Particle-wave Duality for Spatially-localized Complex Fields

I. The Assumptions and some Consequences

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ABSTRACT. Part I derives the form of the nonlinear Schrödinger equations from three natural assumptions. Essentially they are: 1. The fields obey complex Hamiltonian evolution equations. 2. If the Hamiltonian functional is space-translation invariant the corresponding velocity functional must also be space-translation invariant. 3. The Galilei transform of a stationary spatially localized field is a solution of the same equation of which the stationary field is a solution. Part II establishes the existence of particle-wave duality for spatially localized field which satisfy the above three assumptions. This is done by showing that such fields are associated with waves for which de Broglie-type relations hold. Dirac's quantization rules are discussed in relation to the above developments.

1. Introduction

Spatially localized fields, i.e. fields which are appreciably distinct from zero only in a bounded region of space, have been studied as possible *representations* of elementary particles since when the concept of an elementary (discrete) electrical charge became established in physics. Lorentz and Abraham (with their work on the extended electron) were the first to demonstrate that such representations can lead to remarkable results. De Broglie's *Theory of the Double Solution* [1] is perhaps the most prominent attempt to account for some of the postulates of quantum mechanics by replacing the concept of a *point particle* with that of a spatially localized field.

All the results in this paper depend critically on the fact that certain *nonlinear* field equations possess *spatially localized* (also called *soliton-like*) solutions. The existence of such solutions to a large family of *scalar* nonlinear field equations, including the *nonlinear Schrödinger* (NLS) equations, is proved and existence conditions are derived in Berestycki and Lions [2]. The existence of such solutions to *spinor* nonlinear field equations is proved in Cazenave and Vazquez [3]. In addition, there are many publications which demonstrate the existence and illustrate some of the properties of localized solutions by numerical methods. Some additional references are: Lee [4] (Ch. 7), Finkelstein and Fronsdal [5], Bialynicki-Birula and Mycielski [6], Freidberg and Lee [7], Cooperstock and Rosen [8], Enz [9], Bodurov [10, 11, 12, 13, 14]. It should be also pointed out that large number of works in Soliton theory investigate the *solitons* as one-dimensional models of elementary particles. Reference [15] is a collection of such papers.

This paper, which is in two parts, poses and answers the question: Can particle-wave duality be established for spatially localized fields, as representations of elementary particles, without relying on the postulates of QM? Clearly, the spatially localized solutions of certain nonlinear field equations will be far more proper representations of elementary particles if de Broglie-type relations hold for the energy and momentum of such fields. In answering the above question, this paper shows that there are a number of unexpected links between nonlinear field theory and quantum mechanics.

Notation. All functionals are denoted with italic capital letters, all densities — with script capital letters, all linear differential (but not multiplication) operators — with capital letters with a "hat" $\hat{}$, all complex fields — with the Greek letters: ψ, φ and η. The complex conjugate of ψ is ψ^*. The summation convention of repeated indexes is assumed for the entire paper. All integrals' domains are $\mathbb{R}^n$.

2. Complex Hamiltonian evolution equations

The mathematics employed in this paper is commonly known, except for the family of *complex Hamiltonian evolution* (CHE) equations. These equations will be given a special introduction since they play very important role in this paper and since the reader is not expected to have encountered them previously.

Let $\psi_\sigma = \psi_\sigma(x,t)$ be the σ-component of a complex field which may be a spinor, vector, scalar field, or a set of coupled scalar fields defined on the entire Euclidian space $I\!R^n$ whose coordinates are $x = (x_1, \ldots, x_n)$ and t is the time. The CHE equations are

$$\frac{\partial \psi_\sigma}{\partial t} = -i \frac{\delta H}{\delta \psi_\sigma^*}\,, \qquad H = \int_{I\!R^n} \mathcal{H}(x, t, \psi^*, \psi, \partial\psi^*, \partial\psi, \ldots)\, d^n x \qquad (2.1)$$

where $H = H[\psi^*, \psi; t]$ is the *Hamiltonian functional* which must be *real-valued*, $\partial\psi$ denotes all space-derivatives $\partial\psi_\sigma/\partial x_j$ and

$$\frac{\delta H}{\delta \psi_\sigma^*} = \frac{\partial \mathcal{H}}{\partial \psi_\sigma^*} - \frac{d}{dx_k} \frac{\partial \mathcal{H}}{\partial(\partial_k \psi_\sigma^*)} + \cdots$$

is the *variational derivative* of H with respect to ψ_σ^*. For details see Ref. 13. Clearly, equations (2.1) are meaningful only if the Hamiltonian functional H remains finite for all t when evaluated with a solution ψ of (2.1). A necessary condition for this is: ψ_σ are appreciably distinct from zero only in some bounded region of space. Hence, we have the following formal definitions:

