Propositional bases for the physics of the Bernoulli oscillators (A theory of the hidden degree of freedom)

IV - The matter-wave equation and the Newtonian background

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ABSTRACT. In a few previous papers, we developed a theoretical framework displaying the thermodynamic, mechanical and statistical properties of the Bernoulli oscillators - these last are physical entities to which a classical-like, deterministic behaviour is attributed. We provided expressions for the mass-flow theorem and correlated potentials relevant to an ensemble of these oscillators. In this paper, we compare our framework and results with a quantum mechanical context, represented by the Schrodinger wave-equation in the Madelung formulation. By a requirement of consistency with the quantum equations, we are able to find out an expression of the mechanical energy theorem governing the (single-particle) so-called classical degree of freedom. This expression corresponds to a Newtonian-like equation of motion, which seems to us the good candidate to set a bridge between classical and quantum physics. It is (by proposal) interpreted as the physical background underlying the quantum wave-function formalism. We use a "double solution" conjecture to solve our equation, and show that the classical-like densities generated by the solutions can be summed indeed to the corresponding quantum density. Although we remain sometimes within the boundary of a conjectural framework, and limited to the case of translational motion, the possibility to approach a solution to the old problem of inconsistency between classical and quantum mechanics is conclusively displayed in the paper, and discussed as a proposal.

1 Introduction

In the previous papers denoted I-III $[1 \div 3]$ of this work we introduced some thermodynamical, mechanical and statistical properties of the so-called Bernoulli oscillators. We could set up a mechanical-statistical

framework describing the physical behavior of an ensemble of these oscillators, taking into account "local " (i.e. depending on the co-ordinate x within the available space-domain) potentials; and provided expressions for the mass-flow theorem and correlated relevant quantities. More specifically, we found generalized properties and expressions for both the indicated work and the ensemble average of the potential energy Φ_{HDF} - this is the energy provided by the external source (the quantum field) to what we call the classical degree of freedom. This last is constituted by the oscillation center motion of the particle, submitted to the (time-dependent) quantum field action. In the present paper IV, we compare our model equations to the wave-mechanical equations set (hydrodynamic formulation). We use a limit procedure to find the mechanical correspondents - consistent with quantum theory and associated to a single-particle behavior - of the relevant statistical expressions we have made available. By the comparison procedure, we find out the single-particle energy theorem expression consistent with the quantum mechanical statistics. It has the form of a Newtonian-like equation of motion for the classical degree of freedom, which we (by proposal) promote as the representative example of a renovated possibility to affirm a classical concept of particles motion and mechanics. The equation is non-linear in the velocity field and reveals rather stiff for analytical handling; yet - by some conjectures - we are able to set up interesting solutions to our sake of consistency with quantum mechanics. Sometimes, therefore, we remain on a conjectural ground; but an effort is done to give interpretative level to our framework, to the purpose of stimulating critical interest and deepening into the properties we could display.

2 The full quantum case

In this section, we apply the results obtained in paper III to the final task of determining the expressions for $<\Phi_{\mathrm{HDF}}(\mathrm{x},\xi)>|_{\mathrm{x}}$ and $\Phi_{\mathrm{HDF}}(\mathrm{x},\xi)$ itself by the request of consistency of our formalism with the key equation in quantum mechanics, i.e. the Schrödinger equation (hydrodynamic formalism).

2.1 The mass-flow theorem and the Schrödinger equation

Using the P-state equation (73) and the g-state equation (95) in paper III we find

$$\int \mathrm{k}_0(\rho,\kappa)\mathrm{dg}(\rho,\kappa) = \mathrm{G}(\mathrm{P}/\rho,\rho,\kappa) =$$

$$= -\frac{\hbar^2}{4\mathrm{m}}(5-\kappa)\int \rho^{\alpha(\kappa)} d\left(\left(\frac{\rho'}{\rho}\right)' \rho^{-\alpha(\kappa)}\right) + f(\rho) \tag{1}$$

Taking into account the expression of $\alpha(\kappa)$ given in paper III - equation (84), we find

$$\mathrm{G(P/}\rho,\rho,\kappa) = \frac{\hbar^2}{2\mathrm{m}} \frac{(5-\kappa)^2}{15+\kappa} \frac{\rho^{-\frac{\alpha(\kappa)}{2}}{}''}{\rho^{-\frac{\alpha(\kappa)}{2}}} + f(\rho) \tag{2}$$

Now we revisit equations (87),(88) in paper III:

$$\frac{1}{2}\mathrm{mv}_D^2(\rho,0) + <\Phi_{\mathrm{HDF}}(\mathrm{x},\xi_i(\mathrm{x}))> |_{\mathrm{x}} + \mathrm{G(P/}\rho,\rho,0) + \mathrm{I}_D(\nu(\mathrm{x}),0) + \Phi(\mathrm{x}) =$$

$$= \mathrm{Inv} |_0 \tag{3}$$

$$\frac{1}{2}\mathrm{mv}_D^2(\rho,\kappa) + \Phi(\mathrm{x}) + <\Phi_{\mathrm{HDF}}(\mathrm{x},\xi_i(\mathrm{x}))> |_{\mathrm{x}} + \mathrm{G(P/}\rho,\rho,\kappa) + \mathrm{I}_D(\nu(\mathrm{x}),\kappa) +$$

$$+ \int \frac{d\mathrm{P}}{\rho} = \mathrm{Inv} \tag{4}$$

Our purpose is comparing the previous equations with the following ones:

$$\frac{\nabla S^2}{2\mathrm{m}} + \Phi(\mathrm{x}) - \frac{\hbar^2}{2\mathrm{m}} \frac{\sqrt{\rho}''}{\sqrt{\rho}} = \mathrm{E}_n \tag{5}$$

$$\rho \nabla S = \mathrm{const} \tag{6}$$

These last equations represent the standard Madelung hydrodynamic form for the stationary Schrödinger equation. The associated quantum mechanical wavefunction (space-dependent part) is $\Psi(\mathrm{x}) = \pm \sqrt{\rho}$ $\exp(\mathrm{i}S(\mathrm{x})/\hbar)$, with $S(\mathrm{x}) \equiv$ phase function, also entering the continuity equation (6).

2.2 Determination of the HDF potential energy

We solve the set of equations (3)÷(6) as follows:

$$\mathrm{Inv} = \mathrm{Inv}\,|_0 = \mathrm{E}_n \tag{7}$$

$$<\Phi_{\mathrm{HDF}}(\mathrm{x},\xi_i(\mathrm{x}))>\,|_{\mathrm{x}}=$$

$$= \mathrm{E}_n - \frac{1}{2}\mathrm{mv}_D^2(\rho,\kappa) - \Phi(\mathrm{x}) - \mathrm{G}(\mathrm{P}/\rho,\rho,\kappa) - \mathrm{I}_D(\nu(\mathrm{x}),\kappa) - \int \frac{d\mathrm{P}}{\rho} \tag{8}$$

$$<\Phi_{\mathrm{HDF}}(\mathrm{x},\xi_i(\mathrm{x}))>\,|_{\mathrm{x}} = -\frac{1}{2}\mathrm{mv}_D^2(\rho,\kappa) + \frac{\nabla S^2}{2\mathrm{m}} \quad - \quad \frac{\hbar^2}{2\mathrm{m}}\frac{\sqrt{\rho}''}{\sqrt{\rho}} +$$

$$-\mathrm{G}(\mathrm{P}/\rho,\rho,\kappa) - \mathrm{I}_D(\nu(\mathrm{x}),\kappa) - \int \frac{d\mathrm{P}}{\rho} \tag{9}$$

$$<\Phi_{\mathrm{HDF}}(\mathrm{x},\xi_i(\mathrm{x}))>\,|_{\mathrm{x}} = -\frac{1}{2}\mathrm{mv}_D^2(\rho,\kappa) \; - \mathrm{I}_D(\nu(\mathrm{x}),\kappa) \; + \frac{\nabla S^2}{2\mathrm{m}}$$

$$+ (\kappa - 1)\,\frac{\hbar^2}{2\mathrm{m}}\frac{\sqrt{\rho}''}{\sqrt{\rho}} - \frac{\hbar^2}{2m}\frac{(5-\kappa)^2}{15+\kappa}\frac{\rho^{-\frac{\alpha}{2}\,''}}{\rho^{-\frac{\alpha}{2}}} - f(\rho) \tag{10}$$

$$<\Phi_{\mathrm{HDF}}(\mathrm{x},\xi_i(\mathrm{x}))>\,|_{\mathrm{x}} = -\mathrm{G}(\mathrm{P}/\rho,\rho,1) = -\frac{\hbar^2}{2\mathrm{m}}\frac{\rho^{-2''}}{\rho^{-2}} - f(\rho) \tag{11}$$

$$\mathrm{I}_D(\nu(\mathrm{x}),\kappa) + \frac{1}{2}\mathrm{mv}_D^2(\rho,\kappa) = \mathrm{I}_D(\nu(\mathrm{x}),0) + \frac{1}{2}\mathrm{mv}_D^2(\rho,0) = \frac{\nabla S^2}{2\mathrm{m}} \tag{12}$$

