# The path integral approach in the frame work of causal interpretation 

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#### Abstract

We try to clarify the relation of Feynman paths -which appear in the Feynman path integral formalism- with de Broglie-Bohm paths. To accomplish this, we introduce a Bohmian path integral and use it to obtain the propagator of a free particle and a general system.


## 1 Introduction

In the common interpretation of quantum mechanics, there is no picture of paths. In spite of this, the Feynman approach to quantum mechanics[1], uses the concept of path as a mathematical tool to calculate a propagator for the wave function:

$$
\begin{equation*}
K\left(x, t ; x_{o}, t_{o}\right)=\int D[x] e^{\frac{i}{\hbar} S[x]} \tag{1}
\end{equation*}
$$

where $S$ is the classical action and the integral is taken over all possible paths between the two points $(x, t)$ and $\left(x_{o}, t_{o}\right)$. One can show that (1) satisfies the Schrödinger equation and can be used as a substitute for it. This means that, if we have an arbitrary initial wave function, we can obtain it at any other time by the use of (1).

On the other hand, in the causal interpretation of quantum mechanics $[2,3]$, it is claimed that particles move on the paths which are given by

$$
\begin{equation*}
\dot{\vec{x}}=\frac{\hbar}{m} \operatorname{Im}\left(\frac{\vec{\nabla} \psi}{\psi}\right)=\frac{\vec{\nabla} S}{m} \tag{2}
\end{equation*}
$$

where $S$ is the phase of the wave function $\left(\psi=R e^{i \frac{S}{\hbar}}\right)$. In this paper, the authors want to clarify the relation between Feynman paths and the de Broglie-Bohm paths[4]. First, we obtain a path integral formulation of quantum mechanics in the framework of the causal interpretation. We call it Bohmian path integral (BPI). Then, we introduce Fourier-Bohm paths as a tool to calculate BPI. In this way, we are naturally lead to Feynman path integrals (FPI). Our discussion could be considered as an approach for obtaining FPI from the Schrödinger equation.

## 2 The Bohmian path integral

To obtain BPI we consider $\psi$ at two points of a Bohmian path. For simplicity, we first consider two points with infinitesmall distance $d \vec{x}$. Then, we have

$$
\psi(\vec{x}+d \vec{x}, t+d t)=\psi(\vec{x}, t)+\frac{\partial \psi(\vec{x}, t)}{\partial t} d t+\vec{\nabla} \psi(\vec{x}, t) \cdot \overrightarrow{d x}
$$

As a result of our assumption that these two points are on a Bohmian path and using (2), we have $\overrightarrow{d x}=\frac{\vec{\nabla} S}{m} d t$. With the use of this expression for $d \vec{x}$ and the Schrödinger equation for $\frac{\partial \psi}{\partial t}$ we obtain:
$\psi(\vec{x}+d \vec{x}, t+d t)=\left\{1+\frac{i}{\hbar}\left[\frac{(\vec{\nabla} S)^{2}}{2 m}-(Q+V)\right] d t-\frac{\nabla^{2} S}{2 m} d t\right\} \psi(\vec{x}, t)$
where $Q=-\frac{\hbar^{2}}{2 m} \frac{\nabla^{2} R}{R}$ is the so-called quantum potential $(R=|\psi|)$. Now, we can write the expression in $\}$ as an exponential. Then, for the two points $(x, t)$ and $\left(x_{o}, t_{o}\right)$, with a finite distance on a Bohmian path, we have

$$
\begin{equation*}
\psi(\vec{x}, t)=\exp \left\{\frac{i}{\hbar} \int_{\vec{x}_{o}, t_{o}}^{\vec{x}, t} \mathcal{L}_{q} d t-\int_{\vec{x}_{o}, t_{o}}^{\vec{x}, t} \frac{\nabla^{2} S}{2 m} d t\right\} \psi\left(\vec{x}_{o}, t_{o}\right) \tag{3}
\end{equation*}
$$