Definition 2.1. *A scalar, vector or spinor-valued function* $\psi = \psi(x)$ *of the coordinates* $x \in I\!R^n$ *will be called spatially localized, or* ψ*-field, if it is singularity-free and if its norm (squared)* $N = \int \psi_\sigma^* \psi_\sigma \, d^n x$, *with the integral taken over all space, is finite.*

Faddeev and Takhtajan [17] seem to be the first who introduced equations of the form (2.1) in Soliton theory for fields in one space-dimension. In [13] I showed how the CHE equations can be deduced from the classical finite-dimensional Hamiltonian systems, derived some of their most general properties and discussed their significance in non-linear field theory. The family (2.1) contains both linear and nonlinear equations which are entirely independent from the postulates of quantum mechanics.

The Poisson bracket associated with the CHE equations is (see [13] and [17])

$$\{R, S\} = i \int_{I\!R^n} \left(\frac{\delta R}{\delta \psi_\sigma^*} \frac{\delta S}{\delta \psi_\sigma} - \frac{\delta S}{\delta \psi_\sigma^*} \frac{\delta R}{\delta \psi_\sigma} \right) d^n x \qquad (2.2)$$

where $R = R[\psi^*, \psi; t]$ and $S = S[\psi^*, \psi; t]$ are real or complex-valued functionals.

All CHE equations possess the property: *The field's energy density of a* ψ*-field, i.e. the component* $\mathcal{T}_{00}$ *of the stress-energy 4-tensor, is equal to the Hamiltonian density* $\mathcal{H}$. To verify this observe that the CHE equations (2.1) are the same as the Euler-Lagrange equations obtained from the *Lagrangian* density

$$\mathcal{L} = \frac{i}{2}\Big(\psi_\sigma^* \frac{\partial \psi_\sigma}{\partial t} - \frac{\partial \psi_\sigma^*}{\partial t}\psi_\sigma\Big) - \mathcal{H} \ . \tag{2.3}$$

Inserting (2.3) in the $\mathcal{T}_{00}$ component of the stress-energy 4-tensor yields the desired result:

$$\begin{aligned}\mathcal{T}_{00} &= \frac{\partial \psi_\sigma}{\partial t}\,\frac{\partial \mathcal{L}}{\partial(\partial_t \psi_\sigma)} + \frac{\partial \psi_\sigma^*}{\partial t}\,\frac{\partial \mathcal{L}}{\partial(\partial_t \psi_\sigma^*)} - \mathcal{L} \\ &= \frac{i}{2}\Big(\psi_\sigma^* \frac{\partial \psi_\sigma}{\partial t} - \frac{\partial \psi_\sigma^*}{\partial t}\psi_\sigma\Big) - \mathcal{L} = \mathcal{H} \ .\end{aligned} \tag{2.4}$$

For the derivations in Part II we identify the ψ-field total energy with the value of the Hamiltonian functional. Equation (2.4) states that such an identification is well justified.

3. The three assumptions and some of their consequences

Consider a *scalar complex* field $\psi = \psi(x,t)$ with a *single* region of localization in n-dimensional Euclidian space $\mathbb{R}^n$. If we are to study the motion of ψ , regarded as a *whole entity*, we will need to define the position of its *region of localization* in the $\mathbb{R}^n$ space, i.e., the coordinates of its *center of localization*. This is done, as for any other distribution (in the present case the distribution function is $\psi^*\psi$), by the following

Definition 3.1. *If* $\psi = \psi(x,t)$ *is a spatially localized solution with a single region of localization the coordinates of its center of localization are given by the functionals*

$$X_j = \frac{1}{N}\int_{\mathbb{R}^n} \psi^* \psi\, x_j \, d^n x \ , \qquad j = 1, \ldots, n \tag{3.1}$$

where $N = \int \psi^* \psi \, d^n x$ *is the field's norm (squared).*

The field is not normalized since it is a solution of a nonlinear equation. It is self-evident that the field's norm N must never vanish. To assure this we have the following