Concerning the potential $<\Phi_{\mathrm{HDF}}(\mathrm{x},\xi_i(\mathrm{x}))>|_{\mathrm{x}}$, its expression (11) is found identical to the one proposed in equation (107), paper III, by independent analysis. Further discussion about it can be found in the next sections. Concerning equation (12) (cf. equation (91) in paper III), note that in the full quantum case (for closed systems here at hand) the wavefunction $\Psi = \pm\sqrt{\rho}\,\exp(iS/\hbar)$ is generally taken real, so that we are here actually

considering the case with phase function $S = 0 \Rightarrow \text{const} = 0$ in equation (6). Conversely, we may have WKB or quasi-classical cases where $\nabla S^2 \neq 0$ - these ones may support a solution with $\mathrm{I}_{_D} = 0$ and $\nabla S = \mathrm{mv}_{_D}$ which is of rather simple interpretation in our general context and will not to be discussed here. Then we write

$$\mathrm{I}_{_D}(\nu(\mathrm{x}),\kappa) + \frac{1}{2}\mathrm{mv}_{_D}^2(\rho,\kappa) = \mathrm{I}_{_D}(\nu(\mathrm{x}),0) + \frac{1}{2}\mathrm{mv}_{_D}^2(\rho,0) = 0 \qquad (13)$$

To enlighten this equation, we note that it requires the quantity $\mathrm{I}_{_D}(\nu(\mathrm{x}),\kappa)$ to be negative. That this is a possible physical issue has been shown to occur for the model given in paper III, indeed. We rely here on that result to accept equation (13). We do not make any attempt to calculate the potential $\mathrm{I}_{_D}(\nu(\mathrm{x}),\kappa)$ explicitly in the present general case - we have not yet enough indications about its constitutive relationship. This last is likely to depend on the mass effect we have been able to display in our analysis, and on the peculiar effect of "mechanical ordering" of particles motions - we have found it in paper III. Even more important, in our opinion, the $\mathrm{I}_{_D}(\nu(\mathrm{x}),\kappa)$ potential expression might be affected by the possibility - admitted in our model - that the particle jumps from one time-law to another (see a next section entitled to physical interpretation). All of these effects on the $\mathrm{I}_{_D}$ expression remain to be investigated.

By the previous equations, we also find

$$\mathrm{U_B}(\mathrm{x}) = -\frac{\hbar^2}{2\mathrm{m}}\frac{\sqrt{\rho}''}{\sqrt{\rho}} = <\Phi_{\mathrm{HDF}}(\mathrm{x},\xi_i(\mathrm{x}))> |_{\mathrm{x}} + \int \mathrm{k}_0(\rho,\kappa)\mathrm{dg}(\rho,\kappa) + \int \frac{d\mathrm{P}}{\rho} \qquad (14)$$

This equation allows us to give to the quantum Bohm potential a classical-like interpretation by means of the quantities Φ_{HDF}, k_0, g, P we have introduced in our framework. At the same time, equation (12) gives us a classical-like interpretation of the quantum potential $\nabla S^2/2\mathrm{m}$ by the means of the potentials $\mathrm{I}_{_D}(\nu(\mathrm{x}),\kappa)$ and $\mathrm{mv}_{_D}^2/2$.

3 Statistical interpretation

It is not difficult to find out an expression for the function $f(\rho)$ if we assume that the ξ_i are space-parameters additive to the co-ordinate x. In this case we can write:

$$<\Phi_{\rm HDF}({\rm x},\xi_i({\rm x}))> |_{\rm x} \equiv <\Phi_{\rm HDF}({\rm x}+\xi)> |_{\rm x} = -\frac{\hbar^2}{2{\rm m}}\frac{\rho^{-2''}}{\rho^{-2}} - f(\rho) \quad (15)$$

Here the indices i, $|_x$ and the ξ functional dependence on x are omitted by simplicity. By the Mc Laurin series we have (remember $<>|_{\rm x}$is an average over ξ-values in the context of the present paper):

$$<\Phi_{\rm HDF}({\rm x}+\xi)> |_{\rm x} = \Phi_{\rm HDF}({\rm x}) + <\xi> |_{\rm x}\ \Phi^{'}_{\rm HDF}({\rm x}) +$$

$$+\frac{<\xi^2> |_{\rm x}}{2}\Phi^{''}_{\rm HDF}({\rm x}) + <{\rm R}({\rm x},\xi)> |_{\rm x} \quad (16)$$

The quantity R(x,ξ) is the residual function in the series. In this equation, we will take the odd $\xi-$distribution momenta equal to zero :

$$<\xi^{2n+1}> |_{\rm x} = 0 \quad (17)$$

This is to insure space-isotropy with respect to the parameter ξ physical action. We also assume the interesting ansatz

$$<{\rm R}({\rm x},\xi)> |_{\rm x} = \phi({\rm x},\xi)\Phi_{\rm HDF}({\rm x}) \quad (18)$$

Using these equations, we find

$$f(\rho) = -\Phi_{\rm HDF}({\rm x}) - <{\rm R}({\rm x},\xi)> = -\left[1+\phi({\rm x},\xi)\right]\Phi_{\rm HDF}({\rm x}) \quad (19)$$

$$-\frac{\hbar^2}{2{\rm m}}\frac{\rho^{-2''}}{\rho^{-2}} = \frac{<\xi^2>}{2}\Phi^{''}_{\rm HDF}({\rm x}) \quad (20)$$

A solution of this equation is

$$\Phi_{\rm HDF}({\rm x}) = -2\chi\rho^{-2} = -2\delta{\rm m}\nu_0^2\rho^{-2} \quad (21)$$

$$\frac{\hbar^2\rho^2}{2{\rm m}\chi} = <\xi^2> = -\frac{\hbar^2}{{\rm m}\Phi_{\rm HDF}({\rm x})} \quad (22)$$

Here χ is a constant which, by dimensional analysis, takes the form

$$\chi = \delta{\rm m}\nu_0^2 \quad (23)$$

In this equation, the (constant) specific mass-flow ν_0 defined in paper III has been used and δm is the mass coefficient introduced in paper II. It was found there to substantiate a (negative) correction to the particle mass, so that the mass defect $-\delta$m will be called the HDF-induced mass. Now we find

$$f(\rho) = 2\delta \mathrm{m} \nu_0^2 \rho^{-2} \left[1 + \phi(\mathrm{x},\xi)\right] \tag{24}$$

In this expression, the function ϕ(x,ξ) can be neglected for small $<\xi^2>-$values (see equation (18)) but should not be neglected when the variable ξ takes high-values ξ_h. By equation (22) we see that this case occurs in points of space where rather high values of density (ρ_h) and small values of Φ_HDF(x) are attained. Moreover, for $\xi \approx \xi_\mathrm{h}$ we can say that the potential Φ_HDF(x+ξ_h) itself is likely to exhibit some small value because we can give the interpretation (introduced in the next section) that the source of this potential is at a great distance ξ_h from the particle position x. In the regions where $\xi \to \xi_\mathrm{h}$ using equations (16), (20), (22) we can write, therefore :

$$<\Phi_\mathrm{HDF}(\mathrm{x}+\xi_\mathrm{h})> |_\mathrm{x} = \mathrm{c}^*(\mathrm{x}) =$$

$$= \Phi_\mathrm{HDF}(\mathrm{x}) - \frac{\hbar^2}{2\mathrm{m}} \frac{\Phi''_\mathrm{HDF}(\mathrm{x})}{\Phi_\mathrm{HDF}(\mathrm{x})} + <\mathrm{R}(\mathrm{x},\xi_\mathrm{h})> |_\mathrm{x} \tag{25}$$

$$<\mathrm{R}(\mathrm{x},\xi_\mathrm{h})> |_\mathrm{x} = \mathrm{C}(\mathrm{x}) - \Phi_\mathrm{HDF}(\mathrm{x}) \tag{26}$$

By the previous remarks, in these equations we understand that c*(x) at least should be a weak functions of x. It is useful to write C(x) as C(ρ^2) so that we find

$$\lim_{\xi \to \xi_\mathrm{h}} \left[1 + \phi(\mathrm{x},\xi)\right] = \frac{\mathrm{C}(\rho^2)}{\Phi_\mathrm{HDF}(\mathrm{x})} = -\frac{\mathrm{C}(\rho^2)}{2\chi\rho^{-2}} \tag{27}$$

An approximate expression for ϕ(x,ξ) is therefore:

$$\phi(\mathrm{x},\xi) \approx -\frac{\mathrm{C}(\rho^2)\rho^2}{2\chi} \tag{28}$$

