Propositional bases for the physics of the Bernoulli oscillators (A theory of the hidden degree of freedom)

II - Mechanical framework

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ABSTRACT. In a few previous papers, we developed a so-called classical fluctuation model, which revealed remarkably effectual in the task of approaching quantum mechanical results by the means of classical expressions. This paper is the second one of a series of four, introducing developments of the model. In paper I we provided some basic thermodynamic properties of the so-called Bernoulli oscillators: these are classical oscillators perturbed by the action of a "hidden" degree of freedom (HDF). In the present paper II, the mechanical properties corresponding to the thermodynamic model are investigated. HDF is identified physically as an oscillation superimposed to the particle classical degree of freedom. It is driven by an external, time-dependent force taking its origin in the quantum "vacuum", and is submitted to Heisenberg's principle as to a parametric constraint. The principle takes, in our framework, a peculiar classical-like interpretation. HDF perturbs the classical motion, so that an external (time-averaged) potential must be accounted for in the expression of the mechanical energy theorem. We give an analysis of the time-dependent forcing and a HDF-potential expression, as a function of an unknown parametric function whose identity will be made clear in the following paper IV. The HDF-potential is able to drive the particle throughout a barrier jump, thus providing us with a classical-like concept of the tunnelling phenomena. In some following papers, the discussed equations will be compared to both the contexts of our previous models and wave-mechanical physics.

1 Introduction

In the previous paper I [1] we discussed the thermodynamic properties of the Bernoulli oscillators, and showed that they are consistent with a standard classical thermodynamics framework if we assume that a so-called hidden degree of freedom HDF is intermediary between the purely classical, mechanical oscillator and the external physical environment (the quantum vacuum). This last can be assumed as an active (or more properly reactive, as will be advanced in the sequel) medium generating an excitation field ([1]). A few HDF thermodynamic properties have been brought to evidence, and a generalized apparatus has been set up to describe the overall system thermodynamic behavior. In the present paper II, we will not discuss generalities - these last can be found in [1]. Here we will set up a mechanical model for the Bernoulli oscillators, and make advances in the identification of the HDF properties and mechanical behavior. This last will be shown to be the basis to understand the energy transfer mechanism to the particle in a tunnelling process - thus providing a classical-like interpretation of such a phenomenon. As will be displayed in a next section, our program also includes accounting for Heisenberg's indetermination principle (particles space and momentum co-ordinates relation) within a classical-mechanics framework. Accomplishing this task will provide us with a peculiar (proposed) interpretation of the principle itself. As final results in this paper, expressions for the mechanical energy theorem and its statistical counterpart - this last in the form of the flow-mass theorem for a microcanonical particles ensemble - will be given. These expressions are for comparison with the corresponding ones which are given - by independent analyses - both in paper I and in the following paper denoted III. In this last, the comparison will be shown to result into what we believe is a peculiar insight into the physical behavior of systems. The investigation will provide us with an interpretative frame where the possibility of a Newtonian-like background behavior of quantum particles is evidenced - demonstrations and discussion about this last point are deferred to a final paper denoted IV.

2 Theoretical background

We will give here further details about the Bernoulli oscillators. Interest will first be focused into the mechanical properties appearing compatible with the thermodynamic properties we investigated already. Thermodynamics is indeed some sort of statistical mirror for mechanical effects, and the thermodynamic FEOM model [3] we refer as a basis all throughout our work has a definite mechanical counterpart which we are now challenged to find out.

[1] The idea that quantum properties may be explained by a vacuum interaction is first due to Vigier [2].

2.1 The Bernoulli oscillators and the fluctuation field

The Bernoulli oscillators are essentially, in our thinking, classical oscillators, but they are affected by the action of HDF and this last is assumed to be excited by the fluctuation field. Some properties of this action have been discussed already in previous work [$3 \div 5$], taking a mechanical point of view. We found our oscillators, perturbed by the fluctuation field, able to approach a few quantum properties (as the Bohr-Sommerfeld rule displayed [3] and tunnelling capability [5]). Their characteristic physical behavior is, at the present investigation stage, that the oscillators energy fluctuates consequent to the interaction with HDF+the fluctuation field. In different circumstances, we were able to understand some features of this interaction, although we did not know the exact time-law affecting the fluctuation. In order to provide ourselves with some other investigative tools we consider the following.

The fluctuation time-law is likely to display a chaotic behavior. In the present simple framework, we will not be sophistical, and the word "chaotic" will be used to describe up to a very irregular time-behavior - but we will limit to a concept not exceeding the boundary set by a deterministic time-dependence which can be described by ordinary Fourier-transformable functions, just to be definite. It is obvious, therefore, that in such simple type of "chaos" meaningful recurrences can hopefully be found to set up a description matching the thermodynamic and other properties we identified already. More precisely, when appropriate, we can use theoretical instruments we have available as the ergodicity postulate, to extract useful information starting by the displayed statistics - as is clear it may occur, f.i., when investigating slow time-scale behavior of the perturbed oscillators. We believe indeed that either slow, or even fast fluctuations of physical quantities - although by the effect of Heisenberg's principle they are able to escape easily to experimentation as well as to a detailed theoretical analysis - always follow a classical law of motion. If this is so, therefore, what is really defecting to the description of physical reality which is proposed to us by the framework of quantum mechanics is the right way to account for the effects of some oscillation of quantities, causing typical consequences on the particle motion. Some effects, to be specific, like the known, so-called "zitterbewegung" [6] if velocity is, f.i., the relevant fast-perturbed quantity. We believe that these effects are not a priori impossible to describe theoretically, nor incomprehensible in their substance as a restrictive interpretation of the indetermination principle would appear to state. To be practical, it is clear that - as far as long time-scale effects in particle motion perturbed by fluctuations are concerned at least - the ergodicity postulate can help us in displaying them, provided they have proper counterparts which can be identified within the statistical or "ensemble" behavior of systems. We will give in paper III an example of fruitfulness of this point of view. In the present paper II, we will assess the reference framework for the following analysis, by investigating the behavior of a classical oscillator when submitted to the action of external - slow and fast - time-dependent forces.

2.2 The forced classical oscillator and the hidden degree of freedom

The mechanical oscillator we investigate in this paper is by assumption submitted to the following classical motion law

$$-\frac{\mathrm{d}\Phi(\mathrm{x}^*)}{\mathrm{dx}^*} + \mathrm{F}^*(\mathrm{t},\xi(\mathrm{t},\mathrm{x}^*,\mathrm{v}^*,\mathrm{a}^*)) - \alpha^*\mathrm{v}^* = \mathrm{ma}^* \tag{1}$$

Here m is the oscillator mass, x^* is its (linear) space co-ordinate, v^* and a^* are its velocity and acceleration, respectively. In equation (1) we assume that a classical conservative force $-\nabla_{x*}\Phi$ acts on the oscillator. It is clear that the dynamical variables here (and henceforth as well) introduced must be considered as the appropriate components along the relevant x^*-axis of the corresponding precursory vector quantities - we do not need to display the index x^* because our model is uni-dimensional. The hidden degree of freedom is assumed to be effective within this equation, and will be soon identified by the deployment of distinguished terms - which we will call x_z and v_z - within the oscillator space and momentum co-ordinates x^*, v^* set. In our framework, HDF is driven by the time–varying force $F^*(t,\xi(t,x^*(t),v^*(t),a^*(t)))$. This force is assumed to come from the external environment (this is the quantum "vacuum"), much alike some kind of noise pervading the space. We called it, in previous papers, the fluctuation field. It might also be considered as the reaction of the "vacuum" to the assessment of the oscillator motion itself - this interpretation will be made clear in paper IV. The force excites HDF and is responsible for the appearance of quantum-mechanical effects in our framework. Concurrent to this effect, a viscous force $-\alpha^*v^*$ with a (constant, by simplicity) coefficient α^* is included in equation (1) for the sake of completeness. The viscous force also takes its origin in the vacuum. This last is therefore seen as an active medium able to provide, through the time, some extra energy ΔE(t) to the oscillator via the fluctuation force field F*, and to recover it via the friction term - the overall transfer process occurring within the limits allowed by the constraint of Heisenberg's principle. The choice of a friction $-\alpha^*$v* to insure the return energy flow is a standard assumption in linearized forced oscillators models. Note, for the sake of completeness, that the same term might be thought to account (at least approximately) for a radiation reaction effect if the particles we are considering are charged ones - the framework we are developing in these papers is, however, not intended to include specifically the radiation problem.

The overall term F* $-\alpha^*$v* is the net vacuum action on the oscillator and will be here called, to be definite, the quantum field.

A dependence on a hidden, mechanical parametric function ξ(t,x*) whose identity and (time-averaged) expression will be determined in this work (paper IV) is displayed into the expression of the force F*. By assumption, this unknown parameter is taken to depend on the particle kinematic variables set. This last, by completeness, includes position, velocity and acceleration; but the two last variables need not to be displayed in the sequel because they both depend on x*(t). Therefore, the simple notation F*(t,ξ(t,x*)) will be used in the following. Although this forcing is to explain the appearance of quantum mechanical effects - so that, as will be seen later, it is in a peculiar way submitted to the Heisenberg indetermination principle - the time- function F*(t,ξ(t,x*)) is taken to depend deterministically on the initial conditions of motion assigned for the particle. This is consistent with the requirement we have to remain within a classical physics scheme, and also demands for a "classical interpretation" of Heisenberg's principle itself. This last will be found in equations (27)$\div$(30). The initial conditions set of parameters also includes a specification for the hidden parameter ξ . Since the relevant physical apparition of this last will be found to depend on the particle velocity and acceleration space-derivative, this specification will take a peculiar form involving the velocity field derivatives, to be shown in paper IV. The initial conditions to be taken when a particles ensemble is considered are distributed at random according to appropriate statistical laws. Ensemble averaging procedures will be initiated in the final section of this paper, but major performances, in this respect, are reserved to the paper III of this work.

In the present paper II, instead, we want to perform time-averages to the purpose of obtaining an expression of the energy theorem for the particle submitted to the classical potential, plus the quantum field action, in the stationary motion assumption. The final target we want to approach by performing both time and ensemble averages in equation (1) is displaying the expression of the flow-mass theorem for a (micro-canonical) ensemble of particles. This is for the sake of comparison with independent expressions given in papers I and III, and with the stationary Schrödinger equation in paper IV. By means of the consequent analysis, the quantum field physical origin and behavior will be identified and discussed in paper IV.

It is useful to have equation (1) put in the form

$$\frac{1}{2}\mathrm{mv}^{*2} + \Phi(\mathrm{x}^*) \; = \; \mathrm{E(t)} \; = \; \mathrm{E} + \int \; [\mathrm{F}^*(\mathrm{t},\xi(\mathrm{t},\mathrm{x}^*)) - \alpha^*\mathrm{v}^*]\,\mathrm{v}^*(\mathrm{t})\mathrm{dt} =$$

$$= \; \mathrm{E} - \Phi^*(\mathrm{t},\xi(\mathrm{t},\mathrm{x}^*),\alpha^*) \tag{2}$$

In this equation, $\int \mathrm{F}^*(\mathrm{t},\xi(\mathrm{t},\mathrm{x}^*))\mathrm{v}^*(\mathrm{t})\mathrm{dt}$ is the (time-dependent) work performed by $\mathrm{F}^*(\mathrm{t},\xi(\mathrm{t},\mathrm{x}^*))$ on the oscillator and $-\Phi^*(\mathrm{t},\xi(\mathrm{t},\mathrm{x}^*),\alpha^*)$ is the overall work stored by the oscillator when accounted for the energy lost by friction.

