

Propositional bases for the physics of the Bernoulli oscillators

(A theory of the hidden degree of freedom)

III - Mechanical-Statistical framework

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Summary - This paper is the third one of a series of four. In the previous ones, we developed a thermodynamic and mechanical framework introducing the properties of the so-called Bernoulli oscillators. These last are classical oscillators submitted to an external force coming from the quantum vacuum, and driving a distinguished part of the oscillator motion itself which we call the hidden degree of freedom (HDF). In paper II, an expression for the HDF-potential effective in the classical expression of the mechanical energy theorem has been given. In order to show that this expression is consistent with a quantum mechanical context, a few unknown functions must be determined. To this purpose, we set up in this paper a mechanical-statistical framework for the system at hand. By investigating the properties of the reference statistical ensemble of oscillators we are able to find out the generalized state-equations needed in the expression of the mass-flow-theorem we have given in paper I. We produce the constitutive relations for the system pressure, (average) HDF-potential and other relevant statistical quantities. These expressions are for final comparison - to be performed in the following paper IV - with a wave-mechanical context. Discussion and physical interpretation of the framework here introduced are also provided.

1 Introduction

In previous papers denoted I and II [1,2] we introduced a few thermodynamic and mechanical properties of the so-called Bernoulli oscillators. These properties were enlightened by the (proposed) generalized interpretation where the classical laws of mechanics are respected, and the origin of deviations of physical behavior from the standard expected one

(in turn, the origin of quantum effects) is attributed to the so-called fluctuation field. This last is an external source of energy, taking its origin in the quantum "vacuum" and driving a peculiar, distinguished part of the particle motion which we call the hidden degree of freedom HDF. HDF can be viewed as an oscillation (of position and velocity) superimposed to the classical motion of the particle, able to influence this last by the effect of a perturbational potential energy which we call Φ_{HDF} - the HDF potential. Both the HDF oscillation and the oscillation center motion (this last substantiates what we call the classical degree of freedom) are newtonian. The HDF motion is submitted to Heisenberg's indetermination principle, which takes in our framework a (proposed) peculiar "classical" form and interpretation. This form is equivalent to a constraint effective on the mean squared values of the HDF position and velocity co-ordinates, thus evidencing some sort of parametric oscillation. To make a specific point, it can just be noted here that - in case a single frequency (SF) is dominant within this time-behavior, the Heisenberg constraint results into a relation between the oscillation frequency and the (classical degree of freedom) velocity (equation (118), paper II). In the present paper denoted III, we set up a reference statistical ensemble of oscillators, whose properties will allow us to find out - using some induction - the generalized state-equations needed in the expression of the mass-flow-theorem we have made available in paper I. We will provide here the constitutive relations for the so-called "microcanonical temperature", system pressure, (average) HDF-potential and other relevant statistical quantities. The resulting expressions will be compared - in a following paper IV - with a wave-mechanical context, to the final purpose of excerpting the proper mechanical expression of the HDF-potential and investigating its influence on the single-particle, classical energy theorem expression.

2 The microcanonical classical oscillators ensemble

We are going to set up here a model of stationary, statistical ensemble of classical oscillators displaying "local" properties. This will be named the microcanonical classical oscillators ensemble, and will be useful as a reference system.

We want to advertise the reader that here we just start by setting up the ensemble of purely classical oscillators, but this last will evolve throughout the paper up to include an HDF effect (see next equation (61) and associated comment). At that moment, we will call the ensem-

ble "microcanonical Bernoulli oscillators ensemble". Our purpose while investigating the classical/Bernoulli oscillators ensemble properties is essentially to become able to perform the statistical averages invoked in the previous paper II (for further application to the generalized case). This task should be achieved making use of properties as near as possible to those displayed by the FEOM model [3].

2.1 Constitutive equation for the ensemble statistical density

The energy fluctuations effective in the FEOM model [3] have a continuous energy spectrum which is represented by the following probability density (see equations (1), (6) and (10) in paper I):

$$P(E) = \frac{1}{\Delta E} \quad \{E_i \leq E \leq E_f\} \quad (1)$$

$$P(E) = 0 \quad \{E_i > E, E > E_f\} \quad (2)$$

To be definite, the energy variable is simply named E here and we have taken $E_i < E_f$.

Let us consider an ensemble of classical oscillators whose attractive centers have a coincident position in space ($x = 0$). They display different mechanical energy values E , collected in agreement with equations (1),(2). Assume they have their initial phases of motion distributed at random - yet in such a way that the space included in the volume V^* , pertaining to the energy value E , is filled homogeneously and in a stationary way by the class of particles with that energy value. If the energy interval ΔE is not too large, this can be considered a sort of microcanonical ensemble, displaying energy values inside the interval (E_i, E_f) . This particle system can also be referred to as the inhomogeneously energy-broadened system, with energy broadening ΔE . Let us now simply superpose the stationary statistical (classical) densities $\rho_c(E, x)$, pertaining to each of the energy values involved, to obtain the resulting density $\rho(x)$. To do this, we start with the purely classical energy theorem expression, inclusive of the potential energy $\Phi(x)$:

$$\frac{1}{2}mv_c^2 + \Phi(x) = E \quad (3)$$

We define the particles classical (numerical) density $\rho_c(\mathbf{E}, \mathbf{x})$ as follows

$$\rho_c(\mathbf{E}, \mathbf{x}) = \frac{2\nu_c(\mathbf{E})}{v_c(\mathbf{E}, \mathbf{x})} = \frac{\omega_c(\mathbf{E})}{\pi\sqrt{2/m}\sqrt{\mathbf{E} - \Phi(\mathbf{x})}} \quad (4)$$

$$\int_{V^*} \rho_c(\mathbf{E}, \mathbf{x}) d\mathbf{x} = \frac{1}{2} \oint \rho_c(\mathbf{E}, \mathbf{x}) d\mathbf{x} = 1 \quad (5)$$

In these equations, $\nu_c(\mathbf{E}) = \omega_c(\mathbf{E})/2\pi$ is the frequency of the classical motion for a particle with mass m and velocity v_c . The factor 2 appearing there is because the term $\rho_c(\mathbf{E}, \mathbf{x})$ accounts, by definition, for both the two countermoving streams (forward and backward motion) of particles. This choice allows us to correlate the integration over the volume V^* for density normalization to half the circuit integral taken over a full oscillation - what is consistent with the standard normalization we usually find employed in quantum mechanical calculations. Now we define

$$\begin{aligned} \rho(\mathbf{E}_i, \mathbf{E}_f, \mathbf{x}) &\equiv \rho(\mathbf{x}) = \nu_0 \operatorname{Re} \int_{\mathbf{E}_i}^{\mathbf{E}_f} \frac{\rho_c(\mathbf{E}, \mathbf{x}) P(\mathbf{E})}{\nu_c(\mathbf{E})} d\mathbf{E} = \\ &= 2\nu_0 \operatorname{Re} \int_{\mathbf{E}_i}^{\mathbf{E}_f} \frac{P(\mathbf{E})}{v_c(\mathbf{E}, \mathbf{x})} d\mathbf{E} = \frac{2\nu_0\sqrt{2m}}{\Delta\mathbf{E}} \operatorname{Re} \left\{ \sqrt{\mathbf{E}_f - \Phi(\mathbf{x})} - \sqrt{\mathbf{E}_i - \Phi(\mathbf{x})} \right\} \end{aligned} \quad (6)$$

As is clear from this equation, we take an average of the classical densities over the relevant energy domain using the probability density (1) with an additive weight $1/\nu_c(\mathbf{E})$ as a (rough, but effective) instrument to have comparatively uniform distribution of particles on different energy levels in the same regions of space. This is tantamount to insure the local distribution of particles to be stationary. The quantity $-\ln\nu_c(\mathbf{E})$ can be thought as a sort of entropy associated to the distribution $P(\mathbf{E})$.

In these equations, ν_0 is a normalization constant so that

$$\int_{V_f^*} \rho(\mathbf{E}_i, \mathbf{E}_f, \mathbf{x}) d\mathbf{x} = 1 \quad (7)$$

We also have :

$$\rho(x) = \frac{2\nu_0\sqrt{2m}}{\Delta E} \operatorname{Re} \left\{ \sqrt{E_f - \Phi(x)} - \sqrt{E_i - \Phi(x)} \right\} = \frac{2\nu_0}{\Delta E} \operatorname{Re} [p_f(x) - p_i(x)] \quad (8)$$

Here $p_i(x)$ and $p_f(x)$ are the classical expressions of the particles Eulerian momenta corresponding to the initial and final energy values pertinent to the definition interval, respectively. We have furthermore :

$$\begin{aligned} \int_{V_f^*} \rho(E_i, E_f, x) dx &= \frac{\nu_0}{\Delta E} \oint \operatorname{Re} [p_f(x) - p_i(x)] dx = \\ &= \frac{\nu_0}{\Delta E} \left[\oint p_f(x) dx - \oint p_i(x) dx \right] = 1 \end{aligned} \quad (9)$$

If we take $\Delta E = h\nu_0$, $E_i = (n-1/2)h\nu_0$ we recognize in this equation an expression of the Bohr-Sommerfeld rule. This is found here equivalent to the normalization condition (7) for the total density of our statistical ensemble. This finding appears to us as a further, interesting correlations between our fluctuation model and (WKB) quantum mechanics. We will set $\Delta E = h\nu_0$ in the sequel. Now another remark is the following. Expression (6) can also be written:

$$\frac{1}{2} m \frac{4\nu_0^2}{\rho^2(x)} + \Phi(x) + \frac{h^2 \rho^2(x)}{32m} = \frac{E_f + E_i}{2} \quad (10)$$

$$x : \{ \Phi(x) \leq E_i \} \quad \equiv \quad \{ \text{Region I} \}$$

$$\frac{3h^2 \rho^2(x)}{32m} - \frac{\Delta E}{2} + \Phi(x) + \frac{h^2 \rho^2(x)}{32m} = \frac{E_f + E_i}{2} \quad (11)$$

$$x : \{ E_i \leq \Phi(x) \leq E_f \} \quad \equiv \quad \{ \text{Region II} \}$$

These expressions display the form of an energy theorem. They are written by distinguishing two regions of space within the overall volume occupied by the particles ensemble. The Region I is defined as the

space domain where the potential energy $\Phi(x)$ is smaller than E_i , while in Region II the potential values are included into the energy interval (E_i, E_f) . In the two regions of space different analytical expressions will hold. This is due to the fact that in Region I all the particles belonging to the ensemble are present - because all of them have available, by construction, the necessary mechanical energy. At every position of space in Region II, instead, only the particles supported by sufficient mechanical energy will be found. To have better insight into this and other points in our analysis, one may refer to fig.(1). This last has been drawn taking a harmonic potential energy $\Phi(x)$ as a reference, but the framework we set up in this paper is largely independent of the form of the potential - provided it has a somewhat regular form. In case the tested potential $\Phi(x)$ displays large anharmonicity or irregularities, it might be found necessary to develop more appropriate investigation or to apply corrections to some details of the proposed analysis.