where $\mathcal{L}_{q}=\frac{(\vec{\nabla} S)^{2}}{2 m}-(Q+V)$ and the integrals are taken over the Bohmian path which initiates at $\left(\vec{x}_{o}, t_{o}\right)$ and ends at $(\vec{x}, t)$. In fact, we have integrated the Schrödinger equation on a Bohmian path. If we substitute $R e^{i \frac{S}{\hbar}}$ for $\psi$, in the Schrödinger equation, we obtain two equations from
the real and imagenery parts of the resulting equation. The real part is a quantum Hamilton-Jacobi equation which includes $Q$. By integrating this equation on the Bohmian path we obtain the first exponential in (3). The imaginary part is a continuity equation for $R^{2}$. By integrating this equation on the Bohmian path we obtain the second exponential in (3). Now, the first term gives the phase of wave function at $(\vec{x}, t)$ as a result of its phase at $\left(\vec{x}_{o}, t_{o}\right)$. As one may expect, here the classical action of FPI is replaced by the quantum action (which includes $Q$ ). On the other hand, since there is no interference between the paths, in order to have an evolution for $R=|\psi|$, a real exponent is needed. The second term is a proper term for this purpose. As one may expect the divergence of paths $\left(\vec{\nabla} \cdot \frac{\vec{\nabla} S}{m}\right)$ appears in this term.

Note that, BPI can be used to obtain $\psi(\vec{x}, t)$ from $\psi\left(\vec{x}_{o}, t_{o}\right)$, instead of the Schrödinger equation as is the case for FPI. If we have $\psi\left(\vec{x}, t_{o}\right)$ we can obtain $\psi\left(\vec{x}, t_{o}+d t\right)$ by the use of (3) and by repeatation of this process for other $d t$ 's we obtain $\psi\left(\vec{x}, t_{o}+\Delta t\right)$, where $\Delta t$ is finite. That is, we use the word BPI for the relation (3), although it is not a real path integral. Because, the terms in the curly braces that are used to propagate the wave function depend explicitly on the values of the wave function in the region where it is being propagated. Nevertheless, we prefer to use the word BPI for (3). Note that we use the terminology BPI for the relation (3), although it is not a path integral in the usual sense. Because, the terms in the curly braces which are supposedly propagateing the wave function depend explicitly on the value of the wave function in the region where it is being propagated. Nevertheless, we prefer to use the terminology BPI for (3).

In the following section we show that one can obtain $\psi(\vec{x}, t)$ analytically for a free wave packet. We also obtain $K\left(x, t ; x_{o}, t_{o}\right)$ for this case.

## 3 the propagator of a free wave packet

In this section we use BPI to obtain the time evolution of a free wave packet. For this special case, one can do it analytically. Suppose $\psi_{o}(x)$ is a wave packet in one dimension. It can be written as

$$
\begin{equation*}
\psi_{o}(x)=\int \varphi(k) e^{i k x} d k \tag{4}
\end{equation*}
$$

We consider the evolution of this packet as a result of the evolution of its Fourier components $e^{i k x}$. For $e^{i k x}$ states BPI becomes very simple. In fact, for these states we have

$$
\nabla^{2} S=0, \text { and } Q=0
$$

This means that Bohmian paths for the plane waves are straight lines with the gradient $v=\frac{k \hbar}{m}$ in space-time. Furthermore, the quantum of action for BPI changes to the classical action. In this way, our path integral becomes more similar to FPI. We name this paths, the FourierBohm paths (FBP). Now, we try to obtain the wave function at $(x, t)$. For each component $e^{i k x}$ only one FBP has contribution at $(x, t)$, and this initiates from a special point at $\left(x_{o}, t_{o}\right)$. If temporarily we consider $k$ to be discrete, one can write $\psi(x, t)$ as a combination of all such contributions:

$$
\psi(x, t)=\sum_{i} e^{i \frac{S_{i}}{\hbar}} \varphi\left(k_{i}\right) e^{i k_{i} x_{o_{i}}}
$$