Assumption 3.1. *The scalar complex field* $\psi = \psi(x,t)$ *is a solution of some complex Hamiltonian evolution equation whose form is*

$$\frac{\partial\psi}{\partial t} = -i\,\frac{\delta H}{\delta\psi^*}\ , \qquad H = \int_{I\!R^n} \mathcal{H}(\psi^*, \psi, \partial\psi^*, \partial\psi)\, d^n x \tag{3.2}$$

where H *is the Hamiltonian functional which must be real-valued. The space-derivatives* $\partial\psi = (\partial\psi/\partial x_1, \ldots, \partial\psi/\partial x_n)$ *on which the density* $\mathcal{H}$ *depends are of the first order only.*

The CHE equations (3.2) are Hamiltonian only if H is real-valued, since only then the equation $\partial\psi^*/\partial t = i\,\delta H/\delta\psi$ holds. In this case, equations of the form (3.2) are conservative when in addition H is time-independent which is seen from

$$\frac{dH}{dt} = \int \left(\frac{\delta H}{\delta\psi^*}\,\frac{\partial\psi^*}{\partial t} + \frac{\delta H}{\delta\psi}\,\frac{\partial\psi}{\partial t} \right) d^n x = i \int \left(\frac{\delta H}{\delta\psi^*}\,\frac{\delta H}{\delta\psi} - \frac{\delta H}{\delta\psi}\,\frac{\delta H}{\delta\psi^*} \right) d^n x = 0\,.$$

The conservation of the Hamiltonian functional H assures the non-vanishing of ψ which in turn assures the non-vanishing of N, although N may not be constant in time.

With Definition 3.1 and Assumption 3.1 we can obtain the velocity of a localized field as a whole entity, i.e., the *velocity of the localization center*, by simply differentiating the functionals X_j with respect to time

$$V_j = \frac{dX_j}{dt} = \frac{1}{N} \int \left(\frac{\partial\psi^*}{\partial t}\psi + \psi^* \frac{\partial\psi}{\partial t} \right) x_j\, d^n x - \frac{X_j}{N}\,\frac{dN}{dt}\ .$$

Using the field equations (3.2) and integrating by parts this becomes

$$\begin{aligned} V_j = {} & \frac{i}{N} \int \left(\psi\,\frac{\partial\mathcal{H}}{\partial\psi} - \psi^*\,\frac{\partial\mathcal{H}}{\partial\psi^*} + \partial_k\psi\,\frac{\partial\mathcal{H}}{\partial(\partial_k\psi)} - \partial_k\psi^*\,\frac{\partial\mathcal{H}}{\partial(\partial_k\psi^*)} \right) x_j\, d^n x \\ & + \frac{i}{N} \int \left(\psi\,\frac{\partial\mathcal{H}}{\partial(\partial_j\psi)} - \psi^*\frac{\partial\mathcal{H}}{\partial(\partial_j\psi^*)} \right) d^n x - \frac{X_j}{N}\,\frac{dN}{dt} \end{aligned} \tag{3.3}$$

where $\partial_j\psi = \partial\psi/\partial x_j$. When no additional restrictions are imposed on $\mathcal{H}$ the coordinate x_j will remain in the resulting integrands as an explicit argument. Consequently, the value of the velocity functional will depend on the ψ-field position in space even when the Hamiltonian density $\mathcal{H}$ is not an explicit function of $\boldsymbol{x}$. But "$\mathcal{H}$ is not an explicit function of $\boldsymbol{x}$" means that the ψ-field does not interact with any external agents/fields. Hence, the velocity of the localization center must be constant which is in contradiction with the previous conclusion. The only means to avoid this contradiction is to restrict the ψ-field equations (or $\mathcal{H}$) to those for which the integrands in (3.3) are not explicit functions of the coordinates. This is equivalent to saying that the velocity functional V_j must be invariant with respect to space-translations. Note that, a functional $F[u(x)]$ of $u(x)$ is invariant under space-translations if $F[u(x+a)] = F[u(x)]$ for any $a = \text{const}$. Thus, we have

Assumption 3.2. *If the Hamiltonian functional H in (3.2) is invariant under space-translations then the functionals $V_j = dX_j/dt$, which give the velocity of the localization center of a spatially localized solution of (3.2), must also be invariant under space-translations..*