Fig.(??) shows that the inhomogeneously energy broadened system statistical density $\rho(x)$ differs from the classical one $\rho_c(x)$ chiefly by the erasure of poles in the turning-points positions. The density $\rho(x)$ shows evanescent tails at the boundaries. This can be interpreted in the sense that particles are found in regions of space which would be forbidden to classical particles with the mean energy $(E_f + E_i)/2$, thus evidencing some sort of "tunnelling" effect (the point will be discussed with more details in the following). Although $\rho(x)$ does not display any intermediate zero within the definition domain, it looks similar to a quantum-mechanical density. In our framework, we promote the idea that a quantum-mechanical density is also, indeed, the statistical ensemble expression of single-particles (classical degree of freedom) behavior - the inhomogeneity parameter in the quantum case being not the energy but the parameter ξ (or equivalently, η) introduced in paper II. This interpretation, which will be found supported by the general analysis developed in paper IV, looks to us not in contrast with the Copenhagen one (see a remark given in the last section of paper II).

2.2 Volume-flow balance equation

Before processing the previous equations, let us add to them the corresponding volume-flow balance equations. We first write down the expression for the total volume flow $\nu(E_i, E_f, x)$ corresponding to (one out of

the two) particle streams

$$\nu(E_i, E_f, x) \equiv \nu(x) = \nu_0 \int_{E_i}^{E_f} \frac{\rho_c(E, x) v_c(E, x) P(E)}{2\nu_c(E)} dE = \nu_0 \quad \{\text{I}\} \quad (12)$$

$$\begin{aligned} \nu(E_i, E_f, x) &\equiv \nu(x) = \nu_0 \int_{\Phi(x)}^{E_f} \frac{\rho_c(E, x) v_c(E, x) P(E)}{2\nu_c(E)} dE = \\ &= \nu_0 \frac{E_f - \Phi(x)}{\Delta E} = \frac{h\rho(x)^2}{8m} \quad \{\text{II}\} \end{aligned} \quad (13)$$

These equations allow us to calculate the drift velocity of the ensemble by the means of the general relation

$$2\nu(E_i, E_f, x) \equiv 2\nu(x) = \rho(E_i, E_f, x) v_D(x) \quad (14)$$

so that we find

$$v_D(x) = \frac{2\nu_0}{\rho(x)} \quad \{\text{I}\} \quad (15)$$

$$v_D(x) = \frac{h\rho(x)}{4m} = \frac{1}{2} v_c(E, x) \sqrt{\frac{\nu(x)}{\left(\nu(x) - \frac{E_f - E}{h}\right)}} \quad \{\text{II}\} \quad (16)$$

3 The mechanical energy theorem for the classical ensemble

Using equations (10), (11), (15) and (16) we find :

$$\frac{1}{2} m v_D^2 + \Phi(x) + \frac{h^2 \rho^2(x)}{32m} = \frac{E_f + E_i}{2} \quad \{\text{I}\} \quad (17)$$

$$\frac{1}{2} m v_D^2 + \frac{h^2 \rho^2(x)}{16m} - \frac{\Delta E}{2} + \Phi(x) + \frac{h^2 \rho^2(x)}{32m} = \frac{E_f + E_i}{2} \quad \{\text{II}\} \quad (18)$$

In this section we want to show that our equations can be fitted into a general form holding in both regions I and II. To this purpose, first note that

$$\langle K \rangle = \left\langle \frac{1}{2} m v_c^2 \right\rangle = \frac{1}{2} m v_D^2 + \langle K' \rangle =$$

$$\langle K \rangle = \int_{E_i}^{E_f} [E - \Phi(x)] P(E) dE = \frac{E_f + E_i}{2} - \Phi(x) \quad \{\text{I}\} \quad (19)$$

$$\langle K \rangle = \frac{\int_{\Phi(x)}^{E_f} [E - \Phi(x)] P(E) dE}{\int_{\Phi(x)}^{E_f} P(E) dE} = \frac{E_f - \Phi(x)}{2} \quad \{\text{II}\} \quad (20)$$

This is remarked because the following equation is worthy to be displayed (use equations (13)÷(16)) :

$$\langle K' \rangle = \frac{h^2 \rho(x)^2}{32m} \quad \text{Region } \{\text{I+II}\} \quad (21)$$

The quantity $\langle K' \rangle$ is the ensemble average of the particles kinetic energy in the frame of the drifting center-of-mass. Let us now consider the following expression K_{VMRE} (a comment about this quantity will be found in the next section):

$$K_{VMRE} = \int_0^x \left[\frac{m}{2} v_D^2 + \langle K' \rangle \right] d \ln \nu(x) = 0 \quad \{\text{I}\} \quad (22)$$

$$K_{VMRE} = \int_0^x \left[\frac{m}{2} v_D^2 + \langle K' \rangle \right] d \ln \nu(x) = \frac{h^2 \rho^2(x)}{16m} - \frac{\Delta E}{2} \quad \{\text{II}\} \quad (23)$$

Note that in the point x^* at the boundary between the region I and region II we have by continuity

$$\frac{1}{2}mv_D^2(x^*)|_{\text{I}} = \frac{1}{2}m\frac{4\nu_0^2}{\rho^2(x^*)} = \frac{1}{2}mv_D^2(x^*)|_{\text{II}} = \frac{h^2\rho^2(x^*)}{32m} \quad (24)$$

$$\frac{h\rho^2(x^*)}{8m\nu_0} = 1 \quad (25)$$

These last equations enlighten the continuation property by which the value of the integral K_{VMRE} in region II is matched to the value attained at the boundary of region I.

On the other hand, we note that

$$\langle E \rangle = \frac{\int_{E_i}^{E_f} EP(E)dE}{\int_{E_i}^{E_f} P(E)dE} = \frac{E_f + E_i}{2} \quad \{\text{I}\} \quad (26)$$

$$\langle E \rangle = \frac{\int_{\Phi(x)}^{E_f} EP(E)dE}{\int_{\Phi(x)}^{E_f} P(E)dE} = -\frac{h^2\rho(x)^2}{16m} + \frac{\Delta E}{2} + \frac{E_f + E_i}{2} \quad \{\text{II}\} \quad (27)$$

As is clear from the last equations, the ensemble average of the mechanical energy can be given by means of the general expression

$$\langle E \rangle = \frac{E_f + E_i}{2} - K_{VMRE} \quad \{\text{I+II}\} \quad (28)$$

Taking into account the previous equations, we finally find

$$\frac{1}{2}mv_D^2 + \int_0^x \left[\frac{m}{2}v_D^2 + \langle K' \rangle \right] d\ln\nu(x) + \Phi(x) + \langle K' \rangle = \frac{E_f + E_i}{2} \quad \{\text{I+II}\} \quad (29)$$

This equation also writes

$$\frac{1}{2}mv_D^2 + I_D(\nu(x), 0) + \Phi(x) = \frac{E_f + E_i}{2} \quad \{\text{I+II}\} \quad (30)$$

where the potential $I_D(\nu(x),0)$ is defined as

$$I_D(\nu(x),0) = K_{VMRE} + \langle K' \rangle \quad \begin{array}{l} \{I+II\} \\ (31) \end{array}$$

Equation (30) is the energy theorem general expression we will base upon to describe extended model cases, with generalized distributions and different potential forms, in the following. To have insight into a next development, note that the potential $I_D(\nu(x),0)$ will change into $I_D(\nu(x),\kappa)$ when the quantity κ (to be considered later) is assumed different from zero. This potential has been introduced already in papers I and II.

3.1 Theoretical analysis

The equations in the previous section must be commented here in a few details. This is also to acquire a conceptual basis to interpret the generalized framework in the following.

The quantity K_{VMRE} is originated by the fact that the mass-flow is variable (by construction) as a function of x (in the previous example, only in Region II). We will call it the " virtual mass reactive energy " (VMRE). The term describes the fact that the streaming particles across a section in x are found gradually reduced in number as far as x approaches the boundary of the space domain enclosing the system. The mass flow decreases indeed after that part of the molecules have attained their turning points. These points are obviously located, for each particle, at the boundary of the space-domain classically allowed to it by its own energy value E . The particles lost by a running stream will join the countermoving stream afterwards. This behavior of particles is in agreement with the stationary assumption.

We note, by the K_{VMRE} expressions (22) and (23), that it is a zero or negative quantity in the classical ensemble model. Consequently, the quantity $I_D(\nu(x),0)$ is also found able to assume negative values in some region of space (Region II). This is an important remark to be reminded when dealing with the corresponding generalized case in paper IV.

On a physical point of view, the appearance of the term $\langle K' \rangle$ in equation (29) and its precursors means that the internal kinetic energy of the ensemble of particles localized between x and $x+dx$ is an effective potential for the (one particle equivalent) drift translational energy

$(1/2)m\bar{v}_D^2$. It can be seen as the (virtual, because no forces effective between particles have been introduced up to now into the model) reactive effect, on the drifting group of molecules, due to their velocity dispersion. The statistical ensemble is constituted by particles which, one by one, follow the purely mechanical law of motion. Although their motions initial phases are distributed at random, no other thermal-like randomizing effect is included for now. Then we can say that the potential $\langle K' \rangle$ - at the present investigation stage - results from the effect of "mechanically ordered" motions, and that it is "purely mechanical" in its origin. Just to have insight into next developments, we remark here that in some cases when internal forces (i.e. pressure) will be accounted for within the framework, it will be found confluent into the standard expression of the indicated work (these cases are exemplified by the next equations (41) and (51)). We will refer to this occurrence by the term of pressure-induced thermalization of the potential (¹).