2.3 Energy fluctuations, Newton's law and the Kapitza theorem

We have to consider that the time fluctuations of $\Phi^*(\mathrm{t},\xi(\mathrm{t},\mathrm{x}^*),\alpha^*)$ can be strong ones, so that the particle velocity and position can display strong fluctuations in their turn. The fluctuations can be taken responsible for the thermodynamic behavior we described in reference [3], but in this paper we will not deep into this topic and we take a different investigation strategy. We want to focus on mechanical effects.

We will assume first that the fluctuations are fast. We will average them out by integrating the motion equation over a small-scale of time around some space position x, as is generally done within a standard classical framework first introduced by Kapitza [7 − 9]. If the particle velocity can be split into two components - the first one is slowly variable with time (here v, no index) and the second one is the fast-varying part of it (here v_z, z ≡ zitterbewegung) then it is easy to state the following equations :

$$\mathrm{x}^*(\mathrm{t}) = \mathrm{x}(\mathrm{t}) + \mathrm{x}_z(\mathrm{t}) \tag{3}$$

$$v^*(t) = \frac{d}{dt}\left[x\left(t\right) + x_z\left(t\right)\right] = v\left(t\right) + v_z\left(t\right) \tag{4}$$

$$a^*(t) = \frac{d}{dt}\left[v\left(t\right) + v_z\left(t\right)\right] = a\left(t\right) + a_z\left(t\right) \tag{5}$$

The time averages of the fast varying quantities performed over a small time-scale around a co-ordinate value x(t) will be taken equal to zero:

$$< x_z(t) > |_x = < v_z(t) > |_x = < a_z(t) > |_x = 0 \tag{6}$$

We want to perform a time-average of expression (2) and we have :

$$- < \int \left[F^*(t,\xi(t,x^*)) - \alpha^* v^*\right] v^*(t) dt > |_x = < \Phi^*(t,\xi(t,x^*),\alpha^*) > |_x \tag{7}$$

$$< \frac{1}{2} m v^{*2} > |_x + < \Phi(x^*) > |_x + < \Phi^*(t,\xi(t,x^*),\alpha^*) > |_x = E \tag{8}$$

Here E is the relevant mechanical energy constant. In order to provide some specific expressions when we perform these averages, we consider the following topics.

2.4 Constitutive equations for the forcing term

Let us write

$$F^*(t,\xi(t,x^*)) = F^*_s(t,\xi(t,x)) + F^*_z(t,\xi(t,x^*)) \tag{9}$$

The force F^* is assumed to have two components: a (zero order) "slow-varying" one F^*_s, and a "fast-varying" one F^*_z. The slow-varying one is taken not dependent of the fast-varying part of the oscillator co-ordinate x_z, while the fast-varying one depends in principle both on an explicit fast time variation and on x^*. It may therefore display a slow component itself by non-linear coupling. To see this with some useful details, let us consider the following equations. First we can write, by a series development :

$$F^*(t,\xi(t,x^*)) = F^*(t,\xi(t,x)) + \frac{\partial\left(F^*_z(t,\xi(t,x)) + x_z R_{z2}(t,x,x_z)\right)}{\partial x} x_z \tag{10}$$

$$\mathrm{R}_{z2}(\mathrm{t,x,x_z}) = \sum_{n=2}^{\infty} \frac{\partial^{n-1}\,\mathrm{F}_z^*(\mathrm{t},\xi(\mathrm{t,x}))}{\partial \mathrm{x}^{n-1}} \frac{\mathrm{x}_z^{n-2}}{n!} \tag{11}$$

We are also brought to set

$$\mathrm{F}^*(\mathrm{t},\xi(\mathrm{t,x})) = \mathrm{F}_s^*(\mathrm{t},\xi(\mathrm{t,x})) + \mathrm{F}_z^*(\,\mathrm{t},\xi(\mathrm{t,x})) \tag{12}$$

This means that $\mathrm{F}^*(\mathrm{t},\xi(\mathrm{t,x}))$ is given, in its turn, by the sum of slow-varying and fast-varying components :

$$\mathrm{F}_s^*(\mathrm{t},\xi(\mathrm{t,x})) = <\mathrm{F}^*(\mathrm{t},\xi(\mathrm{t,x}))> |_{\mathrm{x}} \tag{13}$$

$$<\mathrm{F}_z^*(\mathrm{t},\xi(\mathrm{t,x}))> |_{\mathrm{x}} = 0 \tag{14}$$

Now our final ansatz will be as follows:

$$\mathrm{F}^*(\mathrm{t},\xi(\mathrm{t,x^*})) = \mathrm{F}_s(\mathrm{t},\xi(\mathrm{t,x})) + \mathrm{F}_z(\mathrm{t},\xi(\mathrm{t,x})) \tag{15}$$

Here $\mathrm{F}_s(\mathrm{t},\xi(\mathrm{t,x}))$ and $\mathrm{F}_z(\mathrm{t},\xi(\mathrm{t,x}))$ are the effective slow and fast parts of the forcing, respectively. Using equation (10) we find

$$\mathrm{F}^*(\mathrm{t},\xi(\mathrm{t,x^*})) = \mathrm{F}_s^*(\mathrm{t},\xi(\mathrm{t,x})) + \mathrm{F}_z^*(\mathrm{t},\xi(\mathrm{t,x})) + \mathrm{x}_z \frac{\partial}{\partial \mathrm{x}} \mathrm{F}_z^*(\mathrm{t},\xi(\mathrm{t,x})) +$$

$$+\mathrm{x}_z^2 \frac{\partial}{\partial \mathrm{x}} \mathrm{R}_{z2}(\mathrm{t,x,x}_z) \tag{16}$$

$$\mathrm{F}_z(\mathrm{t},\xi(\mathrm{t,x})) = \mathrm{F}_z^*(\mathrm{t},\xi(\mathrm{t,x^*})) - <\mathrm{F}_z^*(\mathrm{t},\xi(\mathrm{t,x^*}))> |_{\mathrm{x}} \tag{17}$$

$$\mathrm{F}_s(\mathrm{t},\xi(\mathrm{t,x})) = <\mathrm{F}^*(\mathrm{t},\xi(\mathrm{t,x^*}))> |_{\mathrm{x}} \; = \mathrm{F}_s^*(\mathrm{t},\xi(\mathrm{t,x})) +$$

$$+<\mathrm{x}_z \frac{\partial}{\partial \mathrm{x}} \left(\mathrm{F}_z^*(\mathrm{t},\xi(\mathrm{t,x})) + \mathrm{x_z} \mathrm{R}_{z2}(\mathrm{t,x,x}_z)\right) > |_{\mathrm{x}} \tag{18}$$

$$\mathrm{F}_s(\mathrm{t},\xi(\mathrm{t,x})) = \mathrm{F}_s^*(\mathrm{t},\xi(\mathrm{t,x})) + <\mathrm{F}_z^*(\mathrm{t},\xi(\mathrm{t,x^*}))> |_{\mathrm{x}} \tag{19}$$

These last equations give the effective forcing slow-part. As concerns the fast part, we consider the following. The external fast forcing may generally display a spectrum of frequencies. Within this spectrum, we are first interested to a fundamental pulsation ω_0 able to resonate with a oscillator fundamental frequency, and to its harmonics. By assumption, we will take

$$\frac{d\xi(\mathrm{t,x(t)})}{dt} << \omega_0(\mathrm{x})\xi(\mathrm{t,x(t)}) \tag{20}$$

As is seen in this equation, we let the pulsation ω_0 depend on the coordinate x so that it appears as a "local" characteristic pulsation. Now we see that the fast forcing $\mathrm{F}_z(\mathrm{t},\xi(\mathrm{t,x}))$ is constituted by the fundamental frequency and by the spectrum of its harmonics :

$$\mathrm{F}_z(\mathrm{t},\xi(\mathrm{t,x})) = \sum_{n=1}^{\infty} \mathrm{F}^*_{\varphi_n}(\mathrm{n}\omega_0,\xi(\mathrm{t,x}))\sin\left(\mathrm{n}\omega_0\mathrm{t}+\varphi_n(\mathrm{x})\right) \tag{21}$$

Here the $\varphi_n(\mathrm{x})$ are appropriate phase functions. We have also :

$$\mathrm{F}^*_{\varphi_n}(\mathrm{n}\omega_0,\xi(\mathrm{t,x})) = \frac{\omega_0}{\pi}\int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} \mathrm{F}^*(\mathrm{t},\xi(\mathrm{t,x^*}))\sin\left(\mathrm{n}\omega_0\mathrm{t}+\varphi_n\right)\mathrm{dt} =$$

$$= \frac{\omega_0}{\pi}\int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} \left\{\mathrm{F}^*_z(\mathrm{t},\xi(\mathrm{t,x}))+\mathrm{x}_z\frac{\partial}{\partial \mathrm{x}}\left(\mathrm{F}^*_z(\mathrm{t},\xi(\mathrm{t,x}))+\mathrm{x}_z\mathrm{R}_{z2}(\mathrm{t,x,x}_z)\right)\right\}\sin\left(\mathrm{n}\omega_0\mathrm{t}+\varphi_n\right)\mathrm{dt} \tag{22}$$

Then the comprehensive force $\mathrm{F}^*(\mathrm{t},\xi(\mathrm{t,x^*}))$ can be re-normalized to the action of a fast oscillating force $\mathrm{F}^{\omega_0}_z(\mathrm{t},\xi(\mathrm{t,x}))$ ("resonant component") at the pulsation ω_0:

$$\mathrm{F}^{\omega_0}_z(\mathrm{t},\xi(\mathrm{t,x})) = \mathrm{F}^*_{\varphi_1}(\omega_0(\mathrm{x}),\xi(\mathrm{t,x}))\sin\left(\omega_0\mathrm{t}+\varphi_1\right) \tag{23}$$

plus the action of a fast oscillating force $\sum_{n>1}\mathrm{R}^{n\omega_0}_z(\mathrm{t},\xi(\mathrm{t,x}))$, containing ω_0 multiples (off resonance), due to the rest of $\mathrm{F}^*(\mathrm{t},\xi(\mathrm{t,x^*}))$ x_z-series development :

$$\sum_{n>1}\mathrm{R}^{n\omega_0}_z(\mathrm{t},\xi(\mathrm{t,x})) = \sum_{n=2}^{\infty}\mathrm{F}^*_{\varphi_n}(\mathrm{n}\omega_0,\xi(\mathrm{t,x}))\sin\left(\mathrm{n}\omega_0\mathrm{t}+\varphi_n\right) \tag{24}$$

$$\mathrm{F}_z(\mathrm{t},\xi(\mathrm{t,x})) = \mathrm{F}^{\omega_0}_z(\mathrm{t},\xi(\mathrm{t,x})) + \sum_{n>1}\mathrm{R}^{n\omega_0}_z(\mathrm{t},\xi(\mathrm{t,x})) \tag{25}$$

plus the action of the effective slow-varying force $F_s(t,\xi(t,x))$:

$$F_s(t,\xi(t,x)) = F_s^*(t,\xi(t,x)) + < x_z \frac{\partial}{\partial x} (F_z^*(t,\xi(t,x)) + x_z R_{z2}(t,x,x_z)) > |_x \tag{26}$$

Conclusively, the external force effective on the oscillator can be described by the three distinguished parts : a slow action F_s, a resonant action $F_z^{\omega_0}$ and an off-resonance action $\sum_{n>1} R_z^{n\omega_0}$ depending on the non-linearities of the oscillator behavior.

By the following analyses in this paper, we will be brought to expect that even sub-harmonic terms (with respect to the pulsation ω_0) should be included into the forcing constitutive expressions. It is however not necessary to include them already in the present context. The treatment we are giving here is indeed just a reference framework, to be improved by the means of subsequent considerations.

In a next section we will give elements for the solution of the motion equation (1) submitted to the fluctuation field as described in this section. Before introducing this topic, however, we believe convenient to provide the physical interpretation we give to the distinguished fast and slow parts of the particle motion. This can be found in the next section.