It can be regarded as a reasonable, interpolative expression between the cases with small and high $\xi-$values. Then we find by equation(19) an expression for $f(\rho)$ and we write equation (15) now in the final form

$$<\Phi_\mathrm{HDF}(\mathrm{x},\xi_i(\mathrm{x}))> |_\mathrm{x} = -\frac{\hbar^2}{2\mathrm{m}} \frac{\rho^{-2''}}{\rho^{-2}} - \frac{2\delta \mathrm{m} \nu_0^2}{\rho^2} + \mathrm{C}(\rho^2) \tag{29}$$

4 The Newtonian background

Using the procedure indicated in paper II, equation (121), the following expression for $\Phi_{\mathrm{HDF}}(\mathrm{x},\xi_i(\mathrm{x}))$ is excerpted :

$$\Phi_{\mathrm{HDF}}(\mathrm{x},\xi_i(\mathrm{x})\equiv\xi)=\lim_{\mathrm{I}_D(0,\nu_0)\to 0}^{(i)}<\Phi_{\mathrm{HDF}}(\mathrm{x},\xi_i(\mathrm{x}))>|_{\mathrm{x}}=$$

$$=-\frac{\hbar^2}{2\mathrm{m}}\frac{\rho_c^{-2''}}{\rho_c^{-2}}-\frac{2\delta\mathrm{m}\nu_0^2}{\rho_c^2}+\mathrm{C}(\rho_c^2) \tag{30}$$

$$\Phi_{\mathrm{HDF}}(\mathrm{x},\xi(\mathrm{x}))=-\frac{\hbar^2}{2\mathrm{m}}\frac{\mathrm{v}^{2''}(\mathrm{x})}{\mathrm{v}^2(\mathrm{x})}-\frac{\delta\mathrm{m}}{2}\mathrm{v}^2+\mathrm{C}(\mathrm{v}^2) \tag{31}$$

To perform the last steps of the previous analysis, we have used the (already noted) facts that the $\mathrm{I}_D(\nu(\mathrm{x}),0)\to 0$ limit implies $\nu(\mathrm{x})\to\nu_0=\mathrm{const}$ and that at constant flow ν_0 the density is confluent into the classical quantity $\rho_c(\mathrm{x})=2\nu_0/\mathrm{v}(\mathrm{x})$. Taking the i-th limit as indicated by the procedure is tantamount in these equations to display a characteristic $\xi_i(\mathrm{x})$ parametric function whose definition is

$$\xi_i^2(\mathrm{x})\equiv\xi^2(\mathrm{x})=\frac{\hbar^2}{2\mathrm{m}}\frac{\rho_c^2(\mathrm{x})}{\delta\mathrm{m}\nu_0^2}=\frac{2\hbar^2}{\mathrm{m}\delta\mathrm{m}\mathrm{v}^2(\mathrm{x})} \tag{32}$$

How this expression is identified will be better understood by the means of equations (42)÷(47) in the next section - together with equation (22) they make clear some physical interpretation, and show that the $\xi(\mathrm{x})$ definition originates from the Taylor-series development of the HDF-potential itself. Equation (32) shows that the index i can, however, be dropped off . This is because in the general context it will be clear that $\xi(\mathrm{x})$ is correlated to one out of the possible solutions for the velocity field $\mathrm{v}(\mathrm{x})$ appearing in the final motion equation (36).

Within our general context it is now also easy to identify the expression (31) with the expression

$$\Phi_{\mathrm{HDF}}(\mathrm{x},\xi)=-\frac{\alpha^2(\eta)}{32\mathrm{m}}\frac{\mathrm{v}^2}{4\nu_0^2}+\frac{\mathrm{h}^2}{4\mathrm{m}}\frac{\eta^2(\mathrm{x})-1}{\eta^2(\mathrm{x})}\frac{4\nu_0^2}{\mathrm{v}^2} \tag{33}$$

This one has been found in paper II (equation (104)). Then we can also easily find the correlations between the parameters α, ξ, η, $C(v^2)$. We only consider here the equation

$$\frac{h^2}{4m}\frac{\eta^2(x)-1}{\eta^2(x)}\frac{4\nu_0^2}{v^2} = -\frac{\hbar^2}{2m}\frac{v^{2''}(x)}{v^2(x)} \tag{34}$$

This equation holds if $C(v^2)$ can be neglected, or is associated to the other term in equation (33). Note that $\eta(x)$ can assume a range of values depending on the values of $v^{2''}(x)$, so that a demonstrated consistency of this equation might be taken as a practical evidence of the high-frequency assumption ($\eta>>1$), which we have often used to make calculations in our papers, having been removed within the present framework. The equation will be considered again in a next section. Using now equation (117) in paper II and taking

$$m - \delta m = m_{eff} \tag{35}$$

we finally find out the single-particle energy theorem expression in the form

$$\frac{1}{2}m_{eff}v^2+\Phi(x) - \frac{\hbar^2}{2m}\frac{v^{2''}}{v^2} + C(v^2) = E_n \tag{36}$$

This is our proposed equation for the motion of the particle oscillation center or classical degree of freedom. It has to be coupled with the HDF oscillation equations (112)÷(116) given in paper II for a complete description of the particle motion in our framework. Here, by simplicity, we report only the expressions of those equations written for the case when a single frequency $\omega(x)$ is effective:

$$x_z(t) = x_{z0}(x)\sin(\omega(x)t + \varphi(x)) \tag{37}$$

$$2m \mid x_{z0}v_{z0} \mid = 2m\omega(x)x_{z0}^2(x) \gtrsim h \tag{38}$$

$$\omega(x) \approx \frac{8h\nu_0^2}{mv^2(x)} \tag{39}$$

In order to fit a quantum-mechanical behavior with equation (36), the particle energy must be taken equal to a quantum-mechanical eigenvalue

E_n, as shown indeed. Although by simplicity the index n has only been displayed for the energy eigenvalue, it is clear that in equations (35)÷(39) the mass flow ν_0, the pulsation ω and other relevant quantities must generally be considered dependent on n too.

On the interpretative level, instead, we propose in our model that equation (36) is considered the "classical background" for wave mechanics. This is not only in the sense that many particles, with different time-laws, are able to totalize the quantum mechanical density pertaining to the energy E_n as described by equations (3)÷(6); but also in the sense that every single particle is able to cross over the different time-laws - up to totalize, within observation times "long enough", the same quantum density. The various solutions of equation (36) obviously correspond to different initial conditions sets; the interesting feature of the equation, in this context, is actually just that it can display "discontinuous" solutions for v(x), allowing the particle to jump from one (continuous) solution to another.

In the sequel, we will therefore attempt to set up some (continuous) solutions to our equation, appropriate to the end of demonstrating the possibility of the described, single-particle behavior, by a direct proof. Further analysis we have to let to future work.

Before showing the solution procedures, we give in the next section a physical model directly bringing to equation (36) within a simple and synthetic interpretative frame.

5 Vacuum "mechanical" polarization

Equation (36) can also be found according to the following model. Assume the vacuum is a reactive medium accommodating a number N of reaction centers around a particle whose Euler velocity field is $v_x(x) \equiv dx/dt$. The index x indicates x−components. The position of the i-th center is at a co-ordinate distance ξ_i (positive or negative) from x. Assume the reactive field F_{xi} is conservative, acts on the particle at the co-ordinate x and depends on the distance ξ_i via the equation

$$F_{xi}(x,\xi_i) \equiv F_{xi}(x+\xi_i) = \frac{\delta m}{N} v_x(x+\xi_i) v'_x(x+\xi_i) = \frac{\delta m}{N} a_x(x+\xi_i) \tag{40}$$

The reaction $F_{xi}(x,\xi_i)$ is taken proportional to the particle acceleration field itself, but this last is calculated at the position $x+\xi_i$ so that the

interaction reveals a non-local one. Summing over the distribution of reactive centers in the space, equation (40) brings to the potential energy

$$\Phi_{\mathrm{HDF}}(\mathrm{x},\xi(\mathrm{x})) \equiv \Phi_{\mathrm{HDF}}(\mathrm{x}+\xi(\mathrm{x})) = \sum_{i}^{N} \Phi_i = -\frac{1}{2}\sum_{i}^{N} \frac{\delta \mathrm{m}}{\mathrm{N}} \mathrm{v}_{\mathrm{x}}^2(\mathrm{x}+\xi_i) \Rightarrow$$

$$\Rightarrow -\frac{1}{2}\int_{\Xi} \frac{\delta \mathrm{m}}{\mathrm{N}} \tilde{\rho}(\mathrm{x},\xi^*)\mathrm{v}_{\mathrm{x}}^2(\mathrm{x}+\xi^*)\mathrm{d}\xi^* \tag{41}$$

Here Ξ is the ξ^*-variable definition domain, and $\tilde{\rho}(\mathrm{x},\xi^*)$ is the (normalized) numerical density of centers at a distance ξ^*, effective on the co-ordinate x. We will take

$$\xi = \sqrt{<\xi^{*2}>|_{\mathrm{x}}} \tag{42}$$

This quantity is further specified in equation (47) and is consistent with the final $\Phi_{\mathrm{HDF}}(\mathrm{x}+\xi)$ expression we give in equation (49).