As is clear from our analysis, both the effects described by the term $I_D(\nu(x),0)$ only depend on the mechanical energy distribution inhomogeneity (represented by the quantity $\Delta E = h\nu_0$ in our equations), and do not require any internal force field to be active between particles in order to appear into the energy theorem. Therefore we called them "virtual". In the contrary case, pressure effects and/or induced disorder have to be taken into account in the framework. Then the energy distribution to the particles will become affected by collisions (or other disorder sources), so that a thermodynamic form should be adopted when full thermalization is attained. We are not going, however, to consider distribution forms. We are rather going to investigate the potentials forms. Concerning thermalization, we are interested to investigate intermediate cases, i.e. situations where mechanically ordered internal motions may - at a certain degree - survive the thermalization effects. Therefore, we propose the following analysis.

4 Transition from mechanics to thermodynamics

¹From a general point of view, "thermalisation" of the potential $\langle K' \rangle$ might also be caused by some additional feature of the external interaction not explicitly accounted for in these papers. Note, indeed, that in the generalised context exposed in a next section, the potential $\langle K' \rangle$ is not taken into account within the pressure constitutive equation (74), but is left included into the expression of the potential $I_D(\nu(x),\kappa)$ appearing in equation (87). Whatever the origin, however, pressure-independent thermalisation effects will make $\langle K' \rangle$ ineffective within the context of the mass-flow theorem. They can instead be responsible for a particle being able to jump across different trajectories in our theory (see paper IV).

Within the following sections, we want to show the way we can correlate the energy theorem we found in the classical ensemble case with a thermodynamic formalism. We start by introducing the microcanonical temperature concept and discussing different cases at the interpretative level.

4.1 *The microcanonical temperature*

Let us consider equation (21) again:

$$\langle K' \rangle = \frac{\hbar^2 \rho(x)^2}{32m} \equiv \frac{T_m}{2} \quad \{\text{I+II}\} \quad (32)$$

If $\langle K' \rangle$ resulted from a standard Maxwell-Boltzmann distribution of velocities or energies - i.e. if our system was canonical in a standard thermodynamic equilibrium state - then we would set this potential equal (in the uni-dimensional model) to half the temperature (Boltzmann's constant =1). Here the statistical ensemble is a microcanonical one, the velocity distribution is not at all a Maxwell one and we made clear already that the $\langle K' \rangle$ constitutive parameters correspond to a mechanically ordered situation - but by analogy with the thermodynamic case we will call T_m , the microcanonical temperature, the quantity $2\langle K' \rangle$. The microcanonical temperature is a peculiar parameter of the system energy distribution : for the distribution given in equation (1) we find, as expressed in equation (32), the temperature proportional to $\rho(x)^2$. The coefficient being $\hbar^2/16m$, we are dealing with a "quantum " quantity. Although our position may at present look quite a non-orthodox one, we find comfortable to use this temperature concept for the sake of comparison with standard properties and behavior of known physical configurations of systems. Note indeed - on one hand - that our microcanonical temperature will be found useful as a basic parameter to express state equations or other constitutive relations for the microcanonical ensemble, just in the same way we do with the absolute thermodynamic temperature in standard thermodynamics. On the other hand, instead, it is a quantity different and independent of the absolute temperature - this last is, obviously, the peculiar parameter of the canonical or grand-canonical energy distributions. Using the microcanonical temperature will bring us to a peculiar understanding of the ensemble properties we want to display. In order to appreciate this point we introduce the following sections.

4.2 The purely mechanical, classical interpretative case

The expression (30) holds for the case where no interactive forces are effective between the particles, so that it is the statistical appearance of a purely mechanical case - statistics being here inherent to the inhomogeneous energy broadening effect. In order to approach a thermodynamic formalism, we can write the same expression using the temperature definition (32) and the following remarks :

The quantity $\langle K' \rangle$ has the role of an Helmholtz energy F_0 .

No pressure is effective on the system at the present stage.

According to some general ideas first introduced by Boltzmann, also applied in our previous work [3], entropy can be defined even for a mechanical (non-thermalized) system. This fact allows us to fit our mechanical energy theorem into a thermodynamic formalism. We have to choose, however, a constant value S_0 in the present case. This is tantamount to insure that the particles flow described by equations (29),(30) is an "adiabatic" one.

To be consistent with standard requirements, we can write :

$$\langle K' \rangle \equiv F_0(x) = \frac{T_m}{2} = -T_m \ln z(T_m, V) \tag{33}$$

$$z(T_m, V) = \exp\left(-\frac{1}{2}\right) \equiv \text{const} \tag{34}$$

We also have

$$U = F_0 + T_m S_0 = 0 \tag{35}$$

$$P = 0 \tag{36}$$

U is the thermodynamic energy, the pressure is zero and S_0 is found to be

$$S_0 = -\frac{F_0}{T_m} = -\frac{1}{2} \tag{37}$$

Then we can write

$$\frac{1}{2}mv_D^2 + \int_0^x \left[\frac{m}{2}v_D^2 + F_0(x) \right] d \ln \nu(x) + \Phi(x) + F_0(x) = \frac{E_f + E_i}{2} \quad \{I+II\} \tag{38}$$

The ensemble of these equations features the "thermodynamic" interpretation which - limited to this section - we give to the mechanical energy theorem (29) and associated potentials. It is consistent with taking the microcanonical sum-over-state z as a constant. If our calculation had to evolve towards the classical canonical case, then accounting for a continuous spectrum of mechanical energy values $H_C \equiv [E_f + E_i]/2$ we would find

$$\int_0^\infty z(T_m, V) \exp(-H_C(p, x)/T) dH_C \equiv \text{const} \times T \quad (39)$$

This would be indeed the classical expression (two degrees of freedom case) but we have to note that in our framework it is only a local determination at some position x . By completeness, we should therefore integrate the expression over the x -co-ordinate domain so that the result would be

$$Z(T, V^*) = \int_{V^*} \int_0^\infty z(T_m, V) \exp(-H_C(p, x)/T) dH_C dx \equiv \text{const} \times TV^* \quad (40)$$

The result (40) is however incongruent both with classical physics and with our FEOM model in [3]. The reason is that in this section no account has been taken of the HDF effect. Therefore the previous equations will not be maintained in the following - becoming able to account for the HDF effect and showing congruence of the results is just our purpose. In order to proceed towards the relevant cases to us, we have yet to analyze the case when a pressure is introduced into our equations first. This is shown in the next section.

4.3 The thermalized case

Assume now interactive forces are introduced into our particles system. First, we want to introduce hard-shock, binary collisions. Since the shocks last a time equal to zero and our particles are all identical to each other, introducing such kind of collisions will only make the role of collision partners to exchange locally. No perturbation of the system statistical density or energy potentials will be remarked as a consequence of these collisions. Then our equations (29) and (30) will remain unchanged, and the included parameters will remain the same functions of

x in the equations themselves. Yet we can give different interpretative roles to the energy potentials. If we look indeed to the system as to a "thermalized" one now, then we can invoke the standard flow-of-mass theorem expression and interpret the potential $\langle K' \rangle$ as an indicated work (we are here in a pressure-induced thermalization case).

We can be more specific, and consider intermediate steps bringing the system from the purely mechanical ordered state to the thermodynamic state. In an intermediate situation, we can attribute to some fraction κ of the potential $\langle K' \rangle$ the role of an indicated work. At the same time, this potential $\kappa \langle K' \rangle$ can be thought equal to the thermalized energy fraction or thermodynamic energy U. That is why κ will be called "the thermalization constant". It is a variable parameter, useful to describe the rising of pressure with associated induced thermalization effects in the model. It can assume values between 0 and 1, so that we can set up interpolative expressions between the two extreme cases of a purely statistical-mechanical system configuration with no interactive forces ($\kappa = 0$, no pressure) and the thermodynamic equilibrium configuration ($\kappa = 1$). Then we will write :

$$\kappa \langle K' \rangle \equiv \int \frac{dP}{\rho} \equiv U = \frac{\kappa T_m}{2} \tag{41}$$

Since $T_m \propto \rho^2$ this equation implies

$$P \equiv \rho \beta \kappa T_m = \frac{1}{3} \rho \kappa T_m \tag{42}$$

This last is the state equation holding for the hard-shock, binary collisions case. The pressure is given by a perfect-gas-like expression $\rho \kappa T_m$ modified by a constant factor $\beta = 1/3$.

We also have

$$\langle K' \rangle = F_0(1 - \kappa) + \int \frac{dP}{\rho} = F_0(1 - \kappa) + U = F_0(1 - \kappa) + \frac{\kappa T_m}{2} \equiv \frac{T_m}{2} \tag{43}$$

Here we see that our microcanonical temperature concept starts to detach from the non-orthodox correlation to mechanically ordered motions we used to define it. When κ is taken equal to 1, both the quantities T_m and $\langle K' \rangle$ will attain their full physical meaning as thermodynamic potentials.

The energy theorem expression (30) now takes the upgraded form :

$$\frac{1}{2}mv_D^2 + I_D(\nu(x),\kappa) + \Phi(x) + U = \frac{E_f + E_i}{2} \equiv \text{Inv} \quad (44)$$

$$I_D(\nu(x),\kappa) = \overline{K_{VMRE}} + F_0(1 - \kappa) \quad (45)$$

On the other hand, we want to assume an entropy definition S for the case at hand, and this should be done by formally respecting a First Law expression :

$$-Pd\frac{1}{\rho} = dU - T_m dS \quad (46)$$

Then we obtain

$$U = \int \frac{dP}{\rho} = - \int Pd\frac{1}{\rho} + \int T_m dS = \frac{1}{2} \frac{P}{\rho} + \int T_m dS = \frac{\kappa T_m}{2} = \frac{3}{2} \frac{P}{\rho} \quad (47)$$

$$\frac{P}{\rho} = \frac{\kappa T_m}{3} = \int T_m dS \quad (48)$$

$$S = \frac{\kappa}{3} \ln T_m + S_0 = \frac{\kappa}{3} \ln T_m - \frac{1}{2} \quad (49)$$

We have furthermore

$$z(T_m, V) = T_m^{\kappa/3} \exp \left[-\frac{\kappa h^2 \rho(x)^2}{96mT} - \frac{\kappa}{3} - \frac{1}{2} \right] \equiv \rho(x)^{\frac{2\kappa}{3}} \times \text{const} \quad (50)$$

Yet this equation neither is congruent with known standard limit results. What is important to us, however, is following the evolution of the framework towards the final correct definition. This last will be shown in the next section to be the one where a HDF-dependent potential is taken existent. It will be the term giving congruence to the thermodynamic setting and interpretation associated with the mass-flow theorem expression we are investigating in our papers.