3 Physical interpretation and analytical developments

3.1 The Heisenberg's principle in a "classical interpretation"

In order to attempt a description of quantum phenomena within the present context, as a key point displaying our thinking, we note that the maximum absolute values $|x_{z0}|$, $|v_{z0}|$ assumed by the quantities x_z, v_z lend themselves to be interpreted as indices of the particle position and velocity incertitudes. The slow motion components x,v are instead the particle local oscillation-center position and velocity respectively . We can introduce, more precisely, the following quantities (SF≡ single frequency spectrum) :

$$\Delta x = \sqrt{< (2x_z)^2 > |_x} \rightarrow|_{SF} \approx \sqrt{2} \mid x_{z0} \mid \tag{27}$$

$$\Delta p = m\sqrt{< (2v_z)^2 > |_x} \rightarrow|_{SF} \approx \sqrt{2} m \mid v_{z0} \mid \tag{28}$$

$$\Delta \mathrm{x} \Delta \mathrm{p} = \mathrm{m}\sqrt{<(2\mathrm{x}_z)^2>|_{\mathrm{x}} <(2\mathrm{v}_z)^2>|_{\mathrm{x}}} \rightarrow|_{SF} \approx$$

$$\approx 2\mathrm{m} \mid \mathrm{x}_{z0}\mathrm{v}_{z0} \mid = 2\mathrm{m}\omega \mathrm{x}_{z0}^2 \tag{29}$$

$$\mathrm{A} \equiv \Delta \mathrm{x} \Delta \mathrm{p} \gtrsim \mathrm{h} \tag{30}$$

Equations (27)÷(30) clearly show our philosophy here - it consists in interpreting Heisenberg's principle as a constraint effective on the fast motion components, but not as an a priori epistemological limit. The quantities $\mathrm{x}_z(\mathrm{t})$ and $\mathrm{v}_z(\mathrm{t}) = d\mathrm{x}_z(\mathrm{t})/dt$ are indeed in our frame classical, deterministic time-dependent variables. They can be considered as the position and velocity co-ordinates pertaining to HDF. The HDF physical behavior therefore can be first (but not always, see later) identified with a "zitterbewegung" oscillation excited by the quantum vacuum action, superimposing to the particle slow motion components, and finally resulting in the quantum incertitudes affecting the motion. The quantities x_z,v_z will be called the quantum displacements. The slow motion components will instead be referred to as the classical degree of freedom.

In orthodox quantum mechanics, the incertitudes product $\Delta \mathrm{x} \Delta \mathrm{p}$ is defined $\geq \hbar/2$. In the present framework, instead, we have taken a representative value for the minimal action equal to h. This is not only a matter of quantities definition in the sense that we have chosen non-minimal coefficients. It is clear indeed that, if we want these variances defined equal to the quantum mechanical ones, we are able to re-normalize the equations all our paper throughout, keeping our formal results essentially unperturbed - but the relevant fact is that our incertitudes $\Delta \mathrm{x}$ and $\Delta \mathrm{p}$ are different physical quantities from the standard quantum mechanical ones. They describe the particle fluctuation motion around the oscillation center. The quantum mechanical ones originate instead from (root-mean) ensemble averages, taken by means of the appropriate wave-functions, of squared space and momentum co-ordinates. The interpretation we give in our work to such quantities is that they refer to the classical degree of freedom statistical behavior. We support indeed the hypothesis that a quantum mechanical density is the space distribution of the particles oscillation centers, to which we have attached space and momentum co-ordinates x and v.

To be definite, we called A the quantity $\Delta x \Delta p$. A very simple remark is that when the action A$\rightarrow 0$, then the purely classical case is attained and the HDF motion extinguishes.

In the previous paper I we introduced the quantities Δx and Δp in the sense of thermodynamic parameters. In the present frame, they are instead single-particle mechanical parameters, but no confusion may arise. These parameters will not change their names when taking their statistical appearances. In this context, it is clear that equations (27) and (28) represent time-averaged expressions, featuring the incertitudes Δx and Δp as (twice) the time-variances assumed by the quantum displacements. In the same equations, it is indicated by the symbol $\rightarrow|_{SF}$that, in the limiting case when a sharp frequency spectrum (in practice a single frequency, SF) is exhibited by the zitterbewegung oscillation, the assumed definitions converge into simple expressions proportional to the absolute values of the maximum oscillation amplitudes pertaining to the variables x_z,v_z respectively.

Equations (29)$\div$(30) show that in a single frequency limit the Heisenberg principle takes the form of a parametric constraint on the quantum displacements pulsation ω and the oscillation amplitude $|\, x_{z0} \,|$.

As is quite obvious and will be better seen later on, in order to distinguish between a classical part of the motion x(t) and a "quantum" perturbation x_z(t) we will not limit ourselves to the assumptions that the former is slow-varying and the second fast-varying. We intend to drop off in the following these assumptions, but the equations (27)$\div$(30) - as well as the basic interpretation here introduced - will essentially be maintained. We believe useful, however, to make first clear the fundamentals of our analysis starting with the "zitterbewegung" model.

Moreover, it is clear that the Heisenberg principle "interpretation" here expounded is, by practical constraints in this paper, limited to the case of space-momentum conjugate variables - but it can be generalized to other couples.

It is also obvious that this proposed interpretation of the incertitude relations must be considered, at present, only as some kind of computative simulation, useful to the sake of an attempt; limited to the range of this investigation, and far from being in any way "demonstrated" even within the present framework. We believe that interest in new interpretative schemes may only arise as a consequence of the capability they display to produce positive results. On the other hand, it is clear that even a partial survival of our concept (as a quasi-classical effect) could be usefully accommodated within our knowledge of physical events. Amongst our purposes here therefore, there will be showing that some degree of consistency is displayed by our statements when tested against the deployment of a few aspects of the physical reality as described at present by the quantum mechanical theory .

3.2 A development for the forced oscillator equation

In order to obtain solutions of equation (1), we give here some developments. We have first of all

$$\frac{\mathrm{d}\Phi(\mathrm{x}^*)}{\mathrm{dx}^*} \equiv \Phi^{'}(\mathrm{x}^*) \tag{31}$$

$$\Phi^{'}(\mathrm{x}^*) \equiv \Phi^{'}(\mathrm{x}{+}\mathrm{x}_z) = \Phi^{'}(\mathrm{x}){+}\mathrm{x}_z\Phi^{''}(\mathrm{x}) + \sum_{n=2}^{\infty} \frac{\mathrm{x}_z^n \Phi^{(n+1)}(\mathrm{x})}{n!} \tag{32}$$

$$\sum_{n=2}^{\infty} \frac{\mathrm{x}_z^n \Phi^{(n+1)}(\mathrm{x})}{n!} = \Phi'_{an} \tag{33}$$

Here Φ'_{an} is the rest of the $\Phi^{'}(\mathrm{x})$ series development. This term is non-linear in x_z and is due to the classical potential Φ possibly being anharmonic. It will be found constituted by a slow-varying part $\Phi^{'}_{s}$, a resonant part at the ω_0 frequency $\Phi^{'\omega_0}$, and an off-resonance part $\sum_{n>1} \Phi^{'n\omega_0}$:

$$\sum_{n=2}^{\infty} \frac{\mathrm{x}_z^n \Phi^{(n+1)}(\mathrm{x})}{n!} = \Phi'_{an} = \Phi^{'}_{s}{+}\Phi^{'\omega_0}{+}\sum_{n>1} \Phi^{'n\omega_0} \tag{34}$$

We have:

$$\Phi^{'}_{s} = <\sum_{n=2}^{\infty} \frac{\mathrm{x}_z^n \Phi^{(n+1)}(\mathrm{x})}{n!}> |_{\mathrm{x}} \tag{35}$$

$$\Phi^{'n\omega_0} = \left\{ \frac{\omega_0}{\pi} \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} \sum_{m=2}^{\infty} \frac{\mathrm{x}_z^m \Phi^{(m+1)}(\mathrm{x})}{m!} \sin\left(\mathrm{n}\omega_0\mathrm{t}{+}\psi_n\right) \mathrm{dt} \right\} \sin\left(\mathrm{n}\omega_0\mathrm{t}{+}\psi_n\right) \tag{36}$$

Using the previous equations, the motion equation (1) writes

$$\mathrm{F}^*(\mathrm{t},\xi(\mathrm{t},\mathrm{x}^*)) - \alpha^*(\mathrm{v}+\mathrm{v}_z) - \Phi^{'}(\mathrm{x}) - \mathrm{x}_z\Phi^{''}(\mathrm{x}) - \Phi'_{an} = \mathrm{ma}+\mathrm{ma}_z \tag{37}$$

Taking into account equations (9)÷(19) we find that this last equation can be split into two parts:

$$\mathrm{F}_z(\mathrm{t},\xi(\mathrm{t},\mathrm{x})) - \beta(\mathrm{x})\sum_{n=1}\Phi^{'n\omega_0} - \mathrm{x}_z\left(\mathrm{K}^*(\mathrm{x})+\Phi^{''}(\mathrm{x})\right) - (\alpha^* + \gamma(\mathrm{x}))\,\mathrm{v}_z = \mathrm{ma}_z \tag{38}$$

$$\mathrm{F}(\mathrm{x},\mathrm{t})+\mathrm{F}_s(\mathrm{t},\xi(\mathrm{t},\mathrm{x})) - \Phi^{'}(\mathrm{x}) - \Phi^{'}_s - \alpha^*\mathrm{v} = \mathrm{ma} \tag{39}$$

$$\mathrm{F}(\mathrm{x},\mathrm{t}) = (\beta(\mathrm{x}) - 1)\sum_{n=1}\Phi^{'n\omega_0} + \mathrm{x}_z\mathrm{K}^*(\mathrm{x}) + \gamma(\mathrm{x})\mathrm{v}_z \tag{40}$$

In these equations, we introduce the functions $\beta(\mathrm{x})$, $\mathrm{K}^*(\mathrm{x})$, $\gamma(\mathrm{x})$, $\mathrm{F}(\mathrm{x,t})$ which are not all to be determined here, but reveal useful ansatzs for purposes to be discussed next. Here we have to discuss equation (39) first. This one describes the behavior associated with the slow-varying co-ordinate x(t), so that we recognize that

$$\mathrm{F}(\mathrm{x},\mathrm{t}) = 0 \tag{41}$$

By equation (40) it is clear indeed that we named F(x,t) a combination of fast-varying functions. Now using equations (40), (41) and (38) we find

$$\mathrm{F}_z^{\omega_0}(\mathrm{t},\xi(\mathrm{t},\mathrm{x})) + \sum_{n>1}\mathrm{R}_z^{n\omega_0}(\mathrm{t},\xi(\mathrm{t},\mathrm{x})) - \mathrm{x}_z\mathrm{K}(\mathrm{x}) - \alpha(\mathrm{x})\mathrm{v}_z = \mathrm{ma}_z \tag{42}$$

To write this equation, we took the following ansatzs:

$$\mathrm{K}(\mathrm{x}) = \mathrm{K}^*_\beta(\mathrm{x})+\Phi^{''}(\mathrm{x}) \tag{43}$$

$$\mathrm{K}^*_\beta(\mathrm{x}) = \frac{\mathrm{K}^*(\mathrm{x})}{1-\beta(\mathrm{x})} \tag{44}$$

$$\alpha(\mathrm{x}) = \alpha^* + \tilde{\alpha}(\mathrm{x}) \tag{45}$$

$$\tilde{\alpha}(\mathrm{x}) = \frac{\gamma(\mathrm{x})}{1-\beta(\mathrm{x})} \tag{46}$$

Here we understand that our choice in the splitting (38)÷(39) has made the variable x_z to depend directly on the external fast forcing, which is tantamount to say that the anharmonic term Φ'_{an} is taken into account only for his contribution to the elasticity and friction terms which will finally result effective in equation (42). This is better understood when considering that the result shown in equation (42) is such that the HDF oscillation - i.e. the $\mathrm{x}_z(\mathrm{t})$ time-law - simply turns out as a harmonic oscillation with effective elastic constant given in (43), forced by F_z. On the other hand, we advance here that - within the same assumption (41) plus the following equation (54) - the resulting equation (53) for the slow-varying part x(t) of the oscillator co-ordinate will be found independent on the classical potential anharmonicity Φ'_{an}. This property looks interesting because it is consistent with the fact that, in quantum mechanics, the Bohm potential in the Schr*ö*dinger equation is indeed independent of such very peculiar term.