Since sometimes it is not displayed by simplicity, we recall that the variables ξ_i, ξ are functions of x. By the Taylor series and normalizing integrals we have easily

$$\Phi_{\mathrm{HDF}}(\mathrm{x}+\xi) = -\frac{1}{2}\frac{\delta \mathrm{m}}{\mathrm{N}}\int_{\Xi} \tilde{\rho}(\mathrm{x},\xi^*)\mathrm{v}_{\mathrm{x}}^2(\mathrm{x}+\xi^*)\mathrm{d}\xi^* = \sum_{j} \frac{\Phi_{\mathrm{HDF}}^{(j)}(\mathrm{x})\xi^j}{j!} \tag{43}$$

$$\mathrm{v}_{\mathrm{x}}^2(\mathrm{x}+\xi^*) = \mathrm{v}_{\mathrm{x}}^2(\mathrm{x}) + \xi^*\mathrm{v}_{\mathrm{x}}^2(\mathrm{x})' + \frac{\xi^{*2}}{2}\mathrm{v}_{\mathrm{x}}^2(\mathrm{x})'' + \bar{\mathrm{R}}(\mathrm{x},\xi^*) \tag{44}$$

$$\frac{1}{\mathrm{N}}\int_{\Xi} \tilde{\rho}(\mathrm{x},\xi^*)\mathrm{d}\xi^* = 1 \tag{45}$$

$$\frac{1}{\mathrm{N}}\int_{\Xi} \tilde{\rho}(\mathrm{x},\xi^*)\xi^{*(2n+1)}\mathrm{d}\xi^* = <\xi^{*(2n+1)}>|_{\mathrm{x}} = 0 \tag{46}$$

This equation is analogous to equation (17). Now in order to recover equation (31) we can set

$$\frac{1}{\mathrm{N}}\int_{\Xi} \tilde{\rho}(\mathrm{x},\xi^*)\xi^{*2}\mathrm{d}\xi^* = <\xi^{*2}>|_{\mathrm{x}} = \xi^2 = \frac{2\hbar^2}{\mathrm{m}\delta \mathrm{m}\mathrm{v}^2(\mathrm{x})} \tag{47}$$

$$\frac{1}{N}\int_{\Xi}\tilde{\rho}(x,\xi^*)R(x,\xi^*)d\xi^* = -\frac{2}{\delta m}C(v^2) \tag{48}$$

Then we find from (43)/(44):

$$\Phi_{HDF}(x,\xi(x)) \equiv \Phi_{HDF}(x+\xi) = -\frac{1}{2}\delta m v^2(x) - \frac{\hbar^2}{2m}\frac{v^{2''}}{v^2} + C(v^2) \tag{49}$$

In these equations, the index x for the velocity component is useless and has been omitted, in agreement with the choice we have generally made throughout our papers. On the interpretative level, we might say that the vacuum is a " mechanically " polarizing matter when particle motion v(x) is created. By this statement we mean that the vacuum is able to react to an imposed Eulerian field v(x) by means of "distant" reactive forces. These last originate in points of space which are isotropically distributed (see equation (46)) around the particle current co-ordinate x, but will comprehensively set up a net force on it. We know by paper II that the vacuum reaction is actually time-dependent but it creates the potential Φ_{HDF} as a stationary action. Both forward and backward with respect to the motion direction, reactive centers are perturbed at a characteristic distance $<|\xi^*|>|_x$:

$$<|\xi^*|>|_x \simeq \gamma\sqrt{<\xi^{*2}>|_x} = \gamma\xi = \frac{\gamma\hbar\sqrt{2}}{v(x)\sqrt{m\delta m}} \tag{50}$$

Here γ is a $\tilde{\rho}(\xi^*)-$distribution form factor. In case δm is of the order of m, the distance (50) is of the order of the de Broglie wavelength. Then we believe that the present interpretation may lead to interesting correlations with the de Broglie pilot-wave theory - a brief discussion about this point is in a next section.

We want to note that the quantity $\hbar v$ is homogeneous to an [electrical charge]^2, and that the potential $-\frac{1}{2}\delta m v^2$can be set up by the expression

$$\Phi_{HDF}(x) = -\frac{1}{2}\delta m v^2 = const \times \frac{\hbar v}{<|\xi^*|>|_x} = -\gamma\sqrt{\frac{\delta m}{2m}}\frac{\hbar v}{<|\xi^*|>|_x} \tag{51}$$

This last expression sounds as the basic constitutive equation for the peculiar interaction we have here displayed between the particle and the vacuum. Then we think that interesting correlations might also be found with electricity - perhaps making us able to correlate the "polarization" concept here introduced with the well known electrical correspondent.

The expounded concepts, however, concern the stationary part of the vacuum interaction and have to be coupled to a corresponding time-dependent analysis, accounting for the HDF oscillation equations and time-dependent fluctuation field action, to have a complete description and interpretative frame. This is outside the scope of this work.

6 Particle oscillation-center motion equation and its properties: a "double solution" theory

Taking for a moment apart the problem of determining the quantities $C(v^2)$ and m_{eff}, we have to note first that equation (36) is non-linear in the variable v(x) and rather stiff to handle analytically. In principle, either ad-hoc non-linear analysis, or computer calculations, should be performed in order to provide exhaustive information about the physical behaviors it may originate when the particle is submitted to different potential forms $\Phi(x)$. Then the direct task to provide general properties of our equation by taking (even the simplest) assumptions for $C(v^2)$ and m_{eff} is out of the possibilities of this paper. Notwithstanding, we are able to discuss briefly a few interesting properties of equation (36) and even to give (proposed) solutions: these can be found basing on some peculiar conjectures to be expounded next. In order to introduce these mentioned procedures we start making a few interesting points out here.

Equation (36) has to be solved in the velocity field v(x) with initial conditions depending on the case at hand, and more specifically on the potential energy $\Phi(x)$ form. In case the particle meets a turning point x_0 along its trip, the conditions may be given f.i. in the form

$$\lim_{x\to x_0} v^2(x) = \lim_{x\to x_0} \xi^{-2}(x) = \lim_{x\to x_0} 2v_x(x_0)v_x^{'}(x_0)(x - x_0) = 0 \qquad (52)$$

Here note that $v_x v_x^{'}$ is the "particle" (we may use quotation marks sometimes, to recall that the velocity field v(x) we are referring to is the one relevant to the classical degree of freedom or oscillation center motion) acceleration. This last can generally be assumed different from zero at the turning point x_0. We have also :

$$-\lim_{x\to x_0} \frac{\hbar^2}{2m} \frac{v^{2''}(x)}{v^2(x)} + C(0) = -\lim_{x\to x_0} \frac{\hbar^2}{2m} \frac{\xi^{-2''}(x)}{\xi^{-2}(x)} + C(0) =$$

$$= \lim_{x \to x_0} \frac{h^2}{4m} \frac{\eta^2(x) - 1}{\eta^2(x)} \frac{4\nu_0^2}{v^2} = \pm \frac{P^2(0,1)}{2m} = E_n - \Phi(x_0) \qquad (53)$$

In order to appreciate these equations and the role of the quantity P^2 (0,1), see also equation (110) and correlated comment in paper II. In equations (52) and (53) the initial conditions for the function $\xi(x)$ $\propto v^{-1}(x)$ are also displayed. As a comment here, note that if C(0) can be neglected, in the region near the turning point the potential $-\hbar^2 v^{2''}/2mv^2$ turns out to be negative in the case $v^{2''}(x)>0$; then it is responsible for the extra-energy given to the particle while " tunnelling " outside the classical region defined by the energy value E_n. This case corresponds to the $\eta<1$case discussed in paper II, as is clear by equation (53). The case with $C(0) \neq 0$ could also be easily discussed - provided an expression is assumed for $C(v^2)$.