Now we proceed towards the general case where interactive forces generate every kind of collisions. This is described by the equation

$$\frac{1}{2}mv_D^2 + \Phi(x) + K_{VMRE} + F(\kappa) + \int \frac{dP}{\rho} = \text{Inv} \quad (51)$$

Here the quantities $F(\kappa)$, U and P can be whatever - depending on the constitutive relations we want to choose. To have more insight, suppose we want to define a (generalized) pressure - what is of main concern in this paper. This may be done by an extension of expression (42):

$$P = \rho\beta(\rho)\kappa T_m \quad (52)$$

where the factor β is taken to depend on the density. The temperature definition also might attain a generalized constitutive expression - although we will always refer to equation (32) in our present work.

Note that the term now named (by the sake of generality) $F(\kappa)$ can be interpreted as the residual potential - after (level κ) pressure-induced thermalization - due to some "degree of survival" of mechanically ordered motions as mentioned in a previous section (in practice, a residual fraction of $\langle K' \rangle$). It may look obvious that "full" pressure-induced thermalization, with $\kappa = 1$, should ordinarily cause this potential to disappear. Yet by clarity (and by the sake of general analysis) we want to note, here again, that setting up pressure into our system does not necessarily imply a disappearance of ordered motions. This is because the thermalization concept also requires, specifically, the assessment of a thermodynamic state of disorder. This last might not be fully consequent to the effect of interactive forces, and might require pressure independent thermalization to be established ⁽²⁾. Therefore, $\lim_{\kappa \rightarrow 1} F(\kappa)$ might not be zero in every circumstance. However, in our following framework, the comprehensive quantity $F(\kappa)$ will actually be found zero (but see comments to equation (59)). In the sequel, the potential $F(\kappa)$ will be included into the expression of the potential $I_D(\nu(x), \kappa)$ (as is shown, f.i., in the next equation (55)).

²As is clear, by the use of the parameter κ we actually account for pressure-induced thermalisation in our paper. Whenever it will be necessary to point out the role of a pressure-independent one, we may occasionally use an additional argument κ^* in the functions and/or give some comment in the text.

4.4 The case with HDF (microcanonical Bernoulli oscillators ensemble)

Our discussion has been limited until now to the purely classical case where the effective potential energy on the particles is $\Phi(x)$. If in the start equation (3) we add the HDF contribution this equation becomes

$$\frac{1}{2}mv_c^2 + \Phi(x) + \Phi_{\text{HDF}}(x,\xi) = E \quad (53)$$

We want to set up now a statistical ensemble based on equation (53). We will call it the microcanonical Bernoulli oscillators ensemble. To perform this task, our previous treatment might first be iterated for a single value of ξ , letting $\Phi_{\text{HDF}}(x,\xi)$ undetermined, and averaging over energy values. Actually, we are interested to the case when an average is performed over values of ξ . We remember, yet, from paper II of this work, that averaging over ξ -values has - in a first approximation - the same effect that averaging over energy. Therefore we understand that the average over ξ will result into the following energy theorem expression (pressure is inserted as well):

$$\frac{1}{2}mv_D^2 + \Phi(x) + \langle \Phi_{\text{HDF}}(x,\xi) \rangle |_x + I_D^*(\nu(x),\kappa) + \int \frac{dP}{\rho} = E \equiv \text{Inv} \quad (54)$$

$$I_D^*(\nu(x),\kappa) = K_{VMRE}^* + F^*(\kappa) \quad (55)$$

Here $\langle \Phi_{\text{HDF}}(x,\xi) \rangle |_x$ is an average over ξ -values performed at constant x , and the definition of the potential $I_D^*(\nu(x),\kappa)$ is analogous to what previously expressed. The asterisks on the quantities in equation (55) mean that they are calculated in presence of $\Phi_{\text{HDF}}(x,\xi)$. Actually, all the associated potentials, as well as quantities as pressure, drift velocity etc. in this section should be defined as effective quantities in presence of HDF. In the general case this will be tacitly understood, so that no other index will be displayed - for the specific case here, a further comment about this point is at the end of the section. The important topic now is that in presence of HDF we can assume the congruent sum-over-states expression given by $\rho(x)$, as found in paper I (equation (85)). Thus we can write - using a somewhat general formalism :

$$z(T_m, V \equiv \rho^{-1}) = \exp \left\{ \left[\int P d\frac{1}{\rho} \right] / T_m + \int \frac{dT_m}{T_m^2} \int T_m dS(T_m) \right\} \equiv \rho(x) \quad (56)$$

From this equation, we find the expression of the entropy $S(T_m)$:

$$S(T_m) = |_{T_m \propto \rho^2, \beta = \text{const}} = \frac{1 + \beta\kappa}{2} \ln T_m - \beta\kappa \ln \rho(T_m) + \text{const} = \frac{1 + \beta\kappa}{2} + \ln \rho \quad (57)$$

We also find the expression of the thermodynamic energy :

$$U = \left\{ T_m + \frac{P}{\rho} \right\} \frac{d \ln \rho}{d \ln T_m} = T_m \{1 + \beta\kappa\} \frac{d \ln \rho}{d \ln T_m} \quad (58)$$

$$U = |_{T_m \propto \rho^2} = \frac{\beta\kappa T_m(V)}{2} + \frac{T_m}{2} = \frac{1}{2} \frac{P}{\rho} + c_v T_m \quad (59)$$

The thermodynamic energy is found given by the sum of two terms - the first one is volume-dependent, while the second one looks as an (independently) thermalized part of it, with constant volume specific heat $c_v = 1/2$. In the present context, therefore, the potential $\langle K' \rangle$ is fully thermalized, independently of the κ value. On the interpretative level, this is an indication of the effectiveness of pressure-independent thermalization as we have signalized in previous circumstances. When we take, by equation (56), $z = \rho$ (what we have correlated in paper I to the HDF existence) for the sum-over-states expression, then we find that the potential $F(\kappa)$ is zero ⁽³⁾.

This situation will be found confirmed by the assessment of the following, more general context (exposed in a next section), where full expressions will be used - both for the HDF potential and the pressure - to expound our description of the full quantum case.

Comparing the case at hand with the previous equations (47) and (50), we see that the HDF presence changes drastically the constitutive expressions of the thermodynamic potentials - this situation even occurs when we take $\beta = 0$, which is tantamount to take the quantity

³Still we remark here that modifying, in the present context, the z expression into ρ^{κ^*} would bring us to a c_v -value equal to $\kappa^*/2$ and to a $F(\kappa)$ -value equal to $(1 - \kappa^*)T_m/2$. A very simple renormalisation of all our framework - inclusive of the thermodynamic model in paper I - would therefore allow us to describe the circumstance when partial (level κ^*) pressure-independent thermalisation has to be taken into account. Deepening into this extended description is, however, demanded to future work.

(⁴) $\tilde{\Phi}_{\text{HDF}}(\mathbf{x}, \xi) = 0$ (see next equations (101),(102) or (104),(105) to understand this point: they make clear that the pressure P is originated by HDF itself (⁵)). This case holds indeed when resonance ($\eta = 1$) is attained, so that HDF is ineffective (at the corresponding point in space) into the flow-of-mass theorem expression; but its presence is always signaled by the condition $z = \rho$ which dominates our equations in the present section. We conclude here that the HDF influence on the thermodynamic equations specifies into two independent effects: these last can be represented by the modification of the Gibb's distribution (as remarked in paper I) and of the thermodynamic energy constitutive expressions. Since into this last potential the term $\beta\kappa T_m/2$, which is strictly correlated to the potential P/ρ , is found introduced, we might also say that the distinguished potentials signaling the Φ_{HDF} presence are the Helmholtz energy and P/ρ . This result will be used later on.

By our equations, if we still take a pressure proportional to ρ^3 , the expression of the sum-over-states z for the Bernoulli oscillators ensemble at hand is finally found

$$z(T_m, \rho^{-1}) \equiv \rho(\mathbf{x}) \equiv T_m^{\frac{1}{2}} \exp \left\{ -\beta\kappa \frac{\hbar^2 \rho(\mathbf{x})^2}{32mT_m} - \frac{1}{2} \ln \frac{\hbar^2}{16m} + \frac{\beta\kappa}{2} \right\} \quad (60)$$

The necessary comment to this expression is that it is now - by construction - congruent with the generalized thermodynamic framework we introduced in paper I.

Writing this expression we have, by simplicity, neglected the contributions of the generalized variables directly associated to HDF, i.e. f , $\Delta\mathbf{x}$, g and k as introduced in paper I. The generalized formalism can be restored easily using equation (85) in paper I as well.

Note that the structure of the energy theorem has formally evolved throughout the last sections, according to the improvements we have made while giving interpretation to the different potentials involved in the precursory equation (30) up to the expression (54). Yet if we assume equations (42) and (45), the functional dependences of some key potentials involved ($T_m, P/\rho$, F_0 , I_D) on the density ρ , as well as the function $\rho(\mathbf{x})$ itself, can be maintained exactly the same than required

⁴This quantity can be found defined in equation (62).

