutions separates classes of solutions with the same energy-momentum.

Finally, we may expect that recognizing the structure of the field as a *double vector-component* one through its *duality* properties may open new ways of considerations and may generate new ideas and developments towards an appropriate nonlinearization of classical electrodynamics.

Note.

The Hodge $*$-operator in the 3-dimensional euclidean case in standard coordinates acts as follows:

$$\begin{aligned} *1 = dx\wedge dy\wedge dz, &\quad *dx = dy\wedge dz, \\ dy = -dx\wedge dz, &\quad *dz = dx\wedge dy; \\ (dx\wedge dy) = dz, &\quad *(dx\wedge dz) = -dy, \\ *(dy\wedge dz) = dx; &\quad *(dx\wedge dy\wedge dz) = 1. \end{aligned}$$

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(Manuscrit reçu le 10 mai 2001)