Fig. 1 Several Fourier Components of the Bohmian paths end at x
where $x_{o_{i}}=x-\frac{k_{i} \hbar}{m}\left(t-t_{o}\right)$ (Fig.1), and the summation is taken over all $k_{i}$ 's. On other hand, $S_{i}$ (action on the $i$ th path) is given by

$$
S_{i}=\int_{t_{o}}^{t} \frac{p^{2}}{2 m} d t=\int_{t_{o}}^{t} \frac{k_{i}^{2} \hbar^{2}}{2 m} d t=\frac{k_{i}^{2} \hbar^{2}}{2 m}\left(t-t_{o}\right)
$$

If we replace the summation by integration and replace $x_{o}$ by $x-$ $\frac{k \hbar}{m}\left(t-t_{o}\right)$ we obtain:

$$
\begin{equation*}
\psi(x, t)=\int d k e^{-i \frac{k^{2} \hbar}{2 m}\left(t-t_{o}\right)} \varphi(k) e^{i k x} \tag{5}
\end{equation*}
$$

which is the same thing as one would expect for $\psi(x, t)$ from the Schrödinger equation. If we want to obtain $K\left(x, t ; x_{o}, t_{o}\right)$ for this case, we must consider $\psi_{o}(x)=\delta\left(x-x_{o}\right)$. For this state we have $\varphi(k)=e^{-i k x_{o}} / 2 \pi$. By substituting $\varphi(k)$ in (5) one obtains

$$
K\left(x, t ; x_{o}, t_{o}\right)=\sqrt{\frac{m}{2 \pi i \hbar\left(t-t_{o}\right)}} \exp \left\{\frac{i m\left(x-x_{o}\right)^{2}}{2 \hbar\left(t-t_{o}\right)}\right\}
$$



Fig. 2 Continuous lines show a typical Feynman path ; broken lines indicate Bohmian paths

Note that, here instead of having one Bohmian path at each points $x_{o}$, we have infinite FBP (each path for a definite $k$ ). It is the cost that we pay for the elimination of $\nabla^{2} S$ and $Q$ terms in BPI. In other words, for obtaining information about the evolution of $R=|\psi|$ with the help of phase, we must make use of the infinite FBP at each point $x$, instead of one Bohmian path. This job is done by choosing $\psi_{o}(x)$ as (4), and by taking its evolution as the evolution of its components $e^{i k x}$.

Are Feynman paths different from our FBP? We don't think so. Note that, in both Feynman and Fourier-Bohm path integrals, from each point of the space at time $t_{o}$ infinite paths originate and these have all possible gradients. Furthermore, to each point of the space at time $t$ end infinite paths with all possible gradients. On other hand, at each point of the space, at a time $t_{1}\left(t_{o} \leq t_{1} \leq t\right)$, we have infinite paths with all possible gradients. In fact, since the infinity of Feynman paths and the infinity of FBP are of the same order, we could make each Feynman paths from the combination of several FBP (Fig.2). In other words, the infinite FBP's which appear in the Fourier-Bohm path integral are equivalent with the infinite Feynman paths which appear in the FPI. The only difference is that, in the BPI each path which initiates at $x_{o}$, with the gradient $v_{i}=\frac{k_{i} \hbar}{m}$, is multiplied by one component of $\psi_{o}\left(x_{o}\right)$, i.e. $\varphi\left(k_{i}\right) e^{i k_{i} x}$. But, in FPI the same path is multiplied by all components of $\psi_{o}\left(x_{o}\right)$, i.e. $\sum_{i} \varphi\left(k_{i}\right) e^{i k_{i} x}$. Now, we know that the two methods give the same result for the evolution of $\psi$. Thus, one must be able to show that they are equivalent. The solution is given by the infinity of the number of paths. For each path which initiates with the gradient $v_{i}=\frac{k_{i} \hbar}{m}$ and an action $S$, there are other paths with the same action but with different initial gradients. This means that in the FBP integral we have terms like