Next, some general conclusions will be drawn solely from the above two assumptions. Assumption 3.3 will be stated after that. If we take H to be space-translation invariant, i.e. if we choose the Hamiltonian density $\mathcal{H}$ not to be an explicit function of x, then the above expression for V_j must be also space-translation invariant according to Assumption 3.2. However, the first integral in (3.3) is not translation invariant, because its integrand is a translation invariant expression multiplied by x_j. The third term in (3.3) is not translation invariant, because while $N^{-1}\, dN/dt$ is translation invariant X_j is not.

Hence, V_j will be translation invariant only if the first and the third terms in (3.3) are identically zero. Moreover, they must vanish separately since in the first integral x_j multiplies an expression which contains the derivatives $\partial_j\psi$ and $\partial_j\psi^*$ while in the integrand of X_j no derivatives are present. Observing that ψ is an arbitrary solution of (3.2), whose localization center may be any where in $\mathbb{R}^n$, we see that the first integral in (3.3) vanishes if

$$\psi\,\frac{\partial\mathcal{H}}{\partial\psi} - \psi^*\frac{\partial\mathcal{H}}{\partial\psi^*} + \partial_k\psi\,\frac{\partial\mathcal{H}}{\partial(\partial_k\psi)} - \partial_k\psi^*\,\frac{\partial\mathcal{H}}{\partial(\partial_k\psi^*)} = 0 \qquad (3.4)$$

holds identically. Two important results follow from equation (3.4):

Proposition 3.1. *The Hermitian norm N of ψ is a constant of the motion if ψ is a solution of a CHE equation whose Hamiltonian density $\mathcal{H}$ satisfies (3.4).*

Indeed, taking the time derivative of N, integrating by parts and applying (3.4)

$$\frac{dN}{dt} = i\int\left(\psi\,\frac{\delta H}{\delta\psi} - \frac{\delta H}{\delta\psi^*}\,\psi^*\right)d^nx$$
$$= i\int\left(\psi\,\frac{\partial\mathcal{H}}{\partial\psi} - \psi^*\,\frac{\partial\mathcal{H}}{\partial\psi^*} - \psi\,\frac{d}{dx_k}\frac{\partial\mathcal{H}}{\partial(\partial_k\psi)} + \psi^*\,\frac{d}{dx_k}\frac{\partial\mathcal{H}}{\partial(\partial_k\psi^*)}\right)d^nx$$
$$= i\int\left(\psi\,\frac{\partial\mathcal{H}}{\partial\psi} - \psi^*\,\frac{\partial\mathcal{H}}{\partial\psi^*} + \partial_k\psi\,\frac{\partial\mathcal{H}}{\partial(\partial_k\psi)} - \partial_k\psi^*\,\frac{\partial\mathcal{H}}{\partial(\partial_k\psi^*)}\right)d^nx = 0$$

proves the claim. Consequently, the third term in (3.3) vanishes as required and we obtain

$$V_j = \frac{i}{N}\int\left(\psi\,\frac{\partial\mathcal{H}}{\partial(\partial_j\psi)} - \psi^*\,\frac{\partial\mathcal{H}}{\partial(\partial_j\psi^*)}\right)d^nx \qquad (3.5)$$

for the velocity of the region of localization. The second conclusion following from (3.4) is:

Proposition 3.2. *When Assumptions 3.1 and 3.2 are met the Hamiltonian density* $\mathcal{H}$ *is invariant under the gauge type I transformation* [18]

$$\psi' = \psi\,e^{i\varepsilon}\,, \qquad \psi'^* = \psi^*\,e^{-i\varepsilon} \qquad (3.6)$$

where ε *is a real transformation parameter (independent of the coordinates and time).*

In fact, this is true for any function of ψ, ψ^* and $\partial_j\psi$, $\partial_j\psi^*$ for which equation (3.4) holds. To verify this claim let $\mathcal{H}' = \mathcal{H}(\psi', \psi'^*, \partial\psi', \partial\psi'^*)$ be the transform of $\mathcal{H}$ under (3.6). Then the condition for the invariance of $\mathcal{H}$ under (3.6)

$$\left.\frac{d\mathcal{H}'}{d\varepsilon}\right|_{\varepsilon=0} = i\left(\psi\,\frac{\partial\mathcal{H}}{\partial\psi} - \psi^*\,\frac{\partial\mathcal{H}}{\partial\psi^*} + \partial_j\psi\,\frac{\partial\mathcal{H}}{\partial(\partial_j\psi)} - \partial_j\psi^*\,\frac{\partial\mathcal{H}}{\partial(\partial_j\psi^*)}\right) = 0$$

is, clearly, the same as equation (3.4).

