Particle-wave Duality for Spatially-localized Complex Fields

II. De Broglie-type Relations

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ABSTRACT. Part I derives the form of the nonlinear Schrödinger equations from three natural assumptions. Essentially they are: 1. The fields obey complex Hamiltonian evolution equations. 2. If the Hamiltonian functional is space-translation invariant the corresponding velocity functional must also be space-translation invariant. 3. The Galilei transform of a stationary spatially localized field is a solution of the same equation of which the stationary field is a solution. Part II establishes the existence of particle-wave duality for spatially localized field which satisfy the above three assumptions. This is done by showing that such fields are associated with waves for which de Broglie-type relations hold. Dirac's quantization rules are discussed in relation to the above developments.

1. Introduction

Some of the most prominent physicists like L. de Broglie and A. Einstein have insisted that elementary particles should be represented by spatially localized fields which are solutions of certain nonlinear field equations. Such a representation immediately resolves at least one of the most serious difficulties of the traditional point-like model, which is: The energy of a spatially localized field is finite, while the energy of a point-like electric charge is infinite. However, to be completely viable this representation must be capable of representing not only particles but also the waves associated with these particles. This paper shows that, indeed, there are spatially localized fields which are associated with waves and that the field's energy and momentum obey de Broglie-type relations.

The results in this paper depend critically on the fact that certain *nonlinear* field equations possess *spatially localized* (also called *soliton-like*) solutions, i.e. solutions which are appreciably distinct from zero only in a bounded region of space. The existence of such solutions to a large family of *scalar* nonlinear field equations, including the *nonlinear Schrödinger* (NLS) equations, is proved and existence conditions are derived in Berestycki and Lions [2]. The existence of such solutions to *spinor* nonlinear field equations is proved in Cazenave and Vazquez [3]. Some additional references are: Lee [4] (Ch. 7), Finkelstein and Fronsdal [5], Bialynicki-Birula and Mycielski [6], Freidberg and Lee [7], Cooperstock and Rosen [8], Enz [9], Bodurov [10, 11, 12, 13, 14]. Reference [15] is a collection of works which investigate the *solitons* as one-dimensional models of elementary particles.

2. De Broglie-type relations for spatially localized complex fields

Here it will be shown that there is a wave which is associated with a spatially localized field when this field satisfies the Assumptions 3.1, 3.2 and 3.3 in Part I of this paper. The derivations in the previous part have produced considerably more than the form of the NLS family of equations (I.3.22) and the relation (I.4.4). Namely, they have provided a conceptual and mathematical basis from which de Broglie-type relations for the localized fields obeying NLS equations can be found.

References (X.Y) to mathematical expressions in Pare I will be shown here as (I.X.Y).

The first step in deriving these relations is to find the function $\lambda(\boldsymbol{v}, t)$ which appears in the Galilei transform (I.3.11) and (I.3.13)

$$\psi' = \varphi' e^{i\vartheta} = \varphi(\boldsymbol{x} - \boldsymbol{v}t)\, e^{i\vartheta}\ , \qquad \text{with} \qquad \vartheta = \boldsymbol{k}\cdot\boldsymbol{x} + \lambda(\boldsymbol{v}, t) \tag{2.1}$$

of a stationary spatially localized field $\psi = \varphi(\boldsymbol{x}) \exp(-i\,\omega t)$. This will be done, as explained earlier, by demanding that the Galilei transformed field ψ' (2.1) be a solution of the same NLS equation (I.3.22)

$$i\,\frac{\partial\psi}{\partial t} = -\mu\nabla^2\psi + G(\psi^*\psi)\,\psi \tag{2.2}$$

of which the stationary field ψ is a solution. Accordingly, we substitute the ansatz (2.1) into the above equation to obtain

$$\frac{\partial\lambda}{\partial t}\,\varphi' + i\boldsymbol{v}\cdot\nabla\varphi' = \mu\nabla^2\varphi' + 2i\mu\,\boldsymbol{k}\cdot\nabla\varphi' - \mu k^2\varphi' - G(\varphi'^*\varphi')\varphi' \ . \quad (2.3)$$

The two terms proportional to $\nabla\varphi'$ cancel since $\boldsymbol{v} = 2\mu\boldsymbol{k}$ according to (I.3.21). For the remaining terms we have

$$\frac{\partial\lambda}{\partial t}\,\varphi' = \mu\nabla^2\varphi' - G(\varphi'^*\varphi')\varphi' - \mu k^2\varphi' = -\,\omega\varphi' - \mu k^2\varphi' \quad (2.4)$$

where the second equality takes into account that φ', being a space-translation of φ, obeys the same equation which φ does, namely

$$\omega\varphi' = -\mu\nabla^2\varphi' + G(\varphi'^*\varphi')\varphi' \ . \quad (2.5)$$