⁵The particles interact with each other at their real positions $\mathbf{x}+\mathbf{x}_z$, and the pressure P is the resulting effect on the classical degree of freedom at the position \mathbf{x} .

by the previous equations (32) and (6). To this end, we also have to take the following assumption:

$$\langle \Phi_{\text{HDF}}(x, \xi) \rangle |_{\mathbf{x}} + \mathbf{I}_D^*(\nu(x), \kappa) = \mathbf{I}_D(\nu(x), \kappa) \quad (61)$$

This equation, holding in the Bernoulli oscillators ensemble case, originates from the following remarks. Our mixing procedure was first developed without accounting for Φ_{HDF} , i.e. precisely in the "purely classical" case (note that the Planck's constant was simply inserted in the first model by means of the assumption $\Delta E = h\nu_0$). Now the quantity $\langle \Phi_{\text{HDF}}(x, \xi) \rangle |_{\mathbf{x}} + \mathbf{I}_D^*(\nu(x), \kappa)$ (representing the appearance of the quantum field effects in the context) is instead accounted for. Equation (9), however, can be interpreted as the Bohr-Sommerfeld rule. We are brought to think that this second constitutive procedure for the statistical (Bernoulli oscillators) ensemble must bring to expressions for the energy theorem and potentials coincident with the ones we found in the former case. In other words, if we start again the ensemble constitutive procedure by adding to the potential $\Phi(x)$ a potential $\Phi_{\text{HDF}}(x, \xi)$, then we expect as a result the equation (54); but if we want to keep the same $\rho(x)$ holding, that was found in the classical ensemble case - because it respects the Bohr-Sommerfeld rule and is the most reliable expression at the Bernoulli oscillators stage too - then we are obliged to write equation (61). In this way the functional dependences of the mentioned quantities remain identical to the former case. We might say that the classical ensemble is just one showing a peculiar "degeneration" property: it is able to hid a quantum effect into the corresponding energy theorem and density expressions. By equation (61) we can find the expression for $\mathbf{I}_D^*(\nu(x), \kappa)$ if we have one available for $\langle \Phi_{\text{HDF}}(x, \xi) \rangle |_{\mathbf{x}}$.

From this section, stems indeed the requirement to find an appropriate expression for the quantity $\langle \Phi_{\text{HDF}}(x, \xi) \rangle |_{\mathbf{x}}$, first consistent with the present framework (this is done in the next section) - but, what is most important to us, able to evolve into the more general one. To this end, further investigation is now pursued .

4.5 Ergodicity and the factor 8

Consider equations (96), (102) and (104) in paper II of the present work:

$$\tilde{\Phi}_{\text{HDF}}(x, \xi) \approx \frac{\eta^2(x) - 1}{\eta^2(x)} \frac{1}{2} \text{m} \langle v_z^2 \rangle |_{\mathbf{x}} = \frac{\eta^2(x) - 1}{\eta^2(x)} \frac{h^2 \rho^2(x)}{4m} \quad (62)$$

For the present purposes, in this equation we only take into account the second term in the full Φ_{HDF} expression (this circumstance is expressed by the tilde). The first one will be discussed separately later on, and can be easily recovered in the final result.

In agreement with a previous remark in paper II, in this equation a classical-like density should be considered. Here, however, we are interested to find out an expression for the $\tilde{\Phi}_{\text{HDF}}$ statistical average, taken over an ensemble of ξ -values. We can exploit the remark we have made, that the result of such a kind of averaging should be found equivalent to the issue of an energy-averaging procedure. Therefore an "inhomogeneously energy broadened system density" can be considered to this purpose. Using equation (32) and taking first a high frequency (HF) limit ($\eta \gg 1$) in equation (62) we find :

$$\begin{aligned} \langle \lim_{\eta \gg 1} \tilde{\Phi}_{\text{HDF}}(x, \xi) \rangle |_{\text{x}} &\equiv \lim_{\eta \gg 1} \frac{1}{2} m \langle \langle v_z^2 \rangle \rangle |_{\text{x}} \approx \lim_{\eta \gg 1} \frac{1}{2} m \langle v_z^2 \rangle |_{\text{x}} \approx \\ &\approx \frac{\hbar^2 \rho(x)^2}{4m} \equiv 8 \frac{T_m}{2} \end{aligned} \quad (63)$$

As is clear, the averages are taken over ranges of ξ , η values. In writing down these equations, we used the same reasoning as we used already to find equations (101)÷(104) in paper II, and we will use in the sequel again. It is based on an "easy" technique : we estimate sometimes in our papers single-particle-relevant potentials, or quantities resulting from time-averaging, by the means of comparison with their counterparts resulting instead from an ensemble-averaging procedure (or vice-versa). Major discussion about this point must be provided, and will be found in the sequel. Using equation (63), an approximate expression for $\langle \tilde{\Phi}_{\text{HDF}}(x, \xi) \rangle |_{\text{x}}$ can be given :

$$\langle \tilde{\Phi}_{\text{HDF}}(x, \xi) \rangle |_{\text{x}} \approx 8 \langle \frac{\eta^2(x) - 1}{\eta^2(x)} \rangle |_{\text{x}} \frac{T_m}{2} \quad (64)$$

This expression is some sort of extrapolation we set up by attaching the most reasonable coefficient to the high-frequency limit expression. An improved $\langle \tilde{\Phi}_{\text{HDF}}(x, \xi) \rangle |_{\text{x}}$ expression will be found in the following equation (104) and further comment can be found in the next section, but we want to use expressions (62)÷(64) to give some discussion here.

To find these equations we used the correlations between $\langle x_z^2 \rangle|_x$, Δx^2 and $\rho(x)^2$ displayed in paper II. We can interpret equation (63) saying that when we estimate the mean kinetic energy pertaining to the fast oscillating quantum displacement (or HDF co-ordinate) as discussed in paper II, we find it eight times greater than expected - in the assumption it should be comparable to $T_m/2$. This last assumption might be made if we think to the HDF kinetic-part energy, in the high-frequency limit, as to that pertaining to a thermalized, single degree of freedom - just in agreement with the concept expressed by the equipartition principle in classical thermodynamics. Then we conclude that the HDF kinetic-part energy content is almost one order of magnitude greater than expected if it had to be replenished by the simple effect of a classical statistics. This fact can be considered anyway not surprising, because we attribute the HDF excitation to the mechanical, external quantum field action. At the same time, since in the limit $\eta \gg 1$ the quantity $\frac{1}{2}m\langle v_z^2 \rangle|_x$ equals $\tilde{\Phi}_{\text{HDF}}$, we see that the HDF mechanical action on the residual slow motion part of the particle is a factor 8 times greater than expected by the simple energy broadening effect which is the classical ensemble peculiarity. The interesting thing, here, is that our quantities are found different by a brute factor 8 but are identical as concerns their constitutive expressions, depending on parameters h , m and $\rho(x)$. If we do not care too much about the 8 here, we may say that some "ergodicity" seems to apply. The single-particle, time-averaged quantities turn out proportional to the corresponding statistical ensemble averages. Yet these respective averages are the attributes of distinguished co-ordinate parts (the HDF and the classical degree of freedom co-ordinates) in our model. This fact, however, can be easily understood considering that a key feature of quantum mechanics is just that the particle momentum incertitude Δp is indeed of the same order of the momentum p itself.

Some comment more about our invoked "ergodicity" concept will be found in the next section. Now we have to make some other remarks.

It is clear that if f.i. $\eta^2 = 8/7$ our expression for $\tilde{\Phi}_{\text{HDF}}$ becomes of the order of $T_m/2$ and we can say in this case that $\tilde{\Phi}_{\text{HDF}}$ is the equivalent of a thermalized degree of freedom. Our previous result will however apply to the kinetic energy HDF content $\frac{1}{2}m\langle v_z^2 \rangle|_x$. The major interest for the case $\eta \gg 1$ in this section is because, by the means of equation (63), we have a more reliable expression for $\tilde{\Phi}_{\text{HDF}}$ to discuss than by equation (64).

A distinguished interpretation we can give to the peculiar case with

$\eta^2 \approx 8/7$ is that the Bernoulli oscillators statistical model is equivalent, by "ergodicity", to the case of a particle in a classical potential $\Phi(x)$, driven by a fluctuation field which can be represented by a single pulsation $\omega(x)$ given by the equation

$$\omega(x) \approx \frac{2\hbar\rho(x)^2}{m} \quad (65)$$

In the present context, the expression (65) is a statistical estimation so that $\rho(x)$ should be intended as an "inhomogeneously energy broadened system density" or a quantum-like one. The same expression can be usefully compared to the expression (118) given in paper II - i.e. to the expression we assume to hold for the single-particle case.

The quantum field amplitude might also be estimated considering forced oscillator expressions in the hypothesis of single-frequency excitation, once an expression for the function $\alpha(\xi)$ is assumed. We do not deepen here into this analysis, and only note conclusively that the quantum field determines the HDF kinetic energy content, which last turns out to be of the order of $\hbar\omega(x)$:

$$\frac{1}{2}m\langle v_z^2 \rangle |x \approx \frac{\hbar^2\rho(x)^2}{4m} \equiv \frac{\pi}{4}\hbar\omega(x) \equiv 8\frac{T_m}{2} \quad (66)$$

5 Interpretation of results and investigation strategy

Those who appreciate such a kind of correspondences, will easily note that a factor 8 can be excerpted from the Bohr-Sommerfeld rule when we use it to express the incertitude relation between Δx and Δp . This is shown in the following equations:

$$\begin{aligned} \left| \oint p_f(x) dx - \oint p_i(x) dx \right| &\approx |\langle |p_f(x)| \rangle - \langle |p_i(x)| \rangle| \langle \oint |dx| \rangle \approx \\ &\approx |\langle |p_f(x)| \rangle - \langle |p_i(x)| \rangle| 2(x_{z0i} + x_{z0f}) \approx h \end{aligned} \quad (67)$$

Let us set

$$\Delta x \approx \frac{(x_{z0i} + x_{z0f})}{\sqrt{2}} \quad (68)$$

We can write, furthermore

$$\Delta p \approx | \langle |p_f(x)| \rangle - \langle |p_i(x)| \rangle | \approx m\omega\Delta x \quad (69)$$

These equations are consistent with some corresponding relations used in paper II. Using the previous equations we obtain

$$A^2 \equiv (\Delta p \Delta x)^2 \approx [\langle |p_f(x)| \rangle - \langle |p_i(x)| \rangle]^2 \frac{(x_{z0i} + x_{z0f})^2}{2} \approx \frac{h^2}{8} \quad (70)$$

Comments about this equation will be found accommodated into the following interpretative framework.