As a further comment to equations (42) and (53) (to be introduced next), however, we remark that the overall framework we are setting up here is not to be interpreted as the display of absolute arguments, but rather as the deployment of what we found the attractive, strategical, sometimes conjectural, ones. These are developed in the attempt to provide key indications about the HDF behavior, so that - once we have enough understood about that within some limited framework - then we can promote more rigorous analysis. Therefore, some of the assumptions as the ones we are considering here could also be dropped off in favor of a more general model. For instance, it could be seen that setting the function F(x,t) in equation (40) different from zero would bring our equations to interesting correspondences with both the thermodynamic non-stationary FEOM model and time-dependent quantum-mechanical matter-wave equation.

Equation (42) splits into a number of equations governing the Fourier components $\mathrm{x}_\mathrm{z}^{(n)}$ of the displacement x_z :

$$\mathrm{F}_z^{\omega_0}(\mathrm{t},\xi(\mathrm{t},\mathrm{x})) - \mathrm{x}_z^{(1)}\left(\mathrm{K}_\beta^*(\mathrm{x})+\Phi''(\mathrm{x})\right) - \alpha\mathrm{v}_z^{(1)} = \mathrm{ma}_z^{(1)} \tag{47}$$

$$\mathrm{R}_z^{n\omega_0}(\mathrm{t},\xi(\mathrm{t},\mathrm{x})) - \mathrm{x}_z^{(n)}\left(\mathrm{K}_\beta^*(\mathrm{x})+\Phi''(\mathrm{x})\right) - \alpha\mathrm{v}_z^{(n)} = \mathrm{ma}_z^{(n)} \tag{48}$$

Formally, the solution to equation (42) can be written therefore

$$\mathrm{x}_z(\mathrm{t}) = \sum_n \mathrm{x}_z^{(n)}(\mathrm{n}\omega_0,\varphi_n) \sin(\mathrm{n}\omega_0 t + \chi_n) = \sum_n \mathrm{x}_z^{(n)}(\mathrm{t}) \tag{49}$$

$$\mathrm{v}_z(\mathrm{t}) = \sum_n \mathrm{n}\omega_0 \mathrm{x}_z^{(n)}(\mathrm{n}\omega_0,\varphi_n) \cos(\mathrm{n}\omega_0 \mathrm{t} + \chi_n) = \sum_n \mathrm{v}_z^{(n)}(\mathrm{t}) \tag{50}$$

$$\mathrm{a}_z(\mathrm{t}) = -\sum_n (\mathrm{n}\omega_0)^2\, \mathrm{x}_z^{(n)}(\mathrm{n}\omega_0,\varphi_n) \sin(\mathrm{n}\omega_0 \mathrm{t} + \chi_n) = \sum_n \mathrm{a}_z^{(n)}(\mathrm{t}) \tag{51}$$

On the other hand, equation (39) can be reduced as follows:

$$\mathrm{F}_s^*(\mathrm{t},\xi(\mathrm{t},\mathrm{x})) + \langle \mathrm{F}_z^*(\mathrm{t},\xi(\mathrm{t},\mathrm{x}^*))\rangle |_{\mathrm{x}} - \Phi'(\mathrm{x}) - \Phi_s' - \alpha^*\mathrm{v} = \mathrm{ma} \tag{52}$$

$$\langle \mathrm{x}_z \frac{\partial}{\partial \mathrm{x}} (\mathrm{F}_z^*(\mathrm{t},\xi(\mathrm{t},\mathrm{x})) + \mathrm{x}_z \mathrm{R}_{z2}(\mathrm{t},\mathrm{x},\mathrm{x}_z)) \rangle |_{\mathrm{x}} - \Phi'(\mathrm{x}) = \mathrm{ma} \tag{53}$$

$$\mathrm{F}_s^*(\mathrm{t},\xi(\mathrm{t},\mathrm{x})) = \alpha^*\mathrm{v} + \Phi_s' \tag{54}$$

As remarked already, these equations are consistent with the requirement we have that the net HDF action displayed in the final equation (55) is stationary (F(x,t) = 0) and not explicitly dependent on the classical potential form via the anharmonic term Φ'_{an}. By these means, the energy theorem corresponding to equation (53) can be written indeed

$$\frac{1}{2}\mathrm{mv}^2 + \Phi(\mathrm{x}) + \Phi_{\mathrm{HDF}}(\mathrm{x},\xi(\mathrm{x})) = \mathrm{E} \tag{55}$$

$$\Phi_{\mathrm{HDF}}(\mathrm{x},\xi(\mathrm{x})) = -\int \langle \mathrm{x}_z \frac{\partial}{\partial \mathrm{x}} (\mathrm{F}_z^*(\mathrm{t},\xi(\mathrm{t},\mathrm{x})) + \mathrm{x}_z \mathrm{R}_{z2}(\mathrm{t},\mathrm{x},\mathrm{x}_z)) \rangle |_{\mathrm{x}}\, \mathrm{dx} \tag{56}$$

These equations show our interpretation for the HDF effect. We identify the classical particle motion (as remarked, this is called the classical degree of freedom) into the "slow" motion component and deploy the classical energy theorem form using the variables x,v into the appropriate kinetic and potential terms pertaining to it. Then the assumed existence of HDF brings to add to the classical energy terms a potential energy $\Phi_{\mathrm{HDF}}(\mathrm{x},\xi(\mathrm{x}))$. Conclusively, this comes from the external, time-averaged quantum field exciting the fast-motion component variable x_z, and resulting into a perturbation of the classical motion described by x(t).

To expose clearly our interpretation of physical events and investigation strategy, we add here that - in next developments - to the same force F* constitutive of the fluctuation field we could add perturbation forces coming from further external interactions or, most interesting, from measurements. Then within our frame we might have knowledge of the quantum displacements behavior and correlated influence on the classical motion part when the oscillator is submitted, precisely, to measurements or experimental apparatuses. This task would obviously be, even partially, outside the possibilities of this paper.

The attempt we make in these papers is finding an expression for Φ_{HDF} able to explain some quantum effects at least, and bringing to the Schrödinger equation when statistical averages are performed to describe a particles ensemble behavior.

3.3 The energy theorem obtained by time-averaging

For the sake of completeness, we display here the alternative procedure bringing to the energy theorem by time-averaging after integration. This is also useful in order to better appreciate the assumptions we made on the different quantities behaviors. In order to calculate the energies carried by each term in equation (2) we note the following:

$$<\mathrm{v}^{*2}(\mathrm{t})>|_{\mathrm{x}} \;=<[\mathrm{v}(\mathrm{t})+\mathrm{v}_z(\mathrm{t})]^2>|_{\mathrm{x}} \quad =$$

$$=<\mathrm{v}^2(\mathrm{t})>|_{\mathrm{x}}+<\mathrm{v}_z^2(\mathrm{t})>|_{\mathrm{x}}+2<\mathrm{v}(\mathrm{t})\,\mathrm{v}_z(\mathrm{t})>|_{\mathrm{x}} \tag{57}$$

$$<\mathrm{v}^2(\mathrm{t})>|_{\mathrm{x}}=\mathrm{v}^2(\mathrm{x}) \tag{58}$$

$$<(\mathrm{v}(\mathrm{t})\,\mathrm{v}_z(\mathrm{t}))>|_{\mathrm{x}}=\;<\mathrm{v}_z(\mathrm{t})>|_{\mathrm{x}}\,\mathrm{v}(\mathrm{x})=0 \tag{59}$$

$$<\frac{1}{2}\mathrm{m}\mathrm{v}^{*2}(\mathrm{t})>|_{\mathrm{x}}=\frac{1}{2}\,\mathrm{m}\mathrm{v}^2+\frac{1}{2}\mathrm{m}<\mathrm{v}_z^2(t)>|_{\mathrm{x}} \tag{60}$$

$$<\Phi(\mathrm{x}^*(\mathrm{t}))> |_{\mathrm{x}} = \Phi(\mathrm{x}) + \frac{1}{2} <\mathrm{x}_z^2 \Phi''(\mathrm{x})> |_{\mathrm{x}} + \sum_{n=2}^{\infty} < \frac{\mathrm{x}_z^{n+1} \Phi^{(n+1)}(\mathrm{x})}{(\mathrm{n}+1)!} > |_{\mathrm{x}} \tag{61}$$

$$<\Phi(\mathrm{x}^*(\mathrm{t}))> |_{\mathrm{x}} = \Phi(\mathrm{x}) + \frac{1}{2} <\mathrm{x}_z^2 \Phi''(\mathrm{x})> |_{\mathrm{x}} + < \int \Phi'_{an} \mathrm{dx}_z > |_{\mathrm{x}} \tag{62}$$

As concerns the force $\mathrm{F}^*(\mathrm{t},\xi(\mathrm{t},\mathrm{x}^*))$ we calculate the following (time-averaged) work functions :

$$< \int \mathrm{F}^*(\mathrm{t},\xi(\mathrm{t},\mathrm{x}^*))\mathrm{v}^*(\mathrm{t})\mathrm{dt} > |_{\mathrm{x}} =$$

$$= \int \mathrm{F}_s(\mathrm{t},\xi(\mathrm{t},\mathrm{x}))\mathrm{v}(\mathrm{t})\mathrm{dt} + < \int \mathrm{F}_s(\mathrm{t},\xi(\mathrm{t},\mathrm{x}))\mathrm{v}_z(\mathrm{t})\mathrm{dt} > |_{\mathrm{x}} +$$

$$+ < \int \mathrm{F}_z(\mathrm{t},\xi(\mathrm{t},\mathrm{x}))\mathrm{v}(\mathrm{t})\mathrm{dt} > |_{\mathrm{x}} + < \int \mathrm{F}_z(\mathrm{t},\xi(\mathrm{t},\mathrm{x}))\mathrm{v}_z(\mathrm{t})\mathrm{dt} > |_{\mathrm{x}} \tag{63}$$

$$< \int \mathrm{F}_z(\mathrm{t},\xi(\mathrm{t},\mathrm{x}))\mathrm{v}(\mathrm{t})\mathrm{dt} > |_{\mathrm{x}} = \mathrm{v}(\mathrm{t}) < \int \mathrm{F}_z(\mathrm{t},\xi(\mathrm{t},\mathrm{x}))\mathrm{dt} > |_{\mathrm{x}} = 0 \tag{64}$$

$$< \int \mathrm{F}_s(\mathrm{t},\xi(\mathrm{t},\mathrm{x}))\mathrm{v}_z(\mathrm{t})\mathrm{dt} > |_{\mathrm{x}} = \mathrm{F}_s(\mathrm{t},\xi(\mathrm{t},\mathrm{x})) < \int \mathrm{v}_z(\mathrm{t})\mathrm{dt} > |_{\mathrm{x}} = 0 \tag{65}$$

We also have

$$< \int \Phi'_s \mathrm{dx}_z > |_{\mathrm{x}} = \Phi'_s < \int \mathrm{dx}_z > |_{\mathrm{x}} = 0 \tag{66}$$