Now, the form of $\mathcal{H}$ can be made more specific. Since $\mathcal{H}$ is a scalar under rotations in $\mathbb{R}^n$ it can depend on $\partial_j\psi$ and $\partial_j\psi^*$ only via the forms: $\partial_j\psi\,\partial_j\psi$, $\partial_j\psi^*\,\partial_j\psi$ and $\partial_j\psi^*\,\partial_j\psi^*$. However, only the second of these forms, i.e. $\partial_j\psi^*\,\partial_j\psi = \nabla\psi^*\cdot\nabla\psi$ makes $\mathcal{H}$ gauge I invariant with respect to $\partial_j\psi$ and $\partial_j\psi^*$. Furthermore, $\mathcal{H}$ will be gauge

I invariant with respect to all its arguments, as required by Proposition 3.2, if it is gauge type I invariant with respect to ψ and ψ^*. It is simple to show that any function $F(\psi^*,\psi)$ which is gauge type I invariant depends on ψ and ψ^* only via $\psi^*\psi$. Namely, let $\psi = u+iw$, so that $F(\psi^*,\psi) = \bar{F}(u,w)$. Then the gauge I transformation (3.6) is a rotation in the uw-plane. Hence, $\bar{F}$ is invariant only if $\bar{F}(u,w) = f(u^2+w^2)$, and the above claim follows from

$$F(\psi^*,\psi) = \bar{F}(u,w) = f(u^2+w^2) = f(\psi^*\psi) \ .$$

Consequently, the Hamiltonian density in (3.2) must be of the form

$$\mathcal{H} = \mathcal{H}(\psi^*\psi, \nabla\psi^* \cdot \nabla\psi) \ . \tag{3.7}$$

Definition 3.2. *A complex spatially localized field ψ is stationary if its magnitude $|\psi|$ is not a function of time, i.e. if its form, with ω being a real constant, is*

$$\psi = \varphi(\boldsymbol{x})\, e^{-i\omega t} \tag{3.8}$$

For all ψ with the form (3.8) the equality $H[\psi^*,\psi] = H[\varphi^*,\varphi]$ holds if $\mathcal{H}$ is gauge type I invariant (since there are no time-derivatives in $\mathcal{H}$), i.e. if Assumption 3.2 holds. From this follows that a CHE equation possesses a stationary spatially localized solution if $\mathcal{H}$ is gauge type I invariant and if the associated time-independent CHE equation

$$\omega\varphi = \frac{\delta H[\varphi^*,\varphi]}{\delta\varphi^*} \tag{3.9}$$

possesses a localized solution $\varphi(x)$. The velocity of the localization center of a stationary solution is zero by definition. Hence, according to (3.5) such solution satisfies the condition

$$\int\left(\varphi\,\frac{\partial\mathcal{H}}{\partial(\partial_j\varphi)} - \varphi^*\,\frac{\partial\mathcal{H}}{\partial(\partial_j\varphi^*)}\right)d^n x = 0 \ . \tag{3.10}$$

The last assumption concerns the behavior of a spatially localized field ψ under Galilei transformations $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t$, with $\boldsymbol{v} = \text{const}$. A fundamental requirement for any field equation is that it must be invariant under Galilei or Lorentz transformations, i.e., the Galilei or

Lorentz transform ψ' of ψ must be a solution of the same equation of which ψ is a solution. However, it should be observed immediately that if ψ is a stationary solution (3.8) its Galilei transform cannot be $\varphi(\boldsymbol{x} - \boldsymbol{v}t)\exp(-i\omega t)$. This is so because the velocity functional (3.5) when evaluated with the latter expression is zero, according to (3.10), which is a contradiction. This is reflected in the third

Assumption 3.3. *The field equations are invariant under Galilei transformations and the Galilei transform of a stationary spatially localized field ψ (3.8) which has the form*

$$\psi' = \varphi' e^{i\vartheta} = \varphi(\boldsymbol{x} - \boldsymbol{v}t)\, e^{i\vartheta} \tag{3.11}$$

is a solution of the same equation of which ψ is a solution. $\vartheta = \vartheta(\boldsymbol{v}, \boldsymbol{x}, t)$ is some real-valued function (to be determined) of $\boldsymbol{x}$, t and the velocity $\boldsymbol{v}$.