We are at an investigation stage which, to resume, can be described as follows: when we quit the single particle classical mechanics and introduce the classical/Bernoulli oscillators ensemble models with energy distribution given in equation (1), then the particle density modifies into a statistical pattern $\rho(x)$ which looks similar to a quantum-mechanical one. Although the pattern does not display intermediate zeros, indeed, it shows a behavior analogous to some sort of tunnelling, because the statistical ensemble is able to flow out of the classical region of space which would be reserved to particles having the mean statistical energy $(E_f + E_i)/2$ (see equation (18) and fig.(1)). At the same time, a property identical to the Bohr-Sommerfeld rule is found displayed in equation (9) - although it first explains in our framework as a normalization condition. The result (70) shows us that, at this level, a quasi-classical interpretation would introduce statistical incertitudes Δx and Δp such that an action $\Delta p \Delta x$ is first seen to emerge with a representative value $h/\sqrt{8}$. Then an interesting correspondence between the Bernoulli oscillators ensemble case and the quasi-classical quantum mechanical case is in the fact that the microcanonical temperature value assumed in equation (32) is found 8 times smaller than expected on the basis of the full formulation of Heisenberg's relation.

These considerations still will hold when a hard-shock, "binary" pressure with associated (virtual) thermalization is introduced into the model, or moreover when a HDF effect is added to the potential energy $\Phi(x)$ - provided $\langle \Phi_{\text{HDF}}(x, \xi) \rangle_x$ respects the equation (61).

We think therefore that the microcanonical ensemble of particles we have set up as expressed in equations (8), (17) and (18) is strongly - if only in a first, quasi-classical approximation - correlated to a quantum mechanical configuration.

The microcanonical Bernoulli oscillators ensemble, based on a collection of classically-moving particles submitted to the potential $\Phi(\mathbf{x}) + \Phi_{\text{HDF}}(\mathbf{x}, \xi)$, is a good - although rough - representative of the physical model and interpretation we want to promote in our papers. This is showing that the quantum physics is the formulation of statistical properties of systems whose background behavior is Newtonian (driven by an excitation source called the fluctuation or quantum field, from which $\Phi_{\text{HDF}}(\mathbf{x}, \xi)$ results). The model expounded till now is the prototype one, submitted to the simple energy distribution given in equation (1), able to approach quantum mechanical properties. It looks to substantiate a quasi-classical approximation where the quantum action A effectiveness may be estimated equal to $\hbar/\sqrt{8}$.

We have to remark, however, that the important matter we are facing when attempting a classical description of quantum phenomena is that properties such as the ones we are discussing here - and most important the tunnelling capability - cannot simply be considered as the properties of a statistical ensemble of particles, but must result from the (statistical) behavior of every single particle. In the previous paper II of this paper, we remarked that values of η smaller than unity lead to negative values for $\tilde{\Phi}_{\text{HDF}}(\mathbf{x}, \xi)$, so that tunnelling is recognized indeed in our theory as an effective issue for the physical behavior of single particles.

Our investigations throughout, we meet with the problem of excerpting from the many-particles ensemble case (particles are all identical in our frame) the single-particle behavior and vice-versa. We want to make clear that some steps of our analysis are just carried on by the means of comparisons between time-averaged, single-particle-defined quantities and their statistical-average counterparts. As specific examples, we have used the technique to identify the mechanical oscillation high-frequency limit with the statistical result effective in a many-particles, pressurized system; we use a similar procedure to calculate an effective velocity for momentum transfer in the next equation (71). Using these procedures arises the question how reliable the expressions for $\Phi_{\text{HDF}}(\mathbf{x}, \xi)$ or other quantities which we find are.

Concerning this point, we have the following remarks. We do not use at all in this paper a proper definition of what is called "ergodicity" (see [4] for a handy review) - but use this word when commenting on the previously quoted equivalence. This is simply because the concept is able to illustrate, very peculiarly, our feeling about some physical behavior which we want to bring to evidence. More specifically, we simply

call for ergodicity when we state comparable some ensemble- and time-averages. Our invoked ergodicity provides sometimes rather conjectural results, as occurs in the case of equations (62)÷(66). Waiting for more rigorous analysis, the reader may appreciate or criticize the procedures we use; but we have to note that many of these are only useful to us as indicators - just helpful to find, at each stage, the further investigative steps. These last will take, at the subsequent stage, a rather independent formulation. In this way, we are brought to the key equations (1)÷(14) in paper IV, and these equations with associated interpretation will be found to constitute the basic, conclusive thesis we advance - on quite an independent ground with respect to the previous partial results we have stated before.

By the previous analysis, we are brought to think that statistical ensembles different from the prototype here presented might exist, for which the factor $1/\sqrt{8}$ may transform into unity, the potential $\langle K' \rangle$ takes higher values and the full quantum mechanical case is attained. Here we understand that a very interesting investigation strategy is indicated to us by this conceptual result. As is clear at a glance, however, attempting to find out detailed properties of such ensembles might reveal a very hard matter to be accomplished. This is, first, because the good starting point should not be simply taken as the purely classical case, but detailed account should be taken of the mechanical and statistical influence of HDF into the single-particle energy theorem. Second, because the procedure would imply investigating the effects of many different distributions, including the hidden parameters, up to the end of finding the resulting energy theorem coincident with the expression of the Schrödinger equation.

Whether such an investigation strategy might be pursued by analytical means we do not know at present, but we let the problem of finding generalized forms for ξ -, η - or E-distributions apart and follow, in this paper, a different investigation path. This has been advanced in papers I and II of the work. It consists in determining generalized state equations for a particle system, with pressure or not - then writing down the corresponding expression of the mass-flow theorem and comparing it with the quantum-mechanical hydrodynamic equation. By this comparison, an expression for the single-particle effective HDF energy potential will be excerpted, so that a mechanical energy theorem expression, in Newtonian form, will be found. These investigation steps will be found deployed throughout the next sections.

6 Generalized state equations

A generalized form for the pressure constitutive equation (52) has been introduced already in the present context, and will be specified for the full quantum case we are going to investigate in the next sections. Moreover, we want to introduce a few other equations which we have found by correlating different system parameters, as the elastic function k_0 with the associated pressure g and the $\Phi_{\text{HDF}}(x, \xi)$ expression itself with the pressure P .

6.1 The P -state equation in the full quantum case

We need here to define a pressure field $P(x)$ consistent with a quantum mechanical framework. To do this, we note that in quantum mechanics the momentum field $p_q(x) = -i \hbar \nabla \Psi / \Psi$ is encountered ($\nabla \equiv \frac{d}{dx}$), where $\Psi(x)$ is a wave-function (here taken real so that it is equal in turn to $\pm \sqrt{\rho}$). Unless otherwise specified, $\rho(x)$ will be a quantum-mechanical density in the following. Now a pressure in our uni-dimensional model is dimensionally a force, and can be obtained by a constitutive equation where the characteristic particles velocity v_z^* is multiplied by a fraction $\gamma^* \kappa$ of the quantity $\nabla p_q(x)$. The coefficient $\gamma^* \kappa$ accounts in our framework for the effective coupling between the quantum momentum field and the collision regime, so that $\kappa \gamma^* v_z^* \nabla p_q dx$ is the statistical average of the momentum deposited by the effect of collisions between particles with velocity v_z^* in the space extension dx . Now we take v_z^* equal to the following time-averaged expression (see equation (63)):

$$\begin{aligned} v_z^* &= \langle |v_z| \rangle_x = \frac{\omega}{\pi} \int_0^{\pi} \sqrt{2 \langle v_z^2 \rangle_x} |\sin \omega t| dt = \\ &= \frac{\omega}{\pi} \int_0^{\pi} \sqrt{\frac{16 T_m}{m}} |\sin \omega t| dt = \frac{8}{\pi} \sqrt{\frac{T_m}{m}} = \frac{2}{\pi} \sqrt{\frac{\hbar^2 \rho(x)^2}{m^2}} = \frac{4 \hbar \rho(x)}{m} \end{aligned} \quad (71)$$

This expression accounts for the HDF motion of particles. We write:

$$P = \text{Im} \{ \kappa \gamma^* v_z^* \nabla p_q(x) \} = -\kappa \gamma^* \frac{2 \hbar^2 \rho(x)}{m} \left(\frac{\rho'}{\rho} \right)' \quad (72)$$

This equation provides our definition for the pressure of the system. It has been clearly obtained by generalizing the standard definition of pressure for the case of perfect gases. Now if we take the coefficient γ^* just equal to $1/8$ we find

$$\frac{P}{\rho} = -\kappa \frac{\hbar^2}{4m} \left(\frac{\rho'}{\rho} \right)' \quad (73)$$

This expression is coincident (for $\kappa = 1$) with the one found within various hydrodynamic models [5 – 7]. Equation (73) also writes

$$\int \frac{dP}{\rho} = -\kappa \frac{\hbar^2}{2m} \frac{\sqrt{\rho}''}{\sqrt{\rho}} = \kappa U_B(x) \quad (74)$$

The choice $\gamma^* = 1/8$ has brought us to an expression for the indicated work coincident - for $\kappa = 1$ - with the Bohm potential expression $U_B(x)$ one is used to find into the Madelung formulation of the Schrödinger equation. This last can be written, using the pressure P :

$$\frac{\nabla S^2}{2m} + \Phi(x) + \frac{1}{\kappa} \int \frac{dP}{\rho} = E_n \quad (75)$$

$$\rho \nabla S = \text{const} \quad (76)$$

The associated quantum mechanical wave-function is $\Psi(x) = \pm \sqrt{\rho} \exp(iS(x)/\hbar)$, with $S(x) \equiv$ phase function, also entering the continuity equation (76).

It is very interesting to consider this equation in the special case $\kappa = 1$, because it shows a peculiar way to look at the wave-mechanical equation as to a simple expression of the mass-flow theorem.