Then equation (2) becomes

$$\begin{aligned} &\frac{1}{2}\,\mathrm{mv}^2 + \frac{1}{2}\mathrm{m} < \mathrm{v}_z^2(\mathrm{t}) > |_{\mathrm{x}} + \Phi(\mathrm{x}) + \frac{1}{2} < \mathrm{x}_z^2 > \Phi''(\mathrm{x}) - \int \mathrm{F}_s(\mathrm{t},\xi(\mathrm{t},\mathrm{x}))\mathrm{v}(\mathrm{t})\mathrm{dt} \\ &\quad + < \int \left[\Phi'^{\omega_0} + \sum_{n>1} \Phi'^{n\omega_0} \right] \mathrm{dx}_z > |_{\mathrm{x}} - < \int \mathrm{F}_z(\mathrm{t},\xi(\mathrm{t},\mathrm{x}))\mathrm{v}_z(\mathrm{t})\mathrm{dt} > |_{\mathrm{x}} \\ &\quad + < \int \alpha^*\, \mathrm{v}^2 \mathrm{dt} + \alpha^* \int \mathrm{v}_z^2(\mathrm{t})\,\mathrm{dt} > |_{\mathrm{x}} = \mathrm{E} \end{aligned} \tag{67}$$

Now we integrate the following equation in the variable x_z, and perform the time-average:

$$F_z(t,\xi(t,x)) - x_z\left(K^*_\beta(x)+\Phi^{''}(x)\right) - \alpha(x)v_z = ma_z \tag{68}$$

$$< \int \left\{F_z(t,\xi(t,x)) - x_z\left(K^*_\beta(x)+\Phi^{''}(x)\right) - \left(\alpha^* + \tilde{\alpha}(x)\right)v_z\right\} dx_z > |_x = \\ = \frac{1}{2}m<v_z^2> |_x - \varepsilon(x) \tag{69}$$

Here $\varepsilon(x)$ is inserted because the fast-motion energy constant must be interpreted as a slowly variable function, i.e. a function of the slow component x. Then we find :

$$\frac{1}{2}\,mv^2 + \Phi(x) + < \int \left[\Phi^{'\omega_0} + \sum_{n>1} \Phi^{'n\omega_0} - x_z K^*_\beta(x)\right] dx_z > |_x + \\ - \int F_s(t,\xi(t,x))v(t)dt + \alpha^* \int v^2 dt - <\tilde{\alpha}(x)\int v_z^2\,(t)\,dt> |_x + \varepsilon(x) = E \tag{70}$$

By the sake of congruence with previous analysis, we have to take :

$$\varepsilon(x) = \Phi_s - \langle \int \left[\Phi^{'\omega_0} + \sum_{n>1} \Phi^{'n\omega_0} - x_z K^*_\beta(x)\right] dx_z \rangle |_x + \langle \tilde{\alpha}(x) \int v_z^2\,(t)\,dt \rangle |_x \tag{71}$$

so that by performing the derivatives we find again (use equations (54), (18)):

$$<x_z \frac{\partial}{\partial x}\left(F_z^*(t,\xi(t,x)) + x_z R_{z2}(t,x,x_z)\right) > |_x - \Phi^{'}(x) = ma \tag{72}$$

This equation is identical to equation (53) so that this last is also found submitted to the conditions expressed by equations (59), (64), (66), (71). These last can be appreciated in their sense by the reader itself.

4 Solution procedures

4.1 Solution of the linearized fast oscillation equation (resonant case)

Let us now consider the following simple model, describing the interaction of the particle with the fast force component. We will make use of the forced (harmonic) oscillator classical theory. Assume first that only linear terms in x_z are taken in our equations (small x_z, implying $x_z \equiv x_z^{(1)}$) so that the fast force has a single frequency ω_0 displaying two components in turn, one in phase (p) with the oscillator space co-ordinate and the other one out of phase (q).We have :

$$F_z^{\omega_0}(t,\xi(t,x)) = F^*_{pz}(t,x)) + F^*_{qz}(t,x)) \tag{73}$$

$$F^*_{pz}(t,x) = \bar{F}(x)\sin(\omega_0(x)t - \varphi(x))\cos\varphi(x) \tag{74}$$

$$F^*_{qz}(t,x) = \bar{F}(x)\cos(\omega_0(x)t - \varphi(x))\sin\varphi(x) \tag{75}$$

The quantity $\bar{F}(x)$ is the relevant excitation intensity and φ is an appropriate phase lag between the oscillator co-ordinate and the forcing. We have therefore:

$$F^*_{pz}(t,x) = ma_z + x_z\left(K^*_\beta(x) + \Phi''(x)\right) \tag{76}$$

$$F^*_{qz}(t,x) - \alpha v_z = 0 \tag{77}$$

$$K^*_\beta(x) + \Phi''(x) = m\left(\omega^2(x,\xi) + \alpha^2/4m^2\right) = K(x) \tag{78}$$

Here ω is the natural pulsation corresponding to the oscillation equation (47). Now we have :

$$ma_z = -m\omega_0^2(x)x_z \tag{79}$$

$$x_z = x_{z0}(x)\sin(\omega_0(x)t - \varphi(x)) \tag{80}$$

$$x_z = \frac{F^*_{pz}(t,x))}{K^*_\beta(x) + \Phi''(x) - m\omega_0^2(x)} = \frac{\bar{F}(x)\sin(\omega_0(x)t - \varphi(x))\cos\varphi(x)}{K(x) - K_0(x)} \tag{81}$$

Here we use the standard stationary solution for the oscillation amplitude. By the resonance condition we have

$$-\mathrm{m}\omega_0^2(\mathrm{x})+\mathrm{K}^*_\beta(\mathrm{x})+\Phi^{''}(\mathrm{x}) = \mathrm{K}(\mathrm{x}) - \mathrm{K}_0(\mathrm{x}) = \alpha^2(\mathrm{x})/2\mathrm{m} \tag{82}$$

$$< \mathrm{F}^*_{pz}(\mathrm{t,x}))^2> |_{\mathrm{x}}= <\mathrm{x}_z^2\,(\mathrm{t}) > |_{\mathrm{x}}\ \left[\alpha^2/2\mathrm{m}\right]^2 \tag{83}$$

Note that if - on the contrary assumption - there is not a resonance and $\omega_0 >> \omega$ (to preserve the fast motion hypothesis) then the result would be

$$< \mathrm{F}^*_{pz}(\mathrm{t,x}))^2> |_{\mathrm{x}}= \mathrm{K}_0^2<\mathrm{x}_z^2\,(\mathrm{t}) > |_{\mathrm{x}}= \mathrm{mK}_0<\mathrm{v}_z^2\,(\mathrm{t}) > |_{\mathrm{x}} \tag{84}$$

This expression fits the case studied by Kapitza, discussed in the quoted reference [7].

4.2 Slow motion component energy theorem (resonant case)

Now we calculate the work associated to the slow component (use equations (53), (81), (82)):

$$\int <\frac{\partial\,\mathrm{F}_z^{\omega_0}(\mathrm{t},\xi(\mathrm{t,x}))}{\partial \mathrm{x}}\mathrm{x}_z> |_{\mathrm{x}}\ \mathrm{v(t)dt} = \int <\frac{\partial\,\mathrm{F}^*_{pz}\,(\mathrm{t,x}))}{\partial \mathrm{x}}\mathrm{x}_z> |_{\mathrm{x}}\ \mathrm{v(t)dt} =$$

$$= \int <\frac{1}{\mathrm{K}-\mathrm{K}_0}\frac{\partial \mathrm{F}^*_{pz}}{\partial \mathrm{x}}\mathrm{F}^*_{pz}> |_{\mathrm{x}}\ \mathrm{dx} = <\frac{2\mathrm{m}}{\alpha^2}\int \frac{\partial \mathrm{F}^*_{pz}}{\partial \mathrm{x}}\mathrm{F}^*_{pz}\mathrm{dx}> |_{\mathrm{x}}=$$

$$= \frac{\mathrm{m}<\mathrm{F}^{*2}_{pz}> |_{\mathrm{x}}}{\alpha^2} = \frac{\alpha^2<\mathrm{x}_z^2\,(\mathrm{t}) > |_{\mathrm{x}}}{4\mathrm{m}} \tag{85}$$

Then we have (use equation (56)):

$$\frac{1}{2}\,\mathrm{m\,v}^2(\mathrm{x}) + \Phi(\mathrm{x}) + \Phi^{\mathrm{R}}_{\mathrm{HDF}}(\mathrm{x},\xi) = \mathrm{E} \tag{86}$$

$$\Phi^{\mathrm{R}}_{\mathrm{HDF}}(\mathrm{x},\xi) = -\frac{\alpha^2<\mathrm{x}_z^2\,(\mathrm{t}) > |_{\mathrm{x}}}{4\mathrm{m}} \tag{87}$$

The superscript R means here that the corresponding expression for Φ_{HDF} has been calculated using the resonance condition between the forcing and the oscillator. If equation(84) is assumed we find instead, with $K_0 >> K$:

$$\Phi^{\mathrm{NR}}_{\mathrm{HDF}}(x,\xi) = \frac{m}{2}\omega_0^2 \langle x_z^2 \rangle\,|_x = \frac{m}{2}\langle v_z^2 \rangle\,|_x \tag{88}$$

This expression can be called the Kapitza equation. NR stands for non-resonant.

4.3 Solution of the non-linear fast oscillation component equation (non resonant terms taken into account)

When we use the forced oscillator theory in the general case - which means taking into account equation (48) with n>1 and the non-linear term

$\langle x_z^2 \frac{\partial}{\partial x} R_{z2}(t,x,x_z)\rangle|_x$ in equation (56) - then we find the following equations :

$$x_z^{(n)} = \frac{R_{pz}^{n\omega_0}(t,\xi(t,x))}{K_\beta^*(x)+\Phi''(x) - mn^2\omega_0^2(x)} \approx -\frac{R_{pz}^{n\omega_0}(t,\xi(t,x))}{m\,(n^2-1)\,\omega_0^2(x)} \tag{89}$$

The terms $R_{pz}^{n\omega_0}(t,\xi(t,x))$ are the in-phase forcing parts of the oscillator displacement Fourier components. The contributions to equation (56) due to these terms are as follows in the next section.