$$
\begin{aligned}
\ldots e^{i \frac{S}{\hbar}} \varphi\left(k_{1}\right) e^{i k_{1} x_{o}}+e^{i \frac{S}{\hbar}} \varphi\left(k_{2}\right) e^{i k_{2} x_{o}} & +\ldots+e^{i \frac{S}{\hbar}} \varphi\left(k_{n}\right) e^{i k_{n} x_{o}}+\ldots \\
& =\ldots+e^{i \frac{S}{\hbar}} \sum_{i}^{n} \varphi\left(k_{i}\right) e^{i k_{i} x_{o}} \ldots
\end{aligned}
$$

which is what appears in the FPI.

## 4 the propagator of a general system

Here, we use BPI to obtain propagator for wave function of a system with potential $V(x)$. In this case, we can not consider the evolution of the wave packet $(\psi)$ as a result of the evolution of its Fourier components $\left(e^{i k x}\right)$ for a finite period of time. But for a infinitesmall period of time, it is correct yet. Then, from (3) and (4) for a infinitesmall period of time $\left(t-t_{o}\right)$, we have:

$$
\begin{equation*}
\left.\psi(x, t)=\int \exp \left\{\frac{i}{\hbar}\left\{\frac{\hbar^{2} k^{2}}{2 m}-V(x)\right]\right\}\left(t-t_{o}\right)\right\} \varphi(k) e^{i k x} d k \tag{6}
\end{equation*}
$$

where for Fourier components we take $\nabla^{2} S=0$, and $Q=0$. If we want to obtain $K\left(x, t ; x_{o}, t_{o}\right)$ for this case, we must consider $\psi_{o}(x)=\delta\left(x-x_{o}\right)$ that is $\varphi(k)=e^{-i k x_{o}} / 2 \pi$. By substituting in (6) one obtains:

$$
\begin{align*}
K\left(x, t ; x_{o}, t_{o}\right)=\sqrt{\frac{m}{2 \pi i \hbar\left(t-t_{o}\right)}} & \exp \left\{\frac { i } { \hbar } \left[\frac{m\left(x-x_{o}\right)^{2}}{2\left(t-t_{o}\right)}\right.\right. \\
- & \left.\left.V\left(\frac{x+x_{o}}{2}\right)\left(t-t_{o}\right)\right]\right\} \tag{7}
\end{align*}
$$

This is propagator of the system for the infinitesmall period $t-t_{o}$. If we want to obtain propagator for a finite period of time we must use Feynman path integral method. Here too, we begin with Bohmian paths, but ultimately we come to Feynman paths as a tools.

If we discuss several-particle systems instead of one-particle systems, the wave function exists in configuration space. In this case, in the framework of de Broglie-Bohm theory of quantum mechanics, the question arises as to whether such wave could be a real wave. But we know that the de Broglie-Bohm paths in both versions of this theory are considered as real paths. Note, that the wave function of a several-particle system in the framework of the standard interpretation of quantum mechanics exists in the configuration space. But, Feynman paths (which are not consider as real paths) can be defind in the real space. Then, cosidering a several-particle system to see if it has any new result concerning the relation of de Broglie-Bohm paths and Feynman paths has any new point.

## 5 Conclusion

We introduced Bohmian path integral and used it to obtain the propagator of a free particle. Then, we showed to what extent the FBP are like the Feynman paths and we gave a heuristic argument to show the equvalnce of them. In fact, we introduced FBP as a mathematical tool as is the case for Feynman paths.

Our discussion is an attempt to throw light on the problem of why Feynman's technique works, using Bohm's causal interpretation. Besides, the clarification of the relation between Bohmian trajectories and Feynman's paths may be helpful in the calculation of the propagator for the Schrödinger equation. It may be useful to add that the propagator for some simple systems are calculated recently $[5,6,7]$.

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