Equation (36) is a second-order derivative equation in the (velocity-field) variable v(x). In our model, both its solutions have physical meaning. The solutions will be named $v_{SP}(x, x_{0L}, x_{0R})$ and $v_D(x, x^*_{0L}, x^*_{0R})$. The first one is the single-particle solution, a velocity field defined for $x_{0L} \leq x \leq x_{0R}$, where x_{0L} and x_{0R} are the left and right turning points delimiting the space attainable by the particle. The second one is the (per particle equivalent) drift velocity of an ensemble of many particles - each of them following a $v_{SP}(x, x_{0L}, x_{0R})$-trajectory. For each trajectory, x_{0L} and x_{0R} must be intended as sampled out of the ensemble of possible values (in practice, every point in the space allowed to the particles can be a turning point for some of them). The space domain X allowed to the particles is, by assumption, coincident with the definition space of the quantum density corresponding to the potential $\Phi(x)$. In our model, the drifting ensemble of particles is composed by a representative packet with a variable number of particles at each space point x, flowing from the left extreme boundary in space x^*_{0L} to the right one x^*_{0R} (or vice-versa) . Then the drift velocity $v_D(x, x^*_{0L}, x^*_{0R})$ is defined in all the space X and becomes zero only at the X-boundaries x^*_{0L} and x^*_{0R} ([1]). One can think to the drifting ensemble of particles (the forward or backward beam) as composed - at every position x - by the particles moving through that point from left to right (or vice-versa) at some instant of time. The particles flow is a stationary (Eulerian) one in agreement with the time-independence of equation (36).

[1] When the physical space X available to the quantum particles is all the space, x^*_{0L} and x^*_{0R} will attain their appropriate limits $\pm\infty$.

The solutions have not to be symmetrical functions around the space point x = 0. This is not only because of the potential $\Phi(x)$ eventually lacking this symmetry, but also because of the equation differential properties allowing different kinds of boundary conditions. Therefore - even with a symmetric potential $\Phi(x)$ - we can have, in principle at least, non-symmetrical solutions. In order to better understand the physical context correlated to these last, further investigation should be performed. However, for the sake of simplicity, we will just refer in the sequel to a symmetric case, which is rather exhaustive to our main purposes; on the other hand, in order to set up procedures able to deal with various circumstances, we find anyway comfortable to discuss only "half" velocity field solutions - these are the two parts $v_{SP}(x, x_{0R})$ and $v_{SP}(x, x_{0L})$ of the total field defined, at each time, in the corresponding "half" part of the space X as shown in Fig. (1). One can therefore join together, at an intersection point, any available couple of left and right solutions provided appropriate matching of values, derivatives and correlated quantities can be performed. This task of joining the two parts together is however (conceptually) simple and will be therefore left apart in this paper. Since now, we will indicate by $v_{SP}(x,x_0)$ and $v_D(x,x_0^*)$ these "half" solution (HS) fields, the corresponding turning points to be intended "left" or "right" depending on the circumstance. For the sake of simplicity, however, our next discussions will sometimes only be expounded for the right half solution (RHS) fields. Often indeed, this will be more comfortable that carrying out $\pm$signs or modulus brackets to the purpose of writing general equations through extensive calculations.

In order to solve equation (36), hypotheses should be done about the quantities m_{eff} and $C(v^2)$. We will not take direct physical assumptions however, but rather find out first our solutions by a requirement of consistency with known conditions. This is essentially as shown in the very following. We will then ask to the unknown functions to match the obtained results afterwards.

We start by writing down the double-solution equations

$$\frac{1}{2}\, m_{eff} v_{SP}^2 + \Phi(x) - \frac{\hbar^2}{2m}\frac{v_{SP}^{2''}}{v_{SP}^2} + C({v_{SP}}^2) = E_n \qquad (54)$$

$$v_{SP}(x, x_{0L}, x_{0R}) = 0 \qquad \{x = x_{0L}\ ,\ x_{0R}\} \qquad (55)$$

$$\frac{1}{2}\,\mathrm{m}_{eff}\mathrm{v}_{D}^{2}+\Phi(\mathrm{x})-\frac{\hbar^{2}}{2\mathrm{m}}\frac{\mathrm{v}_{D}^{2''}}{\mathrm{v}_{D}^{2}}+\mathrm{C}(\mathrm{v}_{D}{}^{2})=\mathrm{E}_{n} \tag{56}$$

$$\mathrm{v}_{D}(\mathrm{x},\,\mathrm{x}_{0L}^{*},\,\mathrm{x}_{0R}^{*})=0 \qquad \{\mathrm{x} \;=\; \mathrm{x}_{0L}^{*}\,,\;\mathrm{x}_{0R}^{*}\} \tag{57}$$

We add now to these equations a relationship between v_{SP}^{2} and v_{D}^{2}, which we have found already in paper III, equation (16):

$$\mathrm{v}_{SP}^{2}(\mathrm{x},\mathrm{x}_{0})=4\mathrm{v}_{D}^{2}(\mathrm{x},\mathrm{x}_{0}^{*})\frac{\nu(\mathrm{x})-\nu(\mathrm{x}_{0})}{\nu(\mathrm{x})} \tag{58}$$

Here $\nu(\mathrm{x})$ is the ensemble volume flow, equal to $\rho\mathrm{v}_{D}/2$ ([2]); ρ is the quantum density. This relationship comes from an analysis of classical-like, microcanonical ensembles of particles displaying similarities with quantum systems as discussed in paper III, and looks a very interesting property to refer to. We will show indeed that, if we assume the equation (58) to hold in the present general case too, we will have a consistent physics.

In order to find the solutions $\mathrm{v}_{SP}(\mathrm{x},\,\mathrm{x}_{0L},\,\mathrm{x}_{0R})$ and $\mathrm{v}_{D}(\mathrm{x},\,\mathrm{x}_{0L}^{*},\,\mathrm{x}_{0R}^{*})$ we have first to define two regions in space (as done already in paper III, equations (10) and (11)). Here the Region I is a wide region of space "internal" to X, while the Region II (called the external one) is a boundary region adjacent to the extreme points x_{0L}^{*} , x_{0R}^{*} where the quantum density ρ goes to zero. We discuss the two cases separately in the next subsections.

6.1 Solutions in Region I

In Region I we can find another relationship between the solutions. This is by the simple remark that, in our model, we have just identified the second-order derivative terms as the dominant ones. Note that, if in both

[2] In our papers the factor 2 in the volume flow balance equations is because the density accounts - by definition - for the two counterrunning streams of particles.

the equations (54) and (56) the terms with m_{eff} and C were actually zero, then by the assumption of double solution we could set :

$$v^2_{SP}(x,x_0) \propto v^2_D(x,x^*_0) \left| \int_x^{x_0} \frac{dx}{v^4_D(x,x^*_0)} \right| \qquad \{x \ \epsilon \ \text{Region I}\} \tag{59}$$

Now since generally the mentioned terms are different from zero - but the second-order derivative terms still are the dominant ones, a very good position for our solutions is the following:

$$v^2_{SP}(x,x_0) = g^2(x) \ v^2_D(x,x^*_0) \left| \int_x^{x_0} \frac{dx}{v^4_D(x,x^*_0)} \right| \qquad \{x \ \epsilon \ \text{Region I}\} \tag{60}$$

Here $g(x)$ is a perturbation function to be determined. Explicit expressions for this function and v_{SP}, v_D can be obtained by adding equation (58) to the previous one. We can solve as follows:

$$g(x) \ = \frac{2}{\sqrt{\nu(x)}}\zeta \tag{61}$$

$$\nu(x) - \nu(x_0) = \frac{1}{\zeta} \left| \int_x^{x_0} \frac{dx}{v^4_D(x,x^*_0)} \right| \tag{62}$$

From equation (62) we also get

$$\nu(x) = \left(\frac{5\zeta}{16}\right)^{\frac{1}{5}} \left| \int_x^{x^*_0} \rho^4(x)dx \right|^{\frac{1}{5}} \qquad = \qquad b \left| \int_x^{x^*_0} \rho^4(x)dx \right|^{\frac{1}{5}}$$

$$\{x \ \epsilon \ \text{Region I}\} \tag{63}$$

In these equations, a constant b, whose definition is straightforward, has been introduced by simplicity. Together with the quantity ζ they

are constants, different from zero, which have to be chosen in such a way that the solutions in Region I and Region II can be matched to each other (see the following equations (70), (71)). We have finally

$$\mathrm{v}_{SP}(\mathrm{x},\mathrm{x}_0) =$$

$$= \frac{4}{\rho}\left(\frac{5\zeta}{16}\right)^{\frac{1}{5}}\left|\int_{\mathrm{x}}^{\mathrm{x}_0^*}\rho^4(\mathrm{x})d\mathrm{x}\right|^{\frac{1}{10}}\sqrt{\left|\int_{\mathrm{x}}^{\mathrm{x}_0^*}\rho^4(\mathrm{x})d\mathrm{x}\right|^{\frac{1}{5}} - \left|\int_{\mathrm{x}_0}^{\mathrm{x}_0^*}\rho^4(\mathrm{x})d\mathrm{x}\right|^{\frac{1}{5}}} \tag{64}$$

$$\{\mathrm{x} \ \epsilon \ \mathrm{Region\ I}\}$$

6.2 Solutions in Region II

Here we first note that the same kind of position as given in equation (60) could be seen not to work satisfactorily in Region II; in that same equation, the integrand should also be perturbed. However, in this region we are able to solve instead our equations taking advantage of the simple remark that the limiting behavior of $\mathrm{v}_{_D}$, at the extreme boundary, is just "going to zero". Then we can write

$$\frac{1}{2}\,\mathrm{m}_{eff}\mathrm{v}_{_D}^2 + \mathrm{C}(\mathrm{v}_{_D}{}^2) \to 0 \qquad \{\mathrm{x} \ \to \ \mathrm{x}_{0L}^*\ ,\ \mathrm{x}_{0R}^*\} \tag{65}$$

$$\Phi(\mathrm{x}) - \frac{\hbar^2}{2\mathrm{m}}\frac{\mathrm{v}_{_D}^{2''}}{\mathrm{v}_{_D}^2} \ \to \ \mathrm{E}_n \qquad \{\mathrm{x} \ \epsilon \ \mathrm{Region\ II}\ \} \tag{66}$$