4.4 Slow motion component energy theorem (general case)

Here we have

$$x_z(t) = \sum x_z^{(n)}(t) \tag{90}$$

$$x_z \frac{\partial}{\partial x}\left[F_{pz}^{\omega_0}(t,\xi(t,x)) + \sum_{n>1} R_{pz}^{n\omega_0}(t,\xi(t,x)\right] =$$

$$= x_z \frac{\partial}{\partial x} x_z K(x) + m x_z \frac{\partial}{\partial x}\left[a^{(1)} + \sum_{n=2}^{\infty} a_z^{(n)}\right] =$$

$$= \mathrm{x}_z \frac{\partial}{\partial \mathrm{x}} \left\{ \left[\mathrm{x}_z^{(1)} + \sum_{n>1} \mathrm{x}_z^{(n)} \right] \mathrm{K(x)} - \mathrm{m}\omega_0^2 \left[\mathrm{x}_z^{(1)} + \sum_{n=2}^{\infty} \mathrm{n}^2 \mathrm{x}_z^{(n)} \right] \right\} =$$

$$= \mathrm{x}_z \frac{\partial}{\partial \mathrm{x}} \left\{ \mathrm{x}_z^{(1)} \left[\mathrm{K(x)} - \mathrm{m}\omega_0^2 \right] + \left[\mathrm{K(x)} \sum_{n>1} \mathrm{x}_z^{(n)} - \mathrm{m}\omega_0^2 \sum_{n=2}^{\infty} \mathrm{n}^2 \mathrm{x}_z^{(n)} \right] \right\} =$$

$$= \mathrm{x}_z \frac{\partial}{\partial \mathrm{x}} \left\{ \mathrm{x}_z^{(1)} \frac{\alpha^2}{2\mathrm{m}} + \left[\left(\mathrm{m}\omega_0^2(\mathrm{x}) + \frac{\alpha^2}{2\mathrm{m}} \right) \sum_{n>1} \mathrm{x}_z^{(n)} - \mathrm{m}\omega_0^2 \sum_{n=2}^{\infty} \mathrm{n}^2 \mathrm{x}_z^{(n)} \right] \right\} \approx$$

$$\approx \frac{\alpha^2}{2\mathrm{m}} \mathrm{x}_z \frac{\partial}{\partial \mathrm{x}} \mathrm{x}_z - \sum_{m=1}^{\infty} \mathrm{x}_z^{(m)} \frac{\partial}{\partial \mathrm{x}} \sum_{n=2}^{\infty} \left(\mathrm{n}^2 - 1 \right) \mathrm{x}_z^{(n)} \mathrm{m}\omega_0^2 \tag{91}$$

$$\int < \mathrm{x}_z \frac{\partial}{\partial \mathrm{x}} \left[\mathrm{F}_{pz}^{\omega_0}(\mathrm{t},\xi(\mathrm{t,x})) + \sum_{n>1} \mathrm{R}_{pz}^{n\omega_0}(\mathrm{t},\xi(\mathrm{t,x}) \right] > |_{\mathrm{x}} \, \mathrm{dx} =$$

$$= \frac{\alpha^2}{4\mathrm{m}} < \mathrm{x}_z^2 > |_{\mathrm{x}} - \frac{\mathrm{m}}{2} \sum_{n=2}^{\infty} < \mathrm{n}^2 \mathrm{x}_z^{(n)2} \omega_0^2 > |_{\mathrm{x}} + \frac{\mathrm{m}}{2} \sum_{n=2}^{\infty} < \mathrm{x}_z^{(n)2} \omega_0^2 > |_{\mathrm{x}} =$$

$$= \frac{\alpha^2}{4\mathrm{m}} < \mathrm{x}_z^2 > |_{\mathrm{x}} - \frac{\mathrm{m}}{2} \sum_{n=2}^{\infty} < \mathrm{v}_z^{(n)2} > |_{x} + \frac{\mathrm{m}}{2} \omega_0^2 \sum_{n=1}^{\infty} < \mathrm{x}_z^{(n)2} > |_{\mathrm{x}} - \frac{\mathrm{m}}{2} \omega_0^2 < \mathrm{x}_z^{(1)2} > |_{\mathrm{x}} =$$

$$= \left(\frac{\alpha^2}{4\mathrm{m}} + \frac{\mathrm{m}}{2} \omega_0^2 \right) < \mathrm{x}_z^2 > |_{\mathrm{x}} - \frac{\mathrm{m}}{2} \sum_{n=1}^{\infty} < \mathrm{v}_z^{(n)2} > |_{\mathrm{x}} =$$

$$= \left(\frac{\alpha^2}{4\mathrm{m}} + \frac{\mathrm{m}}{2} \omega_0^2 \right) < \mathrm{x}_z^2 > |_{\mathrm{x}} - \frac{\mathrm{m}}{2} < \mathrm{v}_z^2 > |_{\mathrm{x}} \tag{92}$$

Then we obtain

$$\Phi_{\mathrm{HDF}}(\mathrm{x},\xi) = -\left(\frac{\alpha^2}{4\mathrm{m}} + \frac{\mathrm{m}}{2}\omega_0^2\right) <\mathrm{x}_z^2> |_x + \frac{\mathrm{m}}{2} <\mathrm{v}_z^2> |_x \tag{93}$$

This equation (and the previous equations (87),(88)) show that the stationary effect of the forcing on the oscillator is to introduce into the energy theorem an effective potential $\Phi_{\mathrm{HDF}}(\mathrm{x},\xi)$, which depends on the squared amplitudes of the position and velocity incertitudes. If only the resonant component of the forcing is effective, than $\omega_0^2 <\mathrm{x}_z^2>|_{\mathrm{x}} = <\mathrm{v}_z^2>|_{\mathrm{x}}$ and equation (87) is recovered. If, on the contrary, the effect of the resonance is small, then the term

$$-\left(\frac{\alpha^2}{4\mathrm{m}} + \frac{\mathrm{m}}{2}\omega_0^2\right) <\mathrm{x}_z^2> |_{\mathrm{x}} << \frac{\mathrm{m}}{2} <\mathrm{v}_z^2> |_{\mathrm{x}} \tag{94}$$

will be found negligible and the potential $\Phi_{\mathrm{HDF}}(\mathrm{x},\xi)$ will be given by

$$\Phi_{\mathrm{HDF}}(\mathrm{x},\xi) = \frac{\mathrm{m}}{2} <\mathrm{v}_z^2> |_{\mathrm{x}} \tag{95}$$

As remarked already, this last expression was first given, indeed, by Kapitza for the case of non-resonant coupling between the fast and slow oscillation components of the particle motion.

In all the expressions we have found for $\Phi_{\mathrm{HDF}}(\mathrm{x},\xi)$, the parameter ξ is not displayed explicitly. It will be brought to evidence in the following.

5 Discussion of the results

5.1 A conclusive expression for Φ_{HDF} and the parameter η

Equation (93) can also be written

$$\Phi_{\mathrm{HDF}}(\mathrm{x},\xi) = -\frac{\alpha^2(\eta)}{4\mathrm{m}} <\mathrm{x}_z^2> |_{\mathrm{x}} + \frac{\eta^2(\mathrm{x}) - 1}{\eta^2(\mathrm{x})} \frac{\mathrm{m}}{2} <\mathrm{v}_z^2> |_{\mathrm{x}} \tag{96}$$

This is because we set :

$$\frac{\mathrm{m}}{2} <\mathrm{v}_z^2> |_{\mathrm{x}} = \eta^2(\mathrm{x}) \frac{\mathrm{m}}{2} \omega_0^2 <\mathrm{x}_z^2> |_{\mathrm{x}} \tag{97}$$

Here $\eta(\mathrm{x})$ is a parameter defining the energy content ratio between the kinetic and potential degrees of freedom pertaining to HDF and making thrust on the oscillator. Note that in equation (96) we evidenced a $\alpha-$function form also dependent on this parameter. This is some kind of ansatz simply due to the fact that we are developing a one-parameter theory in this work, and implies no loss of generality. However, the function α itself will be left unspecified here - we will calculate it by the means of congruence in paper IV. This remark also makes clear that the parameter ξ in the Φ_{HDF} expression must be strictly correlated to η. We might have chosen some function of the parameter η itself to replace ξ or vice-versa, but we found more comfortable to keep them distinguished - so that we can give to ξ some physical dimension as will be made clear in the following paper IV.

When the out-of-resonance terms are negligible, we see that $\eta^2(\mathrm{x}) \rightarrow 1$ and equation (87) is recovered. If we take a rough but simple view, we can say that η is the dominant harmonic index value in the fluctuation field spectrum; but note that this last might be largely broadened, so that this interpretation may fail - we understand consequently that η is an effective parameter able to fit our expressions into some equivalent of a single frequency limit, and it might display even values smaller than unity. In our simple treatment, indeed, we assumed a ordinary Fourier spectrum with resonant frequency and harmonics displayed by the fluctuation field; but, in a generalized treatment, either sub-harmonics or any other independent, suitable excitation might be introduced as well. This is precisely a very interesting point in our analysis, because in case η is smaller than unity Φ_{HDF} attains negative values. This case might occur in the neighborhood of the classical motion boundaries - by this effect we might be able to explain the tunnelling phenomena. Actually, negative Φ_{HDF}-values may also be attained depending on the values of the $\alpha(\mathrm{x})$ function represented in equations (45) and (46) - but concerning this specific effect we will give some other interpretation (as a negative mass effect, see equations (106) and (107)) produced in the next section. Here we have to recall, however, that our discussion is to investigative purposes. We keep in mind that, in a tempted treatment as the present must be considered, different effects might be found mixed into complicated or inextricable expressions, so that our purpose is rather trying to identify them for further insight than insure rigour of distinctions (consider f.i. the structure of the function $f(\rho)$ in equation (24), paper IV). Papers III and IV will give us more insight into the previous points.

Now let us refer to equations (29), (30) and write

$$16\mathrm{m}^2\langle \mathrm{v}_z^2\rangle\,|_{\mathrm{x}}\ \langle \mathrm{x}_z^2\rangle\,|_{\mathrm{x}} = \mathrm{A}^2 \equiv \Delta\mathrm{x}^2\Delta\mathrm{p}^2 \gtrsim \mathrm{h}^2 \tag{98}$$

If only one harmonic (SF) is dominant in the fluctuation field spectrum this equation can be written

$$\mathrm{A}^2 = 4\mathrm{m}^2\mathrm{v}_{z0}^2\mathrm{x}_{z0}^2 = 4\mathrm{m}^2\eta^2(\mathrm{x})\omega_0^2(\mathrm{x})\mathrm{x}_{z0}^4 \tag{99}$$

Introducing the variable A into the expression of Φ_{HDF} brings us to write

$$\Phi_{\mathrm{HDF}}(\mathrm{x},\xi) = -\frac{\alpha^2(\eta)}{4\mathrm{m}}\langle \mathrm{x}_z^2\rangle\,|_{\mathrm{x}} + \frac{\eta^2(\mathrm{x})-1}{\eta^2(\mathrm{x})}\frac{\mathrm{A}^2}{32\mathrm{m}\langle \mathrm{x}_z^2\rangle\,|_{\mathrm{x}}} \tag{100}$$

In the full quantum limit we will take $\mathrm{A} = \mathrm{h}$, $\langle \mathrm{x}_z^2\rangle|_{\mathrm{x}} \equiv \Delta\mathrm{x}^2/4$ and we find

$$\Phi_{\mathrm{HDF}}(\mathrm{x},\xi) = -\frac{\alpha^2(\eta)}{16\mathrm{m}}\Delta\mathrm{x}^2 + \frac{\eta^2(\mathrm{x})-1}{\eta^2(\mathrm{x})}\frac{\mathrm{h}^2}{8\mathrm{m}\Delta\mathrm{x}^2} \tag{101}$$

5.2 The high-frequency limit and the perfect transformation

In order to have an insight into our model descriptive capability for physical effects, we have to take now some assumptions about the incertitude Δx - what we want to do is correlating this quantity to the slow velocity field v(x). To obtain such a correlation, we consider the following. In the previous paper I it was shown that, in the thermodynamic limit, a particles system may undergo a "perfect" transformation where the thermodynamic incertitude Δx behaves proportionally to the specific volume. Therefore, we can exploit known correlations between the density and the eulerian velocity field.