Expression (2.4) shows that $\partial\lambda/\partial t = -\,\omega - \mu k^2$. After integration, one finds that

$$\lambda = -(\omega + \mu k^2)t + \lambda_0 \quad (2.6)$$

is the sought function. The integration constant λ_0 is unessential — it can be eliminated by the gauge I [18] transformation (I.3.6) with $\varepsilon = -\lambda_0$. Therefore (2.1) is a solution of the NLS equation (2.2) if the function ϑ is

$$\vartheta = \boldsymbol{k}\cdot\boldsymbol{x} - \omega' t\,, \qquad \text{with} \qquad \boldsymbol{k} = \frac{1}{2\mu}\,\boldsymbol{v} \qquad \text{and} \qquad \omega' = \omega + \mu k^2 \ . \quad (2.7)$$

From (2.1) together with (2.7) follows that a spatially localized field "moving" with a velocity $\boldsymbol{v}$ *exhibits particle-wave duality since such a field is a complex plane wave*

$$\exp i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega' t) \quad (2.8)$$

which is modulated with the spatially localized function $\varphi(\boldsymbol{x} - \boldsymbol{v}t)$.

Next, we show that (2.8) possesses all the characteristics of a de Broglie wave:

a. The *group velocity* of (2.8) obtained from the second and third expressions in (2.7)

$$\frac{\partial\omega'}{\partial k_i} = 2\mu k_i = v_i \ , \qquad i = 1, \ldots, n \quad (2.9)$$

is equal to the velocity $\boldsymbol{v}$ of the localization region of ψ'. It should be observed that no de Broglie-type relations were explicitly used in this simple derivation of (2.9), which is not the case in Quantum Mechanics.

b. *The linear momentum* $\boldsymbol{p} = m\boldsymbol{v}$ *of a spatially localized field is proportional to the wave vector* $\boldsymbol{k}$ *of its associated wave* as seen from (4.4) and the second equation in (2.7)

$$\boldsymbol{p} = m\boldsymbol{v} = \frac{N}{2\mu} \cdot 2\mu\boldsymbol{k} = N\boldsymbol{k}\ . \tag{2.10}$$

c. *The kinetic energy* $E' - E$ *(of translational motion) of a spatially localized field is proportional to the frequency change* $\omega' - \omega$ *of its associated wave.*

To prove this, relying only on our previous results, we identify the field energy with the value of the Hamiltonian functional H, since according to (I.2.4) the field energy density is equal to the Hamiltonian density (I.3.19). Thus, the energy of the stationary field is

$$E = \int \big(\mu\nabla\psi^* \cdot \nabla\psi + \mathcal{G}(\psi^*\psi)\big)\, d^n x = \int \big(\mu\nabla\varphi^* \cdot \nabla\varphi + \mathcal{G}(\varphi^*\varphi)\big)\, d^n x\ . \tag{2.11}$$

The energy of the "moving" field ψ' (2.1) with ϑ given by (2.7) is

$$\begin{aligned} E' &= \int \big(\mu\nabla\psi'^* \cdot \nabla\psi' + \mathcal{G}(\psi'^*\psi')\big)\, d^n x \\ &= \int \big(\mu\nabla\varphi'^* \cdot \nabla\varphi' + \mathcal{G}(\varphi'^*\varphi')\big)\, d^n x + \mu k^2 \int \varphi'^*\varphi'\, d^n x \\ &- i\mu\boldsymbol{k} \cdot \int \big(\varphi'^*\nabla\varphi' - \varphi'\nabla\varphi'^*\big)\, d^n x\ . \end{aligned} \tag{2.12}$$