Now comparing this equation with the matter wave equation (Made lung formulation) we find easily

$$\mathrm{v}_{_D}^2 = \frac{4\nu^2(\mathrm{x})}{\rho^2} \ \approx 4\,const \times \sqrt{\rho} \qquad \{\mathrm{x} \ \epsilon \ \mathrm{Region\ II}\} \tag{67}$$

From this equation we also have

$$\nu(\mathrm{x}) \approx \sqrt{const} \times \rho^{\frac{5}{4}} \; = a\rho^{\frac{5}{4}} \qquad \{\mathrm{x} \;\; \epsilon \text{ Region II}\} \quad (68)$$

so that

$$\mathrm{v}_{SP}(\mathrm{x},\mathrm{x}_0) \approx 4\,a\rho^{\frac{1}{4}} \;\; \sqrt{\frac{\nu(\mathrm{x}) - \nu(\mathrm{x}_0)}{\nu(\mathrm{x})}} \;\; = \;\; 4\,a\rho^{\frac{1}{4}} \left[1 - \left(\frac{\rho(\mathrm{x}_0)}{\rho(\mathrm{x})}\right)^{\frac{5}{4}}\right]^{\frac{1}{2}}$$

$$\{\mathrm{x} \;\; \epsilon \text{ Region II}\} \quad (69)$$

The constant a in these equations must also be taken appropriately to match solutions to each other, at the intersection point between the two Regions. Note here for consistency that Region II must be defined so that it does not exceed the region of space where $\rho(\mathrm{x}) \geq \rho(\mathrm{x}_0)$.

6.3 Matching the Region I and Region II solutions

We call x^{**} the boundary point between the internal (I) and external (II) regions of space. The matching condition can be written in terms of the volume flow functions in the two Regions. Choosing appropriately the ratio between a and b, and the intersection point x^{**} , we can match the functions up to the first space derivative:

$$b \left| \int_{\mathrm{x}^{**}}^{\mathrm{x}_0^*} \rho^4(\mathrm{x}) d\mathrm{x} \right|^{\frac{1}{5}} \qquad = a\rho^{\frac{5}{4}}(\mathrm{x}^{**}) \quad (70)$$

$$b \left\{ \left| \int_{\mathrm{x}}^{\mathrm{x}_0^*} \rho^4(\mathrm{x}) d\mathrm{x} \right|^{\frac{1}{5}} \right\}' \;\; {}_{|\mathrm{x}^{**}} = \frac{1}{5}\frac{b^5}{a^4}\rho^{-1}(\mathrm{x}^{**}) = a \left\{ \rho^{\frac{1}{4}}(\mathrm{x})\rho(\mathrm{x})' \right\} \;\; {}_{|\mathrm{x}^{**}} \quad (71)$$

These conditions determine the values of a/b and x^{**}, and make the corresponding solutions for v_{SP} and v_D continuous in that point up to the first space derivative.

It is not to remark again, however, that our overall procedure to solve equations (54), (56) is affected by the ignorance about the quantities m_{eff} and C; the real assumption we are making about these quantities here is just that they are able to support our solution. Therefore, it is also clear that we can even accept some small perturbation to this last, in order to have analytical solutions everywhere, and still ask for m_{eff} and C to be able to support. At last, the present considerations will bring us to a simple formulation based on an interpolative technique between the expressions in the two regions, to be expounded in a next section.

Finally in this section we have to recall, for the sake of consistency, that the densities corresponding to both parts (64) and (69) of the solution must be integrable, with an appropriate probability distribution function, to result in the quantum density ρ. This property is easily shown to be met with in the next section.

6.4 Summing the classical densities to the quantum density

We consider the following distribution (of turning points, for "half" solutions HS) density:

$$P(x_0) = |\nu'(x_0)|\, T(x_0) \tag{72}$$

Here $T(x_0)$ is the classical period of a particle displaying the velocity field $v_{SP}(x, x_0)$:

$$T\ (x_{0L},x_{0R}) = 2\int_{x_{0L}}^{x_{0R}} \frac{dx}{v_{SP}(x,x_{0L},x_{0R})} \equiv_{[HS]} T\ (x_0) \tag{73}$$

The quantity $\nu'(x_0)$ is a (space) derivative of the particles ensemble volume flow, calculated in x_0. Equation (72) complies with the condition

$$\left| \int_{x}^{x_0^*} \frac{P(x_0)}{T(x_0)} dx_0 \right| = \nu(x) - \nu(x_0^*) = \nu(x) \tag{74}$$

At the extreme X-boundaries x_0^* the volume flow $\nu(x_0^*)$ is indeed zero.

The classical density corresponding to the same field is :

$$\rho_{SP}(x,x_0) = \frac{2}{T(x_0)v_{SP}(x,x_0)} \tag{75}$$

If we integrate this density with the distribution of turning points given by (72), we get for the ensemble average density the result (take the RHS case by simplicity)

$$<\rho> = \left| \int_{\mathrm{x}}^{\mathrm{x}_0^*} \mathrm{P}(\mathrm{x}_0)\rho_{_{SP}}(\mathrm{x},\mathrm{x}_0)d\mathrm{x}_0 \right| = -\int_{\mathrm{x}}^{\mathrm{x}_0^*} \frac{2\nu'(\mathrm{x}_0)}{\mathrm{v}_{_{SP}}(\mathrm{x},\mathrm{x}_0)} d\mathrm{x}_0 = \rho(\mathrm{x})$$

$$\{\mathrm{x} \ \epsilon \text{ Region I+II, RHS } \} \quad (76)$$

When the integration is performed with both the solutions (64) and (69) holding in the two Regions, it is just a straightforward matter showing that the result is always $\rho(\mathrm{x})$ indeed:

$$<\rho> = \int_{\mathrm{x}}^{\mathrm{x}_0^*} \frac{\left(\int_{\mathrm{x}}^{\mathrm{x}_0^*} \rho^4(\mathrm{x})d\mathrm{x}\right)^{-\frac{1}{10}} \rho(\mathrm{x}) \left(\int_{\mathrm{x}_0}^{\mathrm{x}_0^*} \rho^4(\mathrm{x})d\mathrm{x}\right)^{\frac{1}{5}-1} \rho^4(\mathrm{x}_0)}{10\sqrt{\left(\int_{\mathrm{x}}^{\mathrm{x}_0^*} \rho^4(\mathrm{x})d\mathrm{x}\right)^{\frac{1}{5}} - \left(\int_{\mathrm{x}_0}^{\mathrm{x}_0^*} \rho^4(\mathrm{x})d\mathrm{x}\right)^{\frac{1}{5}}}} dx_0 \ = \rho(\mathrm{x})$$

$$\{\mathrm{x} \ \epsilon \text{ Re gion I, RHS}\} \quad (77)$$

$$<\rho> = -\int_{\mathrm{x}}^{\mathrm{x}_0^*} \frac{\rho^{\frac{5}{8}}(\mathrm{x})\,\rho^{\frac{5}{4}\,\prime}(\mathrm{x}_0)}{2\sqrt{\rho^{\frac{1}{2}}\left(\rho^{\frac{5}{4}}(\mathrm{x}) - \rho^{\frac{5}{4}}(\mathrm{x}_0)\right)}} d\mathrm{x}_0 \qquad = \rho(\mathrm{x})$$

$$\{\mathrm{x} \ \epsilon \text{ Region II, RHS}\} \quad (78)$$

6.5 The complete solutions set

By the previous analysis, we conclude that a very simple interpolative solution to equation (54) between the two regions I, II can also be integrated to the quantum density in every point of space in X; this will still be of the form

$$\mathrm{v}_{_{SP}}(\mathrm{x},\mathrm{x}_0) = \frac{4}{\rho(\mathrm{x})}\sqrt{\nu(\mathrm{x})\,(\nu(\mathrm{x}) - \nu(\mathrm{x}_0))}$$