To find the function ϑ and the family of equations for which all three assumptions hold we insert the ansatz (3.11) for the transformed field ψ', whose localization center is moving with the velocity $\boldsymbol{v}$, into the Hamiltonian density (3.7), and take into account that $\nabla\psi' = \big(\nabla\varphi' + i\varphi'\nabla\vartheta\big)e^{i\vartheta}$. Thus, the Hamiltonian density for the "moving" field is

$$\begin{aligned}\mathcal{H}' &= \mathcal{H}\big(\psi'^*\psi',\, \nabla\psi'^*\cdot\nabla\psi'\big)\\ &= \mathcal{H}\big(\varphi'^*\varphi',\, (\nabla\varphi'^* - i\varphi'^*\nabla\vartheta)\cdot(\nabla\varphi' + i\varphi'\nabla\vartheta)\big)\,.\end{aligned} \tag{3.12}$$

Since $\nabla\vartheta$ is a function of $\boldsymbol{x}$ it follows that $\mathcal{H}'$ and hence V_j are not translation invariant, which means that the value of the velocity functional will depend on the position of the localization center. This contradicts the assumption that the velocity $\boldsymbol{v}$ is constant. The contradiction could be avoided by assuming that ϑ does not depend on $\boldsymbol{x}$. But then, from (3.5), (3.10) and (3.12) follows that $V_j = 0$ for any ψ', which is another contradiction. Hence, the only choice is to take

$$\vartheta = \boldsymbol{k}\cdot\boldsymbol{x} + \lambda \tag{3.13}$$

where $\boldsymbol{k} = \boldsymbol{k}(\boldsymbol{v})$ is a vector-valued function of $\boldsymbol{v}$ only, and $\lambda = \lambda(\boldsymbol{v}, t)$ is a scalar-valued function of $\boldsymbol{v}$, t only. In this case (3.12) becomes

$$\mathcal{H}' = \mathcal{H}\big(\varphi'^*\varphi',\, (\nabla\varphi'^* - i\boldsymbol{k}\,\varphi'^*)\cdot(\nabla\varphi' + i\boldsymbol{k}\,\varphi')\big) = \mathcal{H}\big(\varphi'^*\varphi',\, \boldsymbol{\eta}'^*\cdot\boldsymbol{\eta}'\big) \tag{3.14}$$

with the use of the notation $\boldsymbol{\eta}' = \nabla\varphi' + i\boldsymbol{k}\,\varphi'$ for convenience.

The objective now is to make the function $\mathcal{H}\big(\varphi'^*\varphi',\, \boldsymbol{\eta}'^*\cdot\boldsymbol{\eta}'\big)$ as specific as possible by relying only on the Assumptions 2.1, 2.2 and 2.3. When (3.14) is inserted in the functional (3.5) the result is a real-valued function of the constants k_1, k_2, k_3 which must be equal to the velocity of the localization center, i.e.

$$V_j = V_j[\varphi'^*, \varphi'; k] = f(k_1, k_2, k_3) = v_j \;. \tag{3.15}$$

Moreover, the above must hold for the Galilei transform (with parameters v_j) of any stationary spatially localized field. Consequently, the function $f(k_1, k_2, k_3)$ in (3.15) cannot depend on the field φ' in any way. This also shows why k_i cannot be a function of t.