Once the pressure is known, we can find the thermodynamic energy and entropy easily by equation (56). If we keep the temperature $T_m \propto \rho^2$ as given by equation (32), then we find the results (note that $\beta(\rho)$ is a variable function now)

$$S = \frac{1}{2} + \frac{1}{2} \frac{P}{\rho T_m} + \ln \rho = \frac{1}{2} - \kappa \frac{1}{2\pi^2 \rho^2} \left(\frac{\rho'}{\rho} \right)' + \ln \rho \quad (77)$$

$$U = \frac{1}{2} \left[T_m + \frac{P}{\rho} \right] \quad (78)$$

Note here, in agreement with some previous remarks, that the term $T_m/2$ can always be interpreted as the appearance of the pressure-independent-fully-thermalized potential $\langle K' \rangle$.

To these equations, we can add the full expression of the thermodynamic potential given in equation (85), paper I (with $f = 0$), accounting for the variable k :

$$\begin{aligned} \psi &= -\ln z(T_m, \rho^{-1}, k) = \\ &= -\ln \sqrt{\frac{h}{\{m\omega\rho^{-2}\}}} + h(T_m) - \frac{\int P d\rho^{\frac{1}{2}}}{T_m} - \frac{\{PV\} + G_0(T_m)}{T_m} \ln \frac{k}{k_0} \equiv -\ln \rho \end{aligned} \quad (79)$$

These equations contribute to the thermodynamic set characterizing our full quantum case. We are going however to investigate it with further details in the following sections (concerning thermodynamics, we have still to calculate some other quantities as $G_0(T_m)$ and $k_0(\rho)$. This will be done in the following).

Note here finally that in our treatment, the coefficients 4,8,16 are recurrent. They appear to imprint equations (32), (63), (70), (73), the following ones (103)÷(106) and our overall treatment. We have the opinion that these coefficients are statistical weights correlating the present uni-dimensional model to a three-dimensional framework (8 = number of octants). This is in agreement with the remark that quantum mechanics is properly a description of three-dimensional reality, as is dictated by the nature of the Planck's constant h - an angular momentum.

6.2 A generalized property of the pressure and the g-state equation

Consider again the general expression (52) for the P-state equation:

$$P = \rho\beta\kappa T_m \quad (80)$$

Here, yet, we will still be exploiting the (previous) Bernoulli ensemble case, so that the factor β will be taken constant. The important matter we will next infer from this equation will however be found independent of this constant value.

In this context we can write

$$\frac{P}{\rho} = \beta\kappa T_m = 2\beta\kappa \langle K' \rangle \equiv 2\beta\kappa \frac{h^2 \rho(x)^2}{32m} \equiv c\kappa\rho(x)^2 \quad (81)$$

The quantity c is a constant whose definition is clear from the equation itself. We have (also for later use) :

$$\int \frac{dP}{\rho} = -3 \int P d\frac{1}{\rho} = \frac{3}{2} \frac{P}{\rho} = \frac{3}{2} c\kappa\rho^2 \quad (82)$$

Now we define the quantity $G(P/\rho, \rho, \kappa)$:

$$G(P/\rho, \rho, \kappa) = \frac{5 - \kappa}{\kappa} \int \rho^{\alpha(\kappa)} d\left(\frac{P}{\rho} \rho^{-\alpha(\kappa)}\right) + f(\rho) \quad (83)$$

Here we let the function $f(\rho)$, at present, undetermined. The expression of $\alpha(\kappa)$ is

$$\alpha(\kappa) = \frac{15 + \kappa}{5 - \kappa} \quad (84)$$

Using the expression for P/ρ given in equation (81), it is easy to check the validity of the following equation :

$$G(P/\rho, \rho, \kappa) - G(P/\rho, \rho, 0) = - \int \frac{dP}{\rho} \quad (85)$$

This equation will be found not limited to our "prototype" ($\beta = \text{const}$) case, but will assume the role of a generalized property of the pressure in our paper. One can easily check that it is indeed satisfied by our general definition of pressure, equation (73), too. It will be exploited in the following section. It will be shown there that the function $G(P/\rho, \rho, \kappa)$ can be identified with the quantity $\int k_0(\rho, \kappa) dg(\rho, \kappa)$ introduced in paper I of this work. For this reason, taking into account the $G(P/\rho, \rho, \kappa)$ expression (83) we will also be able to write

$$g(\rho, \kappa) k_0(\rho, \kappa) = \frac{5 - \kappa}{\kappa} \frac{P}{\rho} + k_0 \int \frac{df}{k_0} \quad (86)$$

This will be called the system g-state equation.

6.3 The matter-wave interference experiment and the g-state equation

Here we start with equations (93) and (94) established in paper I :

$$\begin{aligned} \frac{1}{2}mv_D^2(\rho, \kappa) + \langle \Phi_{\text{HDF}}(x, \xi_i(x)) \rangle |x + \int k_0(\rho, \kappa) dg(\rho, \kappa) + I_D(\nu(x), \kappa) + \\ + \Phi(x) + \int \frac{dP}{\rho} = \text{Inv} \end{aligned} \quad (87)$$

$$\begin{aligned} \frac{1}{2}mv_D^2(\rho, 0) + \langle \Phi_{\text{HDF}}(x, \xi_i(x)) \rangle |x + \int k_0(\rho, 0) dg(\rho, 0) + \\ + I_D(\nu(x), 0) + \Phi(x) = \text{Inv} |_{P=0} \end{aligned} \quad (88)$$

Among other thing, in these equations we indicate that the quantities v_D , k_0 , g themselves depend on the density and on the thermalization constant κ signaling the pressure "level". We want to treat the full, stationary quantum case, so that we have taken the quantity $f \rightarrow 0$ as remarked in paper I. The framework in paper I calls for a isentropic transformation in the variable k - when this last is calculated along the isentropic we called it k_0 and the same holds in the present context. The corresponding index 0 for g has instead been dropped off for the sake of simplicity. Now we introduce the following and most important remark. Equation (87) represents the expression of the mass-flow theorem holding for an ensemble of particles interacting with each other - this means that we have some pressure in the system so that the indicated work term is effective into the equation. Equation (88) gives instead the corresponding expression holding for essentially the same ensemble, but no pressure is effective between the particles - this means that the equation simply describes the statistical behavior assumed by many non-interacting particles. By the experiment on matter waves interference, we know that the quantum density pattern exhibited as the result of many particles flowing together (these are likely to exercise pressure on each other) is the same that exhibited as the final result of many single-particles individually flowing across the same space, one after one. In

other words, a key feature of quantum physics is that the collective behavior of many particles together is equivalent to that totalized by the iterated action of many of them, although they are at each time effective as singles. On this peculiar behavior the matter-wave concept is based, because each particle is able to behave following interference laws even when it is isolated from the others. Our physics of the "Bernoulli" oscillators, therefore, turns out to promote the appropriate name to indicate this property. Now it is precisely this reported remark which we want to exploit here, to state the concept that equations (87) and (88) must be identical to each other, because the energy theorem expression holding for a single-particle cannot be different from the many-interacting particles expression. This is also tantamount to say that the Inv constitutive expression is independent of κ . Using the quoted equations we obtain :

$$\begin{aligned}
 \frac{1}{2}mv_D^2(\rho,\kappa) + \int k_0(\rho,\kappa)dg(\rho,\kappa) + I_D(\nu(x),\kappa) + \int \frac{dP}{\rho} = \\
 = \frac{1}{2}mv_D^2(\rho,0) + \int k_0(\rho,0)dg(\rho,0) + I_D(\nu(x),0)
 \end{aligned} \tag{89}$$

Equation (89) can be split into the following two ones

$$\int k_0(\rho,\kappa)dg(\rho,\kappa) - \int k_0(\rho,0)dg(\rho,0) = - \int \frac{dP}{\rho} \tag{90}$$

$$\frac{1}{2}mv_D^2(\rho,\kappa) + I_D(\nu(x),\kappa) = \frac{1}{2}mv_D^2(\rho,0) + I_D(\nu(x),0) \tag{91}$$

The first one must be compared to equation (85). In the previous paper I, we have seen that the quantity k_0 is an elastic function pertaining to the hidden degree of freedom, and can be calculated here as a function of density using the indetermination principle in the form (116) given in paper II - in the high-frequency limit at least. Since we have calculated the relevant frequency already (see f.i. equation (65)), and we can think the (high-frequency) zitterbewegung model confluent into a situation with $\kappa = 1$, we have

$$k_0(\rho, 1) = m\omega^2(\rho, 1) = \frac{(2h)^2 \rho(x)^4}{m} \tag{92}$$

This is an expression of the statistical elastic function relevant to our system for the case $\kappa = 1$. Now we remark that equation (90) is of the same form of equation (85). This last equation has first been written for the case when $P=c\kappa\rho^3$ (equation (81)) but since it holds for the quantum pressure (73) as well, we find easily

$$\int k_0(\rho,\kappa)dg(\rho,\kappa) = G(P/\rho,\rho,\kappa) = \frac{5-\kappa}{\kappa} \int \rho^{\alpha(\kappa)} d\left(\frac{P}{\rho} \rho^{-\alpha(\kappa)}\right) + f(\rho) \quad (93)$$

From this equation, it is inferred that

$$k_0(\rho,\kappa) = m\omega^2(\rho,\kappa) = \frac{(2h)^2 \rho_0^{[4-\alpha(\kappa)]} \rho(x)^{\alpha(\kappa)}}{m} \quad (94)$$

where ρ_0 is a characteristic constant parameter, and

$$gk_0 = -\frac{P}{\rho} + \frac{5}{\kappa} \frac{P}{\rho} + k_0 \int \frac{df}{k_0} \quad (95)$$

These equations give the generalized expressions for the elastic function $k_0(\rho)$ and correlated g-state equation, respectively.

We introduce here some comment about these two results.

If we insert the expression (94) for the HDF oscillation frequency into the Heisenberg constraint equation (116) in paper II, we are brought to an expression for $\langle mv_z^2 \rangle \propto \rho^{\alpha(\kappa)/2}$. For $\kappa \neq 1$, this expression behaves differently from the quantity $T_m \propto \rho^2$. We might say that in this case we have not "ergodicity", and note that the P-state equation (73) (which must be maintained even for $\kappa \neq 1$) shall now be definitely based on the use of the statistical ensemble average T_m , also appearing in the primary equation (71).