$$\{\mathrm{x} \ \epsilon \text{ Region I+II } \} \tag{79}$$

where

$$\nu(\mathrm{x}) = \ b(\mathrm{x}) \left| \int_{\mathrm{x}}^{\mathrm{x}_0^*} \rho^4(\mathrm{x}) d\mathrm{x} \right|^{\frac{1}{5}} \ + \ a(\mathrm{x})\, \rho^{\frac{5}{4}}(\mathrm{x})$$

$$\{\mathrm{x} \ \epsilon \text{ Region I+II}\} \tag{80}$$

Here $a(\mathrm{x})$ and $b(\mathrm{x})$ are complemental functions to each other, changing "fast enough" from 0 to the constant values a and b respectively, in the appropriate zones of Regions I and II - in such a way that the limiting behaviors (63) and (68) are reproduced. So the mass flow $\nu(\mathrm{x})$ has been written here as an interpolative expression, accounting for both the behaviors we have found in the two regions. The drift velocity field $\mathrm{v}_{_D}(\mathrm{x},\mathrm{x}_0^*)$ can also, consequently, be written

$$\mathrm{v}_{_D}(\mathrm{x},\mathrm{x}_0^*) = 2\frac{\nu(\mathrm{x})}{\rho(\mathrm{x})} = 2\, b(\mathrm{x})\, \rho^{-1}(\mathrm{x}) \left| \int_{\mathrm{x}}^{\mathrm{x}_0^*} \rho^4(\mathrm{x}) d\mathrm{x} \right|^{\frac{1}{5}} \ + 2 \ a(\mathrm{x})\, \rho^{\frac{1}{4}}(\mathrm{x}) \tag{81}$$

$$\{\mathrm{x} \ \epsilon \text{ Region I+II}\} \tag{82}$$

As is clear, the solution (79)$\div$(82) is integrable to ρ, according to equation (76), all the space of interest throughout. The solution is everywhere analytical and the constraints (70) and (71) can be dropped off within this context.

This last remark actually calls for the question whether (and which ones) other physical constraints could be admitted within our model in order to determine the functions $a(\mathrm{x})$ and $b(\mathrm{x})$. They are likely to depend on the effective mass m_{eff} in equation (54). Investigating this

point is, however, outside the scope of this paper; but is important here to note that a primary constraint to be satisfied, which we have not mentioned yet, is just normalizing to unity the probability density (72) all throughout the available space of turning points.

A qualitative view of equations (79)÷(82) is given in the same fig.(1). A very simple comment to this plot is that in Region I the oscillation center velocity field goes to ∞ in points of space where the density $\rho(x)$ is zero. This circumstance brings no interpretative problem within our present framework but it might call for more accurate modelling on the assumption of a relativistic point of view.

7 Physical interpretation

In this section, we will give brief discussions of the most important points concerning physical interpretation. It is not to say here that the problem of finding consistency between classical and purely quantum models of physical phenomena cannot however, neither in principle, be reduced only to the topics investigated in this paper ([3]). These topics are yet of primary importance to open a path towards the mentioned main research target.

7.1 Statistical appearance of many single-particle trajectories

Equations (72)÷(82) together with previous ones in the paper substantiate the view that it is possible to find Newtonian-like trajectories which, in agreement to a classical model, can be ensemble-averaged to result in the appearance of particles in space with quantum density $\rho(x)$. As far as a many-particles case is considered, interpreting the equations is straightforward - because the density (72) can be directly translated into a relative number of particles following a particular trajectory $v_{SP}(x,x_0)$ with turning point x_0. Then $v_D(x,x_0^*)$ is, precisely, the (Eulerian-like) group velocity characterizing the ensemble flow. As far as a single-particle behavior has to be framed instead, our interpretation is that the same density is just the probability to find the particle on a given trajectory $v_{SP}(x,x_0)$: on one hand, since it is proportional to the period

[3] We have to recall here that the main formal difficulty to be faced by supporters of a causal interpretation of quantum mechanics is extricating the problem of the so-called quantum distribution function [4 − 6], or finding consistent joint probability densities for physical variables. Whether our model developments may bring to positive results in this domain we cannot know at present, and have to let the answer to future work.

of that trajectory, we may assume that the persistence in time of this last is long enough that the trajectory is indeed maintained for a time of the order of the period; on the other hand, we assume that - being on a trajectory - the particle is always allowed to jump to another one, after (statistically) a time of the order of the persistence. Then, statistically as well, all the ensemble of trajectories will be covered by a particle during the time, so that its probabilistic appearance in space is finally given by the ensemble average of densities, $\rho(\mathrm{x})$ and by the ensemble average of velocities, $\mathrm{v}_{D}(\mathrm{x},\mathrm{x}_0^*)$.

7.2 Wave-corpuscle duality and complementarity principle

A discussion of the role that these topics assume in a physical theory where the wave-mechanical background is Newtonian can only be given here a slight indication. In our framework, the quantum density is a statistical density, either describing a many-particles system or, identically, the statistical result of iterated single-particle apparitions - each of them submitted to one specified initial motion condition within the ensemble of possible values of the quantity $\mathrm{P}^2(0,1)/2\mathrm{m}$. In the first appearance, it is obvious that wave-behavior can be supported by a particles ensemble. As concerns the second appearance, in order to understand how the single-particles are able to behave accounting for the wave effects (as - most important - interference and diffraction) we promote the following interpretation. In our framework the particles are point-like ones, but are submitted to an external forcing which is found to depend strictly on its motion state itself (the velocity field $\mathrm{v}(\mathrm{x})$), so that this interaction appears as a non-local, reactive expression of the quantum "vacuum". The interaction distance $<|\xi^*|>$ given in equation (50) has the form of a de Broglie wavelength, what brings us to the interpretation that the vacuum is polarized by the particle motion at this average distance - so that its reaction is, in some way, triggered by the particle itself. The particle would then be able to move "self-piloting" [7] and taking account - in a sense - of the distant vacuum conditions, i.e. following non-local initial conditions as the ones we have shown in equation (53). These last depend indeed on the second derivative of the kinetic energy field. Motion non-local initial conditions and distant vacuum interactions are responsible for the particle to follow a peculiarly anomalous path, and displaying - according to the set of possible conditions - velocity fields and statistical densities summable to the characteristic wave-behavior. Moreover, we remark that the pressure field in the many-particle system has been

shown in paper III, equation (89), to be ineffective in determining behavior variations with respect to the single-particle statistics. Then we come to the interpretation that the quantum mechanical matter-wave behavior - as far as it is obtained as the statistical result of many single-particle iterations - is however, indeed, the feature of a single particle being able to feel distance conditions, because the many particle system behavior is then nothing else that the total addition of all the single behaviors.

It is not to say that "self piloting" and "feeling distance conditions" are folkloristic expressions to mean the peculiar effects of a distance interaction.

The wave-corpuscle duality takes, as a result of the present framework and previous discussion, the simple interpretation that distant vacuum interactions are able to address point-like particles to trajectories whose statistical superposition amounts to wave-like patterns. On the other hand, concerning the complementarity principle, it is clear that our theory is based on the complete, classical description of the two particle motion components : HDF and the classical degree of freedom. The Heisenberg constraint ceases, in our framework, to support the dominant wave-particle dualism philosophy - being brought back to the role of a parametric constraint between classical quantities, the HDF space and velocity co-ordinates.

We have finally to note, yet, that the promoted interpretation enters an investigation domain of effects, as interference and diffraction, which is peculiar of at least two-dimensional space physics. It is then clear that we have to wait for further developments in order to make conclusive points out by the means of direct proofs.

7.3 The "surfing" particle model and a mechanistic concept restored

The de Broglie concept of pilot wave turns out to be the very interesting one to (by attempt) provide a schematic description of the particle behavior we evidenced in this paper. We want only briefly, in this section, promote indeed the following model for further investigation. In our framework, the vacuum exchanges energy with the particle and may be represented as the couple (one forwards, the other backwards the particle position) of reactive points at a distance $<|\xi^*|>$ from the particle. If the vacuum-particle interaction is responsible for a vacuum excitation represented by a wave, then the wavelength λ_w has to be of the order of

some fraction ϑ of $<|\xi^*|>$ (see equation (50)), and we can write :

$$\lambda_w \approx \vartheta\gamma\xi = \frac{\vartheta\gamma\hbar\sqrt{2}}{\mathrm{v(x)}\sqrt{\mathrm{m}\delta\mathrm{m}}} \approx \frac{\mathrm{h}}{\sqrt{\mathrm{m}\delta\mathrm{m}}\ \mathrm{v(x)}} \tag{83}$$

By dimensional considerations, the frequency ν_w would be (see equation (51)):

$$\nu_w \approx \frac{\mid \Phi_{\mathrm{HDF}}(\mathrm{x}) \mid}{\mathrm{h}} = \frac{\delta\mathrm{m}\mathrm{v(x)}^2}{2\mathrm{h}} \tag{84}$$

Then the wave velocity c_w will be

$$c_w = \frac{\partial\nu_w}{\partial 1/\lambda_w} \approx \sqrt{\frac{\delta\mathrm{m}}{\mathrm{m}}}\mathrm{v(x)} \tag{85}$$

In the classical limit, $\delta\mathrm{m}\rightarrow 0$ so that $c_w \rightarrow 0$ as well. In the "full quantum limit" instead - if we assume that $\delta\mathrm{m}\approx\mathrm{m}$, then we have $c_w = \mathrm{v(x)}$ which means that the "particle" and the (forwards) wave travel together. Now for $\delta\mathrm{m} = \mathrm{m}$ we have $\mathrm{m}_{eff} = 0$, i.e. there is no (classical) inertia and this explains by the fact that the classical kinetic energy field $\frac{1}{2}\mathrm{mv}^2$ is totally provided by HDF - i.e. in turn by the vacuum itself. Then we may say (schematically) that the "particle" makes surf on the vacuum waves.