Condition (3.15) imposes, via (3.5), a severe restriction on the Hamiltonian density. To find all $\mathcal{H}$ which satisfy (3.15) insert (3.14) into the velocity functional (3.5) for ψ'

$$\begin{aligned} V_j &= \frac{i}{N}\int\left(\psi'\,\frac{\partial\mathcal{H}'}{\partial(\partial_j\psi')} - \psi'^*\frac{\partial\mathcal{H}'}{\partial(\partial_j\psi'^*)}\right)d^nx \qquad (3.16)\\ &= \frac{i}{N}\int\left(\varphi'\,\frac{\partial\mathcal{H}'}{\partial(\partial_j\varphi')} - \varphi'^*\frac{\partial\mathcal{H}'}{\partial(\partial_j\varphi'^*)}\right)d^nx \\ &= \frac{2k_j}{N}\int\frac{\partial\mathcal{H}'}{\partial|\boldsymbol{\eta}'|^2}\,\varphi'^*\varphi'\,d^nx + \frac{i}{N}\int\frac{\partial\mathcal{H}'}{\partial|\boldsymbol{\eta}'|^2}\left(\varphi'\partial_j\varphi'^* - \varphi'^*\partial_j\varphi'\right)d^nx \end{aligned}$$

where $|\boldsymbol{\eta}'|^2 = \boldsymbol{\eta}'^*\cdot\boldsymbol{\eta}'$ and the factor $\partial\mathcal{H}'/\,\partial|\boldsymbol{\eta}'|^2$ is a function of $\varphi'^*\varphi'$ and $\boldsymbol{\eta}'^*\cdot\boldsymbol{\eta}'$. Since $\int\psi'^*\psi'\,d^nx = N$, the first term becomes independent of φ' and φ'^*, as required by (3.15), only if

$$\frac{\partial\mathcal{H}'}{\partial|\boldsymbol{\eta}'|^2} = \mu = \text{const.} \tag{3.17}$$

where μ is some real constant. An integration of (3.17) yields

$$\begin{aligned} \mathcal{H}' &= \mu\,\boldsymbol{\eta}'^*\cdot\boldsymbol{\eta}' + \mathcal{G}(\varphi'^*\varphi') \\ &= \mu\,(\nabla\varphi'^* - i\boldsymbol{k}\,\varphi'^*)\cdot(\nabla\varphi' + i\boldsymbol{k}\,\varphi') + \mathcal{G}(\varphi'^*\varphi') \end{aligned} \tag{3.18}$$

where $\mathcal{G} = \mathcal{G}(\varphi'^*\varphi')$ is some function of $|\varphi'|^2$, at least once differentiable. In addition, $\mathcal{G}(0) = 0$ must hold in order for H to be finite when evaluated with a localized field. Written in terms of ψ' (3.18) is

$\mathcal{H}' = \mu\, \nabla\psi'^{*} \cdot \nabla\psi' + \mathcal{G}(\psi'^{*}\psi')$. From this we see that for the first term in (3.16) to be independent of φ' , φ'^{*} the Hamiltonian density must be

$$\mathcal{H} = \mu \nabla\psi^{*} \cdot \nabla\psi + \mathcal{G}(\psi^{*}\psi) \ . \tag{3.19}$$

With (3.17) the second term in (3.16) vanishes

$$\begin{aligned}\int \frac{\partial \mathcal{H}'}{\partial |\boldsymbol{\eta}|^2} \left(\varphi' \partial_j \varphi'^{*} - \varphi'^{*} \partial_j \varphi'\right) d^n x &= \mu \int \left(\varphi' \partial_j \varphi'^{*} - \varphi'^{*} \partial_j \varphi'\right) d^n x \\ &= \mu \int \left(\varphi\, \partial_j \varphi^{*} - \varphi^{*} \partial_j \varphi\right) d^n x = 0\end{aligned}$$

since when $\mathcal{H}$ is given by (3.19) the integral $\int \left(\varphi\, \partial_j \varphi^{*} - \varphi^{*} \partial_j \varphi\right) d^n x$ is proportional to the localization center velocity (j-component) of the stationary field, which is zero by definition. See (3.10). Therefore, (3.16) reduces to

$$V_j = \frac{2\mu k_j}{N} \int \varphi'^{*} \varphi'\, d^n x = \frac{2\mu k_j}{N} \int \varphi^{*} \varphi\, d^n x = 2\mu\, k_j \tag{3.20}$$

which together with the condition (3.15) produces

$$\boldsymbol{k} = \frac{1}{2\mu}\, \boldsymbol{v} \ . \tag{3.21}$$

The most general form of the field equations which are compatible with Assumptions 3.1, 3.2 and 3.3 is obtained by inserting the Hamiltonian density (3.19), just found, into the form of the CHE equations (3.2):

$$i\, \frac{\partial \psi}{\partial t} = -\mu \nabla^2 \psi + G(\psi^{*}\psi)\psi \tag{3.22}$$

where $G(\rho) = d\mathcal{G}(\rho)/d\rho$. This is the family of nonlinear Schrödinger equations.