We can first consider that the incertitude Δx (remember here it is a time-averaged quantity calculated for a single-particle) in a deep quantum limit (small quantum numbers) is comparable to the specific volume. We will then set it in equation (101) equal to $1/\sqrt{2}$ times the inverse of the statistical density $\rho(\mathrm{x})$ (in a single frequency limit, this is consistent with taking $1/\rho(\mathrm{x}) \approx |\ 2\mathrm{x}_{z0}\ |$). Here we are estimating a single-particle parameter by the means of a comparison with the correspondent statistical ensemble average quantity. Major comment about such a kind of procedure will be found in the next paper III. We remark here that this mentioned case would just be attained when pressure is effective on the system, i.e. when this last is an ensemble of interacting particles in thermodynamic equilibrium, and Δx is the statistical expression of the incertitude. Note, yet, that although the present treatment is for a single-particle, we are in the limit of fast fluctuating forces effective on it - the case can therefore be thought homologous to a situation where some kind of thermal pressure or hard-shock collisions are present, thus providing some consistency to our reasoning. We will write:

$$\Phi_{\mathrm{HDF}}(\mathrm{x},\xi) \approx -\frac{\alpha^2(\eta)}{32\mathrm{m}}\frac{1}{\rho^2(\mathrm{x})} + \frac{\mathrm{h}^2}{4\mathrm{m}}\frac{\eta^2(x)-1}{\eta^2(x)}\rho^2(\mathrm{x}) \tag{102}$$

In this equation, however, we are going now to consider not a quantum-mechanical-like density, but rather a classical one. This is for the same reason than before - our energy theorem here is for the single particle, while in the present context we would rather look to the quantum density as to an ensemble density (the quantum density is a single-particle density only in the probabilistic sense given by the orthodox Copenhagen interpretation). The classical density is however a statistical quantity and is correlated to the Eulerian (single particle) velocity field v(x) by the continuity equation, which we will write in the form

$$\rho_c(\mathrm{x})\mathrm{v}(\mathrm{x}) = 2\nu_0 \tag{103}$$

Here ν_0 is the (constant) mass flow exhibited by a stationary ensemble of all-identical, non-interacting particles oscillating around the same center, so that is equal to the inverse of period. The factor 2 is to account for the effective density constituted by particles traveling both in the forward and backward directions of motion during the oscillation. Using equation(103) in equation(102) we get

$$\Phi_{\mathrm{HDF}}(\mathrm{x},\xi) = -\frac{\alpha^2(\eta)}{32\mathrm{m}}\frac{\mathrm{v}^2}{4\nu_0^2} + \frac{\mathrm{h}^2}{4\mathrm{m}}\frac{\eta^2(\mathrm{x})-1}{\eta^2(\mathrm{x})}\frac{4\nu_0^2}{\mathrm{v}^2} \tag{104}$$

and the energy theorem (55) can be written

$$\frac{1}{2}\,\mathrm{m}_{eff}(\eta)\mathrm{v}^2 + \Phi(\mathrm{x}) + \frac{\mathrm{h}^2}{\mathrm{m}}\frac{\eta^2(\mathrm{x})-1}{\eta^2(\mathrm{x})}\frac{\nu_0^2}{\mathrm{v}^2} = \mathrm{E} \tag{105}$$

$$m_{eff}(\eta) = m\left[1 - \frac{\delta m(\eta)}{m}\right] \tag{106}$$

$$\frac{\delta m(\eta)}{m} = \frac{\alpha^2(\eta)}{16m^2 4\nu_0^2} \tag{107}$$

We can also write, using a new variable P(v,η):

$$\frac{1}{2}\, m_{eff}(\eta)v^2 + \Phi(x) \pm \frac{P^2(v,\eta)}{2m} = E \tag{108}$$

$$\pm\frac{P^2(v,\eta)}{2m} = \frac{\hbar^2}{2m}\,\frac{\eta^2(x) - 1}{\eta^2(x)}\,\frac{2\pi^2 4\nu_0^2}{v^2} \tag{109}$$

The $\pm$ sign in these equations is taken to keep the cases $\eta^2(x) \gtrless 1$ distinguished. Discussion about these equations is reserved to the next section.

Here note that, by the previous analysis, the single-particle position incertitude Δx has been conclusively set equal to $v(x)/\sqrt{8}\nu_0$. This setting is quite different from the one intrinsic to the matter-wave theory, represented - in the quasi-classical case - by the de Broglie relationship $\Delta x \approx \lambda \approx h/mv$. As it will be made clear in the following paper IV, indeed, in our framework the quantities Δx and the de Broglie wavelength h/mv are different physical quantities, and take different interpretative roles.

5.3 The quantum field - sustained energy theorem and the tunnelling phenomena

In this section, we want to discuss the result (105)/(108) we obtained in the form of the energy theorem for a single-particle submitted to the quantum field. First of all, we remark that the term depending on the $\alpha-$function in the Φ_{HDF} expression converges into an effective-mass term appearing in equations (105) and (106), thus substantiating a previous remark. The induced mass coefficient $-\delta m$ is negative, so that an important effect of the quantum field is reducing the particle inertia. We recall here that the function α is not only linked to some friction coefficient value we may find effective into our equations, but also to non-linearities (see equations (45) and (46) for the definition of

α) of the oscillator behavior under the fluctuation field action. Note that the effective mass found in equation (106) can even, in principle, attain negative values, what might reveal not unrealistic at all. The negative-mass concept first appeared in quasi-classical approximations made in quantum solid-state physics [10], and is easily accommodated into a classical theory because (as is exactly the case here) it is simply due to some peculiarity of the external action on the particle that we can easily appreciate. We will discuss again about the mass effect in paper IV - but the possibility of a negative mass will not be investigated in the present papers. Here we have to say that the mass effect cannot be thought able to account for tunnelling phenomena, because the negative kinetic term becomes negligible when the velocity goes to zero, i.e. just in the neighborhood of the particle turning point - thus appearing unable to provide the energy contribution necessary for tunnelling.

Tunnelling phenomena are of major interest here, and we expressed already the opinion that they can be attributed to the second term in the Φ_{HDF} expression given in (100) or following equivalents. This can be seen by means of equation (105)/(108) itself when we search for the position x_0 of the particle turning point - we can obtain it taking a zero limit value for the velocity :

$$\lim_{\mathrm{x}\to\mathrm{x}_0} \Phi_{\mathrm{HDF}}(\mathrm{x},\xi) = \lim_{\mathrm{v}\to 0,\mathrm{x}\to\mathrm{x}_0} \frac{\mathrm{h}^2}{\mathrm{m}} \frac{\eta^2(\mathrm{x}) - 1}{\eta^2(\mathrm{x})} \frac{\nu_0^2}{\mathrm{v}^2} =$$

$$= \pm \lim_{\mathrm{v}\to 0,\mathrm{x}\to\mathrm{x}_0} \frac{\mathrm{P}^2(\mathrm{v},\eta)}{2\mathrm{m}} = \pm \frac{\mathrm{P}^2(0,1)}{2\mathrm{m}} = \mathrm{E} - \Phi(\mathrm{x}_0) \tag{110}$$

From this equation, it is clear that the meaningful limit value for $\eta(\mathrm{x})$ is $\eta(\mathrm{x}_0) = 1$ - what can also be intended as the initial condition relevant to the hidden parameter η itself. For an easy comparison, here the corresponding purely classical equation is reported:

$$\mathrm{E} - \Phi(\mathrm{x}_{0c}) = 0 \tag{111}$$

In this equation, x_{0c} is the classical turning point. Equation (110) displays both the possibilities that $\lim\Phi_{\mathrm{HDF}}$, when the turning point is attained, is a positive or a negative quantity. Since Φ_{HDF} is the extra-potential in the energy theorem with respect to the pure classical case, we see that the turning point position x_0 turns out to be different from the classical one, and is found in the classical region if the limit is positive (the particle stops before attaining the classical position x_{0c}) while it is found outside the classical position (the particle enters the "forbidden" region, what we call tunnelling) if the limit is negative. In both cases we will ask therefore - for the sake of consistency - that the limiting value of η is unity; but the tunnelling case is attained when starting with η values smaller than unity in some point of space near x_0.

By the previous discussion, we have seen that the quantum field is able to modify the expression of the classical motion energy theorem by adding to it the term Φ_{HDF}. This last distinguishes into two peculiar terms - the first one introduces an effective mass of the particle, which can be smaller than the real mass or even negative; the second one introduces a perturbation whose major importance, at present, is in the discussed possibility of describing tunnelling phenomena. We do not attempt here to determine (by the means of further assumptions we could take about the fluctuation field time-law) the effective mass function, neither the effective additive potential $\mathrm{P}^2/2\mathrm{m}$ - they depend on complicated non-linearities displayed both by the excitation source and oscillator behavior; but the interpretation we have given to these terms will survive when developing the next steps of our analysis. In order to have more information about the functions Φ_{HDF} and η, we have indeed to evolve towards a more extended framework. The major constraint we want to drop off is that the quantum displacement x_z is a fast oscillating quantity. Since the purely mechanical framework to be set up in this case reveals too much complicated for analytical investigation, we will follow another investigative path, to be shown in paper III. This will consist in forming microcanonical ensembles of our particles, while taking their mechanical properties as resulted from this investigation - and fitting the result into the properties displayed by the thermodynamic framework expounded in paper I. On the other hand, we will also ask to our microcanonical framework to be consistent with a quantum physics apparatus constituted by the Schrödinger equation in the Madelung formulation. The results, to be expounded in papers III and IV, will then include a definite expression for Φ_{HDF} - so that it will be conclusively produced the statement (submitted to the limits and incertitudes of this investigation) that the wave mechanics is possibly the microcanonical ensemble appearance of a Newtonian substrate. This last will be identified into the equations of motion finally expressed in the next section.

6 Newton equation of motion for the Bernoulli oscillators

We resume here, for the sake of clarity, the complete set of motion equations we attribute to our Bernoulli oscillators, in the assumption that the function F(x,t) (40) is zero, inclusive of the superimposed HDF oscillation, and of the Heisenberg constraint expressed in the form (98) :

$$\mathrm{x}^*(\mathrm{t}) = \mathrm{x}(\mathrm{t}) + \mathrm{x}_z(\mathrm{t}) \tag{112}$$

$$\mathrm{v}^*(\mathrm{t}) = \frac{\mathrm{d}}{\mathrm{dt}}\left[\mathrm{x}(\mathrm{t}) + \mathrm{x}_z(\mathrm{t})\right] = \mathrm{v}(\mathrm{t}) + \mathrm{v}_z(\mathrm{t}) \tag{113}$$

$$\mathrm{x}_z(\mathrm{t}) = \sum_{n,n_s} \mathrm{x}_z^{(n)}(\mathrm{t}) = \sum_{n,n_s} \mathrm{x}_z^{(n)}(\mathrm{n}\omega_0(\mathrm{x}),\varphi_n(\mathrm{x}))\sin\left(\mathrm{n}\omega_0(\mathrm{x})\mathrm{t} + \varphi_n(\mathrm{x})\right) \rightarrow|_{SF}\approx$$

$$\approx \mathrm{x}_z^{(\eta)}(\mathrm{t}) = \mathrm{x}_z^{(\eta)}(\eta\omega_0(\mathrm{x}),\varphi_\eta(\mathrm{x}))\sin\left(\eta\omega_0(\mathrm{x})\mathrm{t} + \varphi_\eta(\mathrm{x})\right) \tag{114}$$

$$\mathrm{v}_z(\mathrm{t}) = \sum_{n,n_s} \mathrm{v}_z^{(n)}(\mathrm{t}) = \sum_{n,n_s} \mathrm{n}\omega_0\mathrm{x}_z^{(n)}(\mathrm{n}\omega_0(\mathrm{x}),\varphi_n(\mathrm{x}))\cos\left(\mathrm{n}\omega_0(\mathrm{x})\mathrm{t} + \varphi_n(\mathrm{x}\right) \rightarrow|_{SF}\approx$$

$$\approx \mathrm{v}_z^{(\eta)}(\mathrm{t}) = \eta\omega_0\mathrm{x}_z^{(\eta)}(\eta\omega_0(\mathrm{x}),\varphi_\eta(\mathrm{x}))\cos\left(\eta\omega_0(\mathrm{x})\mathrm{t} + \varphi_\eta(\mathrm{x})\right) \tag{115}$$

$$16\mathrm{m}^2 < \mathrm{v}_z^2 > |_{\mathrm{x}} < \mathrm{x}_z^2 > |_{\mathrm{x}} = \mathrm{A}^2 \rightarrow_{\text{quantum limit}} \rightarrow \mathrm{h}^2 \tag{116}$$

$$\frac{1}{2}\mathrm{mv}^2 + \Phi(\mathrm{x}) + \Phi_{\mathrm{HDF}}(\mathrm{x},\xi(\mathrm{x})) = \mathrm{E} \tag{117}$$

In equations (114)÷(115) we extended our sums to some special non-integer indices n_s - not included in the previous equations set (49)÷(51) - to take into account the (now recognized) possibility of sub-harmonic terms being raised by the fluctuation field in the oscillator response. We also show the peculiar form taken by the HDF oscillation in the single frequency approximation characterized by a value of the parameter η. More has to be stated however (in papers III and IV) about the expressions of the different quantities ($\Phi_{\mathrm{HDF}}(\mathrm{x},\xi(\mathrm{x})),\xi(\mathrm{x}),\omega_0(\mathrm{x})$ etc.) entering this set of equations, so that further discussion about the Newtonian framework here promoted is reserved to these next papers. Here we have to remark that the pulsation $\omega_0(\mathrm{x})$ can be written, in the full quantum limit :

$$\omega_0 = \frac{\mathrm{h}}{\mathrm{m}\eta(\mathrm{x})\Delta\mathrm{x}^2} \approx \frac{8\mathrm{h}\nu_0^2}{\mathrm{m}\eta(\mathrm{x})\mathrm{v}^2(\mathrm{x})} \tag{118}$$

Here we used equation (29) and the fact that $\eta(\mathrm{x})$ is the effective harmonic index in the SF interpretative scheme. Equation (118) characterizes the parametric constraint the HDF oscillation is submitted to as a consequence of Heisenberg's principle. Note - just to make a specific point - that the equation indicates that for particles at rest ($\mathrm{v} = 0$) the pulsation $\omega \equiv \eta\omega_0$ goes to infinity so that the HDF oscillation becomes very fast (with zero amplitude however).