Concerning equation (95), this last can be usefully compared with the corresponding expression we found in paper I (equation (47)) for the FEOM model case. We have to take care, however, that the FEOM model is a purely thermodynamic one, intended for a canonical ensemble with $\kappa = 1$, $T \equiv$ absolute temperature etc. Here note that in a next section it will be found that the function $f \approx \rho^{-2}$ so that the explicit expression for gk_0 turns out to be

$$gk_0 = -\frac{P}{\rho} + \frac{5}{\kappa} \frac{P}{\rho} + \frac{2}{\alpha(\kappa) + 2} f(\rho) \quad (96)$$

Using equation (47) in paper I we would find by analogy

$$G_{0m}(T_m) = -PV - gk_0 = -\frac{5}{\kappa} \frac{P}{\rho} (T_m) - \frac{2}{\alpha(\kappa) + 2} f(\rho(T_m)) \quad (97)$$

Here the Gibbs potential $G_{0m}(T_m)$ is a microcanonical correspondent of the quantity defined in paper I. This expression may be useful for next extensions of our model.

In this section, some comment has been made about the wave behavior of particles. Major comment will be found in a next section.

Concerning equation (91), it will be revisited and discussed in the following paper IV.

6.4 The HDF-state equation

Let us consider here again equation (64) :

$$\langle \tilde{\Phi}_{\text{HDF}}(x, \xi) \rangle |x \approx 4 \left\langle \frac{\eta^2(x) - 1}{\eta^2(x)} \right\rangle |x T_m = \frac{h^2}{4m} \left\langle \frac{\eta^2(x) - 1}{\eta^2(x)} \right\rangle |x \rho^2(x) \quad (98)$$

As remarked already, this is only an approximate expression we could set up by rough induction - because it is able to display the correct high-frequency limit expression; but it lends itself to handling for improvements. Using the state equation

$$\frac{P}{\rho} = \beta \kappa T_m \equiv c \kappa \rho(x)^2 \quad (99)$$

we find

$$\beta \kappa \left\langle \frac{\eta^2(x)}{\eta^2(x) - 1} \right\rangle |x \langle \tilde{\Phi}_{\text{HDF}}(x, \xi) \rangle |x \approx 4 \beta \kappa T_m = 4 \frac{P}{\rho} \quad (100)$$

Note that, if the assumption $\beta = \text{const}$ is abandoned and a variable factor $\beta(\rho)$ is considered for generality, we may have the case

$$\beta(\rho(x)) = \left\langle \frac{\eta^2(x) - 1}{\eta^2(x)} \right\rangle |x \quad (101)$$

This expression for β is interesting to us - first because in the high-frequency limit we expect $\beta \rightarrow 1$ indeed, and second because it makes the pressure only dependent on $\langle \tilde{\Phi}_{\text{HDF}}(x, \xi) \rangle|_x$. We have:

$$\kappa \langle \tilde{\Phi}_{\text{HDF}}(x, \xi) \rangle|_x \approx 4 \frac{P}{\rho} \quad (102)$$

From a previous remark, however, we remember that one should be able to express the $\tilde{\Phi}_{\text{HDF}}(x, \xi)$ action by the means of two thermodynamic potentials, which should be strongly correlated to the P/ρ potential and to the Helmholtz energy. Now this last can be represented by the quantity $\int P d(1/\rho)$. In the prototype case with $P \propto \rho^3$ we take into account equation (82) and are able to fit the pressure expression into the following one:

$$4 \frac{P}{\rho} = c_1 \frac{P}{\rho} - c_2 \int P d \frac{1}{\rho} = \left[c_1 + \frac{c_2}{2} \right] \frac{P}{\rho} = -4 \frac{P}{\rho} - 16 \int P d \frac{1}{\rho} \quad (103)$$

Here we have introduced a linear combination of the relevant potentials with coefficients c_1 and c_2 , and chosen moreover $c_2 = c_1^2$ - what finally brings to the values displayed in the last term.

At the present investigation stage, we are brought to the mentioned choice by the requirement that the coefficients are correlated to the number of octants 8, and by some inspection of the mathematical relations implied into the constitutive expression of the Bohm potential via the pressure (see equations (73) and (74)); yet we cannot "demonstrate" here that it is the good choice. It can just be said, at present, the fortunate one: in the following paper IV we will instead demonstrate - by an independent analysis - that the general expression for $\langle \tilde{\Phi}_{\text{HDF}}(x, \xi) \rangle|_x$, argued here already, is indeed, definitely:

$$\langle \tilde{\Phi}_{\text{HDF}}(x, \xi) \rangle|_x = -\frac{4}{\kappa} \frac{P}{\rho} - \frac{16}{\kappa} \int P d \frac{1}{\rho} \quad (104)$$

This equation shows that there is a sound correlation between Φ_{HDF} and the pressure P . It can be viewed and referred to as the HDF-state equation.

We want to remark that some of the equations ((81)÷(82) and (98)÷(103)) displayed until now in this section reflect the specific properties of our prototype model, with $P \propto \rho^3$ and $T_m \propto \rho^2$ (concerning equation (32), it is assumed by us to hold in the general case however).

All the other equations appearing in the section, from (71) to (107) - except the quoted ones - will instead be found to survive at the end of the (independent, and more general) analysis given in the next paper IV.

Equation (104) has been said already the HDF-state equation. By the peculiar way we have found it, we can argue that it is independent of the high-frequency approximation. This opinion will be consolidated by the analysis in paper IV; but concerning this specific point we have to remark that further analysis - to ascertain the proper validity domain of the expression - is out of the possibilities of this work, and is still waited for.

Actually, one might prefer to look at equation (104) (reversed) as just the constitutive relation for P, which can be written

$$P = -\frac{\kappa}{4}\rho^5 \int \rho^{-4} d\langle \tilde{\Phi}_{\text{HDF}}(x,\xi) \rangle |_{\text{x}} \quad (105)$$

As noted before, this equation directly shows that the pressure in a quantum system is "transmitted" to the classical degree of freedom by HDF.

Equations (98)÷(105) have however been written by using only the $\tilde{\Phi}_{\text{HDF}}$ part of the potential. In order to have a complete expression for $\langle \Phi_{\text{HDF}}(x,\xi) \rangle |_{\text{x}}$, we have still to add to the expression (104) a term associated to the mass effect we encountered in paper II. At present we will simply name this term as $-f(\rho)$, thus identifying this function with the one introduced already in equation (83). By inspection of the following equations the consistency of the procedure is easily recognized. An explicit expression for $f(\rho)$ will be found in paper IV so that the general expression relevant to this section can now be written

$$\langle \Phi_{\text{HDF}}(x,\xi) \rangle |_{\text{x}} = -\frac{4}{\kappa} \frac{P}{\rho} - \frac{16}{\kappa} \int P d\frac{1}{\rho} - f(\rho) \quad (106)$$

If now we use our expression (73) for the quantum pressure P we can have an expression for the potential $\langle \Phi_{\text{HDF}}(x,\xi) \rangle |_{\text{x}}$ associated to the full quantum case. Performing calculations, this turns out to be:

$$\langle \Phi_{\text{HDF}}(x,\xi_i(x)) \rangle |_{\text{x}} = -\frac{\hbar^2}{2m} \frac{\rho^{-2''}}{\rho^{-2}} - f(\rho) \quad (107)$$

This expression is just the final one to rely on in this paper - is remarked again that we have further analysis to deploy in order to confirm this result. It will be given in the next paper IV.

7 Conclusion

In this paper, we introduced a reference statistical ensemble whose properties lend themselves to simple analytical discussion. These properties are used as a basis to develop some induction and provide extended expressions for the state-equations relevant to a generalized model. Specifically, we were able to find out expressions for the pressure, the statistical elastic function, the HDF-potential fields and some other associated potentials and quantities. A discussion of the approximations we have used to find out results, as well as of the key ideas governing our investigative path, has been given: it includes an analysis of the wave-behavior featuring a quantum mechanical system and of its consequences within our framework context. Our purpose is evidencing a renovated possibility to affirm a classical concept of particle motion. To this end, a detailed comparison of our resulting framework with a quantum mechanical framework will be demonstrated fruitful in the following paper IV of this work.

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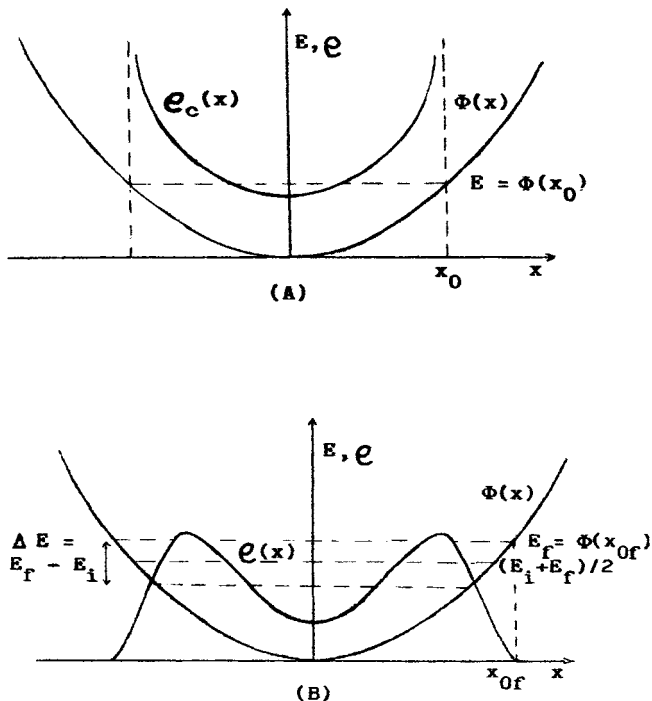


Figure 1. (qualitative) behavior of the classical density $\rho_c(x)$ for a system of particles with energy E in a harmonic potential. It exhibits poles at the turning points (x_0 's). (B): (qualitative) behavior of the inhomogeneously energy broadened system density $\rho(x)$ for a system of particles with energies E included in the interval E_i, E_f . It exhibits an undulating shape and tails at the boundaries. All quantities are in arbitrary units.