The first integral in (2.12) is the energy of the stationary field (2.11)

$$\int \big(\mu\nabla\varphi'^* \cdot \nabla\varphi' + \mathcal{G}(\varphi'^*\varphi')\big)\, d^n x = \int \big(\mu\nabla\varphi^* \cdot \nabla\varphi + \mathcal{G}(\varphi^*\varphi)\big)\, d^n x = E$$

since $\varphi' = \varphi(\boldsymbol{x} - \boldsymbol{v}t)$. The last integral in (2.12) vanishes

$$\int \big(\varphi'^*\nabla\varphi' - \varphi'\nabla\varphi'^*\big)\, d^n x = \int \big(\varphi^*\nabla\varphi - \varphi\nabla\varphi^*\big)\, d^n x = 2\int \varphi^*\nabla\varphi\, d^n x = 0$$

since it is proportional to the velocity functional (I.3.5) evaluated with the stationary field ψ, which is zero by definition. Consequently (2.12) reduces to

$$E' - E = \mu k^2 N = \frac{N}{4\mu}\, v^2 \tag{2.13}$$

with $\boldsymbol{k} = (1/2\mu)\,\boldsymbol{v}$ from (2.13) and $\int \varphi'^* \varphi' \, d^n x = \int \varphi^* \varphi \, d^n x = N$. This confirms the claim that $E' - E$, being proportional to v^2, is the kinetic energy associated with the translational motion of the localized field. In addition, (2.13) shows, in agreement with (I.4.4), that the field's mass is $m = N/2\mu$. The proof is completed by eliminating k^2 from (2.13) in favor of $\omega' - \omega$ using the third expression in (2.7)

$$E' - E = \mu k^2 N = N(\omega' - \omega) \ . \tag{2.14}$$

The expressions (2.10) and (2.14) are identical to de Broglie's relations except in them the Planck's constant $\hbar$ is replaced by the field's norm N. Again! Of course, one cannot expect to find the value of $\hbar$ by calculating the norm N of some spatially localized solution of some NLS equation.

3. On Dirac's quantization rules

The expression (I.4.4) which relates the ψ-field mass to the constants μ and N was derived in Part I from the Poisson bracket conditions (I.4.3). The correctness of the result tells us that we should attempt to extend the method to more general relations, namely, relations which are the counterpart of Dirac's quantization rules in QM. For this purpose let ψ_σ be the σ-component of some multi-component field ψ and $\psi^\dagger = (\psi^*)^{\mathrm{t}}$ be the transpose of ψ^*. The following theorem is essential for the discussion in this section

Theorem 3.1. *The infinite-dimensional Poisson bracket of two normalized bilinear real or complex-valued functionals Q and P which are associated with the linear matrix-differential operators $\hat{\mathbf{Q}}$ and $\hat{\mathbf{P}}$, with entries $\hat{Q}_{\rho\sigma}$ and $\hat{P}_{\rho\sigma}$, according to the "rule"*

$$Q = \frac{1}{N}\int \psi^\dagger\, \hat{\mathbf{Q}}\, \psi \, d^n x \ , \qquad P = \frac{1}{N}\int \psi^\dagger\, \hat{\mathbf{P}}\, \psi \, d^n x \tag{3.1}$$

is a normalized bi-linear functional which is associated precisely by the same "rule"

$$\{Q,P\} = \frac{1}{N}\int \psi^\dagger\, [\hat{\mathbf{Q}},\hat{\mathbf{P}}]_{\text{QM}}\, \psi\, d^n x\ , \qquad [\hat{\mathbf{Q}},\hat{\mathbf{P}}]_{\text{QM}} = \frac{1}{iN}\,(\hat{\mathbf{Q}}\hat{\mathbf{P}} - \hat{\mathbf{P}}\hat{\mathbf{Q}})\ . \tag{3.2}$$

with the linear matrix-differential operator $[\hat{\mathbf{Q}},\hat{\mathbf{P}}]_{\text{QM}}$.

Proof: The Poisson bracket of the norm $N = \int \psi^\dagger\, \psi\, d^n x$ with any bilinear functional $F = \int \psi^\dagger\, \hat{\mathbf{F}}\psi\, d^n x = \int \psi^*_\rho\, \hat{F}_{\rho\sigma}\, \psi_\sigma\, d^n x$ is zero according to

$$\begin{aligned}
\{N,F\} &= \frac{1}{i}\int \left(\frac{\delta N}{\delta\psi_\sigma}\,\frac{\delta F}{\delta\psi^*_\sigma} - \frac{\delta F}{\delta\psi_\sigma}\,\frac{\delta N}{\delta\psi^*_\sigma} \right) d^n x \\
&= \frac{1}{i}\int \big(\psi^*_\sigma\, \hat{F}_{\sigma\rho}\, \psi_\rho - (\hat{F}^\dagger_{\rho\sigma}\, \psi_\rho)^*\, \psi_\sigma\big)\, d^n x \\
&= \frac{1}{i}\int \big(\psi^*_\sigma\, \hat{F}_{\sigma\rho}\, \psi_\rho - \psi^*_\rho\, \hat{F}_{\rho\sigma}\, \psi_\sigma\big)\, d^n x = 0
\end{aligned}$$