The assumption (41) appears here to limit practically the validity of these equations to the case of fast oscillations; but note that even when F(x,t) is different from zero then we can - either perform a time-average on the energy spectrum provided by this residual field to attain a stationary behavior description - or, in principle, taking hypotheses about F(x,t), assume a time-law for the energy E and solve the motion equations within non-stationary circumstances. We understand therefore that the setting (112)$\div$(118) with associated interpretation can conceptually survive the removal of some specific assumptions we have done by simplicity. The important matter is indeed that we attribute a deterministic time-behavior to the overall set of parameters. On the other hand, the very important question we have to solve to affirm the usefulness of this framework (and self-consistency first; comprehensive consistency with quantum physics is a much wider subject) is whether the Heisenberg principle really tolerates our classical interpretation and formalism within some recognized, meaningful domain of physical circumstances. At the same time, one would ask the more specific question whether the proposed motion equations, with associated solutions, are always compatible with the Heisenberg constraint as expressed by equation (116). As far as the first question is regarded, it is obvious that we will be able to attain some precise definition of this domain only by future work. Concerning the second one instead, we advance the opinion that the answer is linked to the possibility one has to define meaningful integration time-intervals for the averages shown in equation (98) or

(116) when the quantum displacements variables x_z and v_z display slow time-variations. In these equations, indeed, the time-intervals for integration are small - over the time-scale of interest - only when the fast motion approximation is taken. The real question actually is - is that a possible issue, and what happens if we take v_z equal to a stationary zero ?

Attempting an answer to both the questions would however bring us to analyze very specific time and space correlations, inclusive of the action of external constraints as measurements apparatuses. But concerning the very last question stated above, one may be brought to answer simply : that is not a possible issue, and if any other physical constraint imposed will push towards it, then the system will pop out of quantum world to become classic and the Heisenberg constraint will be broken. Analyzing even partially the spectrum of the possibilities is obviously outside the limits of the present work, but we want to give evidence to the fact that investigating such very specific matter would be a difficult task - both theoretically and experimentally - even if our classical interpretation of the Heisenberg principle would consolidate and maintain true in the deep of quantum phenomena. This is simply because, in any case, the theoretical or experimental probes we could use to perturb our system - in order to trigger its reaction - would be submitted to the same Heisenberg constraint. This is a quite standard argument for those believing in such a kind of absolute barrier set to our scientific knowledge. However, we want to promote the opinion that if the classical interpretation could be maintained in the deep, then it would be clear that the real difficulty in making successful calculations or interpreting experiments would just be due to the (perhaps inextricable ?) complication of action-reaction effects and energy-momentum transfer between the system and the probe; but dealing with such kind of problem within a classical framework is quite a different matter than surrender to the statement of an absolute limit imposed by nature as an impossible barrier to overcome. We definitely want to add our efforts to those promoting renovated investigations against the frustrating interpretation of the absolute limit.

In order now to set up a background apparatus, useful for the task to be done in the next papers, we will give in the following section elements concerning the $\xi-$values distribution effective on the statistical average we want to perform over the energy theorem expression.

7 Elements for statistical averaging

7.1 The microcanonical distributions of ξ, $\eta-$values and the statistical energy theorem expression

In our framework, the functions ξ(x) and η(x) must be considered statistical variables because they depend on initial conditions to be assigned on the basis of an appropriate distribution of values in some position x_0- much alike the Eulerian velocity field v(x) is a function depending on the initial condition value v(x_0) attributed to it. Within this context, in order to match the quantum-mechanical case, we should just take the energy value E in equation (117) equal to a quantum-mechanical "eigenvalue" E_n. In equation (110) it is seen easily that assigning the initial condition for η is tantamount to assume a value of the quantity P^2(0,1) once the position of one of the turning points has been given. Using a probabilistic distribution $P_{\eta=1}(x_0)$ of turning points is therefore the tool involved within the task to assemble a microcanonical system of our particles. This distribution is unknown at the present investigation stage; but we can equally have information about the result - because this last has to match quantum-mechanical properties. To understand more about this point, we note the following.

Taking one out of such distributions $P_{\eta=1}(x_0)$ should bring us practically (i.e., in a first approximation) to the same result that we would obtain if - having not fixed the E value into a E_n value yet - a corresponding distribution of energy values P(E) was assigned, at a constant P^2(0,1) value. It can be thought that such a kind of computative performance could be obtained provided - as is obvious - one knew the correspondence between the distributions themselves. This is because the turning point position depends equally, and strongly, on the energy of the particle. We understand therefore that distributing initial conditions for η is practically equivalent to distribute mechanical energy values E - according to a sort of microcanonical distribution around some average value E_n. The statistical ensemble we will have available as a consequence of such a mixing will display different properties, depending on the kind and measure of the interactions we may assume effective amongst our particles. The statistical result will then be found submitted to the effect of the generalized pressures f, g, P we evidenced constitutive of a thermodynamic state in paper I - the effect of these variables being displayed into the flow-of-mass theorem expression given in the conclusive part of the same reference. Another interesting property of the mixing can be forecast on the basis of the observation - also made in paper I - that the Gibbs distribution of our system takes into account entropy terms $-Tln\rho_n$ attached to the microcanonical ensemble average energy value E_n. We can assume, therefore, that distributing initial conditions for η results - in the average - in statistical states homologous to the ones we would obtain by an energy broadening effect, and in introducing the quoted entropy terms into the Boltzmann factors pertaining to the microcanonical ensembles. In the following paper III, we will set up a simple statistical model accounting for the mentioned effects; here we want to give evidence to the case when interactions (and pressures) are zero, so that the main effect we account for, while forming our ensembles, is simply an inhomogeneous energy broadening. By the previous considerations, this effect will imply considering a peculiar flow-of-mass rate law ν(x), determined by the inhomogeneity of the turning points within the ensemble. As a consequence, at zero generalized pressures the mean statistical energy theorem (normalized to one-particle) displayed by our ensemble will contain both the König term or center-of-mass kinetic potential and the reactive thrust potential due to the variable-mass-equivalent effect accounting for the variable mass flow. Both these effects will be accounted for, in our framework, by the peculiar term we called $I_D(\nu(x),0)$ in the previous paper I, and will be calculated with details for the case of the model introduced in paper III. Here we only make the important remark that the transition from the single-particle energy theorem expression to the statistical average expression in these conditions is characterized by the appearance of the potential $I_D(\nu(x),0)$. To be clear, we write :

$$\frac{1}{2}\mathrm{m}\mathrm{v}_i^2 + \Phi(\mathrm{x}) + \Phi_{\mathrm{HDF}}(\mathrm{x},\xi_i(\mathrm{x})) = \mathrm{E}_n \qquad \{\text{single i-th particle}\} \tag{119}$$

$$\frac{1}{2}\mathrm{m}\mathrm{v}_D^2 + \Phi(\mathrm{x}) + \langle \Phi_{\mathrm{HDF}}(\mathrm{x},\xi_i(\mathrm{x})) \rangle \,|_{\mathrm{x}} + \mathrm{I}_D(\nu(\mathrm{x}),0) = \mathrm{E}_n \qquad \{\text{many particles}\} \tag{120}$$

We have used the unspecified expression for Φ_{HDF} and it is clear that what has been said in terms of η is transferred on the variable ξ_i. In the last equation, the average $\langle\rangle|_{\mathrm{x}}$is performed over an ensemble of particles with different initial conditions values for ξ_i; v_D is the center-of-mass velocity of an ensemble of particles at position x. The use we will make of equation (120) is explained by the following consideration. Whenever we have available - by a statistical model - an expression for $<\Phi_{\rm HDF}({\rm x},\xi({\rm x}))>|_{\rm x}$ consistent with that equation, then it is clear that we can determine the corresponding expression $\Phi_{\rm HDF}({\rm x},\xi_i({\rm x}))$ effective on a single-particle by the following procedure :

$$\Phi_{\rm HDF}({\rm x},\xi_i({\rm x})) = \overset{(i)}{\lim_{{\rm I}_D(\nu_0,0)\to 0}} <\Phi_{\rm HDF}({\rm x},\xi_i({\rm x}))> |_{\rm x} \tag{121}$$

The limit expressed in this equation is affected by a superscript (i) because is one of the possible issues. It implies the limit $\nu({\rm x})\to \nu_0 = $const due to the property of constant mass-flow assumed in the all-identical particles, purely homogeneous statistical case. Use of the last equation is reserved to the final paper IV.

7.2 An interpretative remark

As is also clear by the previous discussions, the main research strategy we want to pursue within the following papers is comparing equation (120) with the basic quantum-mechanical wave equation, to find an expression of the potential $\Phi_{\rm HDF}({\rm x},\xi_i({\rm x}))$ effective on a single-particle. Here an interpretative remark must be evidenced: equation (120) is a many-particles equation, while the quantum-mechanical wave equation is known to hold even for a single particle. In order that equation (119) (it will be finally found specified into the form (36) in paper IV) may be able to represent what we call "the Newtonian background" below the wave-matter context, we will have to assume, conversely, that every single particle is able to cross over many time-laws (different ξ_i-valued solutions of the equation), up to totalize, along the time, the quantum-mechanical density.

Major remarks about this point will be found in the following papers.

8 Conclusion

We developed in this paper a mechanical model for the Bernoulli oscillator, i.e. a classical oscillator forced by an external action which we call the quantum field. We showed that this oscillator behavior can be described by the evolution of two distinguished parts of both its space and momentum co-ordinates, called the quantum displacements (or HDF) and standard classical co-ordinates respectively. The influence of the quantum displacements, driven by the quantum field, on the classical co-ordinates is displayed - first within the assumption of fast HDF oscillation. Expressions are found for the energy theorem pertaining to the classical part in order to show its evolution towards quantum properties. The main findings in this paper are represented by the indicated possibility of a classical interpretation of Heisenberg's indetermination principle and a classical explanation of tunnelling effects, but the overall framework results into an apparatus useful to accommodate an extended description. This is expected, by further independent analysis to be developed in papers III and IV, to drop off the assumption of fast HDF oscillation and provide generalized expressions and interpretation by the means of a direct comparison of our results with a standard quantum-mechanical framework.

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