4. The mass of a space-localized field

Substituting $\mathcal{H}$ from (3.19) into (3.5) one finds the velocity functional

$$V_j = \frac{i\mu}{N} \int \left(\psi\, \partial_j \psi^{*} - \psi^{*} \partial_j \psi\right) d^n x = \frac{2\mu}{iN} \int \psi^{*} \frac{\partial \psi}{\partial x_j}\, d^n x \tag{4.1}$$

whose arguments, now, are the spatially localized solutions of the NLS equations (3.22).

The mass m of a localized ψ-field can be defined by requiring the classical relation $\boldsymbol{p} = m\,\boldsymbol{v}$ to hold between the velocity V_j (4.1) and the momentum functionals P_j :

$$P_j = m V_j = \frac{2\mu m}{iN} \int \psi^* \, \frac{\partial \psi}{\partial x_j} \, d^n x \ . \tag{4.2}$$

One way to express m in terms of μ and N (alternative derivation can be found at the end of Sec. 2, Part II) is to require, in complete correspondence with classical mechanics, that the values of the functionals X_j (3.1) and P_k (4.2) are *canonically conjugate* variables, i.e., that they satisfy the following Poisson bracket relations:

$$\{X_j\,, P_k\} = \delta_{jk} \ , \quad \{X_j\,, X_k\} = 0 \ , \quad \{P_j\,, P_k\} = 0 \ . \tag{4.3}$$

Since X_j and P_j are functionals, the brackets are of the infinite-dimensional type (2.2). While the second and third relations hold identically, the first one produces

$$\begin{aligned} \{X_j\,, P_k\} &= \frac{1}{i} \int \left(\frac{\delta X_j}{\delta \psi} \, \frac{\delta P_k}{\delta \psi^*} - \frac{\delta P_k}{\delta \psi} \, \frac{\delta X_j}{\delta \psi^*} \right) d^n x \\ &= -\frac{2\mu m}{N^2} \int \left(\psi^* x_j \, \frac{\partial \psi}{\partial x_k} + \frac{\partial \psi^*}{\partial x_k} \, x_j \, \psi \right) d^n x = \frac{2\mu m}{N} \, \delta_{jk} \ . \end{aligned}$$

Hence, (4.3) will hold if we take the field's mass to be

$$m = \frac{N}{2\mu} \ . \tag{4.4}$$

With this relation the Hamiltonian density (3.19) becomes

$$\mathcal{H} = \frac{N}{2m} \nabla \psi^* \cdot \nabla \psi + \mathcal{G}(\psi^* \psi) \tag{4.5}$$

and the NLS equation (3.22) becomes

$$i \, \frac{\partial \psi}{\partial t} = -\frac{N}{2m} \nabla^2 \psi + G(\psi^* \psi) \psi \ . \tag{4.6}$$

If we set $G = 0$ in (4.6) we obtain Schrödinger's equation for a free particle with mass m except *the field's norm N appears in place of Planck's constant $\hbar$.* The dimensionality of N is $[action]$ which, indeed, is the dimensionality of Planck's constant. To see this, observe that according to (2.4) the dimensionality $[\mathcal{H}]$ of the Hamiltonian density is $[energy]\,/\,[volume]$. Then, from the form of the CHE equations

$$i\,\frac{\partial\psi}{\partial t} = \frac{\delta H}{\delta\psi^*} = \frac{\partial\mathcal{H}}{\partial\psi^*} + \cdots$$

the above claim follows:

$$\begin{aligned}[N] &= [\psi^*\psi]\cdot[volume] = [\mathcal{H}]\cdot[time]\cdot[volume]\\ &= [energy]\cdot[time] = [action]\end{aligned}$$

where $[a]$ denotes the dimensionality of the variable a.

The statement "... the norm N appears in place of Planck's constant $\hbar$" is not meant to imply that the value of $\hbar$ can be calculated by calculating the norm N of some spatially localized solution of some nonlinear field equation.

Part II of this paper utilizes the results obtained here to show that the spatially localized fields, discussed above, obey de Broglie-type relations.

References

All references are listed at the end of Part II of this paper.

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