where $\hat{F}^\dagger_{\rho\sigma}$ is the adjoint of the $\rho\sigma$-entry $\hat{F}_{\rho\sigma}$ of the matrix-operator $\hat{\mathbf{F}}$. Note that $\hat{F}^\dagger_{\rho\sigma} \neq \big(\hat{\mathbf{F}}^\dagger\big)_{\rho\sigma}$. Hence, in the following calculation of $\{Q,P\}$ the norm N can be treated as a numerical constant (rather than as a functional) even when N is not constant in time

$$\begin{aligned}
\{Q,P\} &= \frac{1}{i}\int \left(\frac{\delta Q}{\delta\psi_\tau}\,\frac{\delta P}{\delta\psi^*_\tau} - \frac{\delta P}{\delta\psi_\tau}\,\frac{\delta Q}{\delta\psi^*_\tau} \right) d^n x \\
&= \frac{1}{iN^2}\int \Big(\big(\hat{Q}^\dagger_{\rho\tau}\, \psi_\rho\big)^*\, \hat{P}_{\tau\sigma}\, \psi_\sigma - \big(\hat{P}^\dagger_{\rho\tau}\, \psi_\rho\big)^*\, \hat{Q}_{\tau\sigma}\, \psi_\sigma \Big)\, d^n x \\
&= \frac{1}{iN^2}\int \psi^*_\rho\, \big(\hat{Q}_{\rho\tau}\, \hat{P}_{\tau\sigma} - \hat{P}_{\rho\tau}\, \hat{Q}_{\tau\sigma}\big)\, \psi_\sigma\, d^n x \\
&= \frac{1}{N}\int \psi^\dagger [\hat{\mathbf{Q}},\hat{\mathbf{P}}]_{\text{QM}}\, \psi\, d^n x \qquad \square
\end{aligned}$$

This theorem states a very significant relationship which is the counterpart of the *correspondence between classical variables and linear operators in QM.* While in QM this is a postulated correspondence, here we have a derived mathematical identity (3.2).

As seen in Part I the position, velocity and linear momentum of a localized ψ-field are given (in Cartesian coordinates) with bilinear functionals whose form is same as that of the *expectation values* of the corresponding linear operators in QM. The difference is: Unlike the wave functions in QM, here the ψ-fields, being solutions of nonlinear equations, cannot be normalized and remain solutions of the same equations. In stead, the above bilinear functionals are normalized by dividing the appropriate integrals by N. The above observation holds regardless of the type of coordinates used to describe the position and the motion of a localized field as a whole. For example, the *radial* position and the *angular* position of a scalar ψ-field in $\mathbb{R}^3$:

$$R = \frac{1}{N}\int \psi^*\psi\, r\, d^3x \ , \qquad \Theta_k = \frac{1}{N}\int \psi^*\psi\, \theta_k\, d^3x$$

with $r = \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$ and $\theta_k = \tan^{-1}(x_j/x_i)$, are bilinear functionals, by definition. Then, a straightforward calculation, using (2.2), shows that the corresponding radial velocity and angular velocity functionals are also bilinear:

$$\begin{aligned}\frac{dR}{dt} &= \frac{2\mu}{iN}\int \psi^*\left(\frac{\partial\psi}{\partial r} + \frac{\psi}{r}\right) d^3x \\ \frac{d\Theta_k}{dt} &= \frac{2\mu}{iN}\int \frac{\psi^*}{x_i^2 + x_j^2}\left(x_i\,\frac{\partial\psi}{\partial x_j} - x_j\,\frac{\partial\psi}{\partial x_i}\right) d^3x \ .\end{aligned}$$

This is so because the function $G(\psi^*\psi)$ in (2.2), as a multiplication operator, commutes with r and θ_k. Accordingly, the radial and angular momenta of a localized field are given by bilinear functionals.

Now, suppose we have found a set of $2n$ real-valued bilinear functionals ($k = 1,\ldots,n$)

$$Q_k = \frac{1}{N}\int \psi^\dagger\, \hat{\mathbf{Q}}_k\, \psi\, d^nx \ , \qquad P_k = \frac{1}{N}\int \psi^\dagger\, \hat{\mathbf{P}}_k\, \psi\, d^nx \tag{3.3}$$

where $\hat{\mathbf{Q}}_k$, $\hat{\mathbf{P}}_k$ are self-adjoint matrix-differential operators and $\psi = \{\psi_\sigma\}$ is a multi-valued field, such that the Poisson bracket relations

$$\{Q_j\,, P_k\} = \delta_{jk} \ , \quad \{Q_j\,, Q_k\} = \{P_j\,, P_k\} = 0 \ , \quad j,k = 1,\ldots,n \tag{3.4}$$

hold identically. The brackets, of course, are of the infinite-dimensional type (I.2.2). Then, the infinite and the finite-dimensional Poisson brackets of any two at least once-differentiable functions $R = R(Q,P)$ and $S = S(Q,P)$ of the above $2n$ functionals are equal, i.e.