The same concept of particle inertia m_{eff} receives some light from this model: this effective mass is the fraction of mass which is not lifted by the wave. The particle velocity field v(x), in this context, also takes a peculiar physical meaning - as of a velocity assumed, in the quantum limit, within a sort of vacuum structural speedway constituted by the wave velocity field c_w.

This model - and our comprehensive framework indeed - both appear to bring us backward to a mechanistic point of view, because of the vacuum taking a well defined role in expressing a deterministic, reactive force law. Although mechanicalism has been plainly defeated by modern physics, this last might perhaps find itself to face some resumption of this concept if the causal interpretation of quantum mechanics, to which our ideas are intended to contribute, will progress.

8 A few other remarks

Within the proposed interpretative framework, we will certainly meet with the criticism that the interaction radius $<|\xi^*|>$ may assume infinite

values also when the velocity field v(x) goes to zero, perhaps unphysically. Concerning this point, we remark that the real physical actor in the frame is the force

$$-\nabla\Phi_{\mathrm{HDF}}(\mathrm{x}+\xi_i^*) = \frac{1}{2}\delta\mathrm{m}\nabla\mathrm{v}^2(\mathrm{x}+\xi_i^*) =$$

$$= \frac{1}{2}\delta\mathrm{m}\left\{\nabla\mathrm{v}^2(\mathrm{x}) + \xi^*\nabla\mathrm{v}^2(\mathrm{x})' + \frac{\xi^{*2}}{2}\nabla\mathrm{v}^2(\mathrm{x})'' + \nabla\bar{\mathrm{R}}(\mathrm{x},\xi^*)\right\} \tag{86}$$

By iteration of procedures previously introduced we find another expression for $\Phi_{\mathrm{HDF}}(\mathrm{x}+\xi)$:

$$\Phi_{\mathrm{HDF}}(\mathrm{x}+\xi) = -\frac{1}{2}\delta\mathrm{m}\mathrm{v}^2(\mathrm{x}) - \frac{1}{2}\delta\mathrm{m}\int\frac{<\xi^{*2}>}{2}\mathrm{v}^2(\mathrm{x})'''d\mathrm{x} + \mathrm{C}(\mathrm{v}^2) \tag{87}$$

Comparing with equation (49) we find

$$-\frac{1}{2}\delta\mathrm{m}\int\frac{<\xi^{*2}>}{2}\mathrm{v}^{2'''}d\mathrm{x} = -\frac{\hbar^2}{2\mathrm{m}}\frac{\mathrm{v}^{2''}}{\mathrm{v}^2} \tag{88}$$

This equation shows that the physically meaningful interaction radius might better be defined as

$$<|\xi^*|> \simeq \gamma\sqrt{<\xi^{*2}>} = \frac{\gamma\hbar\sqrt{2}}{\sqrt{\mathrm{m}\delta\mathrm{m}}}\sqrt{\frac{1}{\mathrm{v}^{2'''}}\left(\frac{\mathrm{v}^{2''}}{\mathrm{v}^2}\right)'} \tag{89}$$

However, requiring that this expression is finite when $\mathrm{v}(\mathrm{x}) \to 0$ arises the serious question whether the solutions $\mathrm{v}_{_{SP}}(\mathrm{x})$ we have found are compatible with this further constraint imposed to equation (36). The question makes, in turn, clear again the utility to perform non-linear analysis, also looking forward to determine consistent values of different parameters providing a physical solution. We note indeed, for instance, that in this paper we have come to a constant expression for m_{eff} - but, in a more advanced development, this might be found to assume some specific dependence on the density or the x-co-ordinate. The consistency requirements then might be satisfied by "eigen-mass" functions to be determined. At the same time, however, renormalization of the framework might be found necessary to account for a variable $\delta\mathrm{m}(\mathrm{x})$. The overall problem concerning consistency and physical meaning of the

various parameters is of greater purport than what can be afforded in this paper.

We note - as final remarks - that if in equation (36) we are allowed to take $m_{eff} = 0$ then in a further, three-dimensional development, we might perhaps explain orbit zero-angular-momentum states while keeping the oscillation center azimuth velocity field different from zero. Orbit zero-angular-momentum states can however, in our framework, also be explained by the assumption that the particle oscillation center velocity field is zero, so that the (fast) HDF oscillatory motion is the only part of motion remaining to the particle ([4]). Then the azimuth angular velocity time-law may display a zero average consistent with the observation of S-states. If, however, an "unsaturated" expression for the oscillation frequency as given in equation (39) would still hold in more advanced developments, then a relativistic model would be required for a complete description of such a contingency.

It is not to say that these statements or remarks are at present only indications or conjectures useful to stimulate further criticism and investigation.

9 Conclusion

In the present and correlated papers, by some physical/mathematical considerations we address a proposed path to solve the problem of identifying a classical-like context subtending the matter-wave physics. We find out an energy theorem form for the particle classical degree of freedom, and we give a proposed solution to the equation - complying with our primary requirement to result back in the appropriate quantum density after statistical averaging.

The physical model expounded in our comprehensive work can be resumed as follows. Classical oscillators become similar (as far as the properties we have analyzed are concerned) to quantum ones when they are submitted to a peculiar perturbation due the quantum vacuum. This perturbation could be called a "generalized" Kapitza one. The resulting time-averaged, generalized Kapitza theorem for the particle oscillation center has a form where a couple of unknown functions (m_{eff} and C) can have some role - but the dominant term is the one displaying a space

[4] Oscillatory motion due to a fast perturbation is described, to the end of classical-like interpretation of a quantum particle motion in three-dimensional space, as early as in reference [7].

second derivative in the (squared) velocity field. We solve the equation accounting for various requirements so that our final interpretation in the paper is as expounded in the previous sections: considering equations (79)÷(82) as a possible issue for the still debated problem of identifying a classical-like view of quantum effects - at least as a starting point towards more extensive, next analysis.

We believe conclusively having contributed a physical framework aimed to provide new stimulus towards an investigative path looking at a re-statement of a classical concept of particle motion and mechanics. We discussed our equations with reference to the most important properties we identified, relevant to the purpose of demonstrating self consistency and worthiness of further analysis. The intrinsic limit of our framework consisting in the boundary of uni-dimensional, stationary analysis, we could only give slight indications concerning the rotational motion and the wave behavior. Yet we believe that generalized developments might provide epistemological advances towards the end of restoring the causality principle and increasing our capabilities in technological control on the physical matter.

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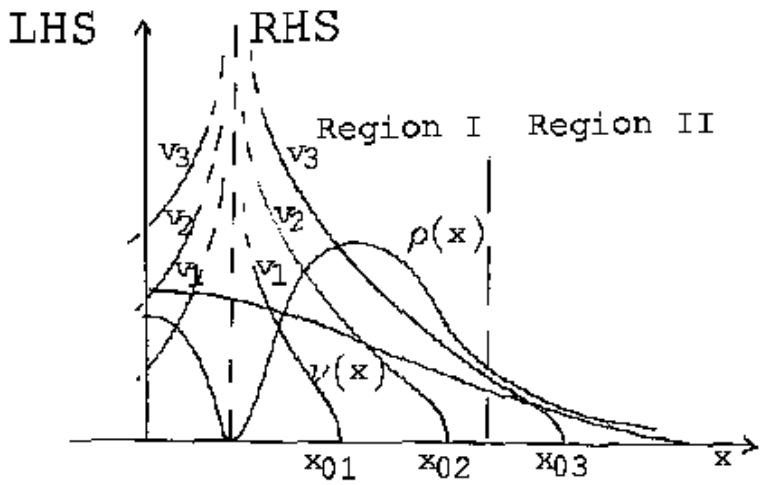

Figure 1. (Qualitative) behaviours of the solutions $v_{SP}(x,x_0)$ for a few values of x_0 and of the corresponding volume flow $\nu(x)$ for the case of the n = 3 level of a quantum harmonic oscillator. $\rho(x)$ is the quantum density; all the quantities are in arbitrary units.