$$\{R, S\} = i\int\left(\frac{\delta R}{\delta\psi_\sigma^*}\,\frac{\delta S}{\delta\psi_\sigma} - \frac{\delta S}{\delta\psi_\sigma^*}\,\frac{\delta R}{\delta\psi_\sigma}\right) d^nx = \frac{\partial R}{\partial Q_k}\,\frac{\partial S}{\partial P_k} - \frac{\partial S}{\partial Q_k}\,\frac{\partial R}{\partial P_k} \ . \tag{3.5}$$

This is verified by a direct calculation using (3.4) and the identities

$$\frac{\delta R}{\delta \psi_\sigma} = \frac{\partial R}{\partial Q_j}\frac{\delta Q_j}{\delta \psi_\sigma} + \frac{\partial R}{\partial P_j}\frac{\delta P_j}{\delta \psi_\sigma}\ , \qquad \frac{\delta S}{\delta \psi_\sigma} = \frac{\partial S}{\partial Q_k}\frac{\delta Q_k}{\delta \psi_\sigma} + \frac{\partial S}{\partial P_k}\frac{\delta P_k}{\delta \psi_\sigma} \quad (3.6)$$

while observing that the derivatives $\partial R/\partial Q_j$, $\partial R/\partial P_j$, $\partial S/\partial Q_k$ and $\partial S/\partial P_k$ are functions of functionals and thus they do not depend on the coordinates x :

$$\{R,S\} = \frac{\partial R}{\partial Q_j}\frac{\partial S}{\partial Q_k}\{Q_j,Q_k\} + \frac{\partial R}{\partial P_j}\frac{\partial S}{\partial P_k}\{P_j,P_k\} + \left(\frac{\partial R}{\partial Q_j}\frac{\partial S}{\partial P_k} - \frac{\partial S}{\partial Q_k}\frac{\partial R}{\partial P_j}\right)\{Q_j,P_k\} = \frac{\partial R}{\partial Q_k}\frac{\partial S}{\partial P_k} - \frac{\partial S}{\partial Q_k}\frac{\partial R}{\partial P_k}\ .$$

Hence, the values of the functionals Q_k and P_k , when evaluated on solutions, can be treated as classical canonical variables. Next, we see from Theorem 3.1 that sufficient conditions for (3.4) to hold are the commutation relations

$$[\hat{\mathbf{Q}}_j, \hat{\mathbf{P}}_k] = iN\delta_{jk}\ , \quad [\hat{\mathbf{Q}}_j, \hat{\mathbf{Q}}_k] = [\hat{\mathbf{P}}_j, \hat{\mathbf{P}}_k] = 0\ , \quad j,k = 1,\ldots,n \quad (3.7)$$

which have exactly the same form as *Dirac's quantization rules* except that Planck's constant $\hbar$ is replaced by the ψ-field norm N , as already observed in a number of previous, less general, relations.

4. Concluding discussion

This paper showed that there are waves associated with certain spatially localized fields which are in complete correspondence with the de Broglie waves associated with the point-like particles. This result together with the corresponding de Broglie-type relations were derived from three natural assumptions, which are completely independent from the postulates of quantum mechanics.

It should be recognized that the same results can be derived from an alternative, less general, set of two assumptions, namely:

(a) *The spatially localized field ψ is a solution of a NLS equation.*

(b) *Assumption 3.3, in Part I.*

Such an approach would be just as valid mathematically and would make the paper considerably shorter. However, from physics point of view, this alternative will not be as satisfying as the presented approach. Had we used the form of the NLS equations as a starting point we would have implicitly linked the results to the form of Schrödinger's equation and thus to some of the assumptions of quantum mechanics. In stead, we derived in Part I the form of the NLS equations from the same assumptions from which we derived in this Part II the de Broglie-type relations.

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18. The term "gauge type I transformations" is used in order to distinguish from the "gauge transformations" of electromagnetic potentials and from the "gauge type II transformations" which act simultaneously on the wave-function and on the electromagnetic potentials. This terminology was adopted from Goldstein's book[16] .

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