

On the physical meaning of the gauge conditions of Classical Electromagnetism : the hydrodynamics analogue viewpoint

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ABSTRACT. Based on an analogy between Fluid Mechanics and Electromagnetism, we claim that the gauge conditions of Classical Electromagnetism are not equivalent contrary to the common belief. These "gauges" are usually considered as mathematical conditions that one must specify in order to solve any electromagnetic problem. Here, the author shows that these conditions are physical constraints which can be interpreted as electromagnetic continuity equations. As a consequence, light cannot be considered as a pure transverse wave in vacuum from the point of view of the potentials. We discuss the (lack of) meaning of gauge transformations.

1 Introduction

In Classical Electromagnetism, the generalized momentum \mathbf{p} of a particle with mass m and charge q moving at a velocity \mathbf{v} in a vector potential \mathbf{A} is [1] : $\mathbf{p} = m\mathbf{v} + q\mathbf{A}$. Hence, the vector potential can be seen as the electromagnetic impulsion (per unit of charge) of the field. For example, induction phenomena are due to the transfer of momentum from the field to the charge via the vector potential. Simply speaking, the vector potential is a kind of velocity up to a factor q/m . There is a long history of papers ([2, 3] and references therein) and of books which advocate forcefully a physical interpretation to the vector potential ([4, 5, 6, 7] and references therein). One can define the vector potential at a point M as the mechanical impulsion that an external operator must furnish to a unit charge in order to bring it from infinity (where the vector potential vanishes far from the currents) to the point M . The generalized energy ϵ of the same particle in a scalar potential V is [1] : $\epsilon = mv^2/2 + qV$. Hence, the scalar potential can be seen as the potential energy (per

unit of charge) of the field. For example, an electron is accelerated in an electron gun and its gained energy per unit charge is the scalar potential. Of course, the potentials are defined up to a constant and the experimentalist sets by convention the scalar potential of a plate in the electron gun to zero for instance. Similarly, one can define the scalar potential at a point M as the mechanical energy that an external operator must furnish to a unit charge in order to bring it from infinity (where the scalar potential vanishes far from the charges) to the point M.

Usually, in order to solve a problem in Electromagnetism, one must specify what is called a gauge, that is a supplementary condition which is injected in the Maxwell equations expressed in function of the electromagnetic potentials. Two gauge conditions were introduced in Classical Electromagnetism [1] : $\nabla \cdot \mathbf{A} = 0$ which is the Coulomb gauge, used for example in magnetostatics, and : $\nabla \cdot \mathbf{A} + 1/c_L^2 \partial_t V = 0$ which is the Lorenz gauge. Here, c_L is the velocity of light [1] : $c_L = \sqrt{1/\mu_0 \epsilon_0}$ where μ_0 and ϵ_0 are respectively the permeability and the permittivity of the vacuum. It is common to say that these gauge conditions are mathematical conveniences that lead to the same determination of the electromagnetic field. In this context, the choice of a specific gauge is motivated from its conveniences in calculations !

We would like to underline that these gauges may not be equivalent :

- From the mathematical point of view, the Coulomb gauge is the approximation of the Lorenz gauge in the stationary case -which is a well known result - but also when the velocity of light is taken to be infinite (what this paper will demonstrate...).
- From the physical point of view, the gauges can be seen as electromagnetic continuity equations. To understand this last point, one can use the following analogy with hydrodynamics.

2 The analogue proof of the non-equivalence between the gauge conditions

In order to solve a problem in fluid mechanics, one must specify a physical constraint which tells us if the fluid flow is compressible or not. The incompressibility constraint reads [8] : $\nabla \cdot \mathbf{u} = 0$ whereas the compressibility constraint is [8] : $\nabla \cdot \mathbf{u} + 1/\rho D_t \rho = 0$ where $D_t = \partial_t() + (\mathbf{u} \cdot \nabla)()$

is the so-called total derivative, \mathbf{u} is the velocity of a fluid particle and ρ its density. If the flow is not stationary and if one considers acoustic waves which are perturbations of the pressure, the density and the velocity of the fluid around a basic state (subscript 0) $p = p_0 + \delta p$, $\rho = \rho_0 + \delta \rho$ and $\mathbf{u} = \mathbf{0} + \delta \mathbf{u}$, one can evaluate the velocity of sound by the following formula [8] : $c_S = \sqrt{\partial p / \partial \rho} = \sqrt{1 / \rho \kappa}$ where κ is the compressibility of the fluid. The compressibility constraint becomes : $\nabla \cdot \delta \mathbf{u} + 1/c_S^2 \partial_t (\delta p / \rho_0) = 0$ which has a form equivalent to the Lorenz gauge. If the velocity of sound tends to infinity, one recovers the incompressibility constraint. The new result is that the Coulomb gauge would imply that the velocity of light tends to infinity in a time dependent problem when propagation is absent as in hydrodynamics [9]. Moreover, if the flow is stationary, the compressibility constraint reduces to the incompressibility constraint which is analogous to the Coulomb gauge.

3 The experimental proof of the non-equivalence between the gauge conditions

Now, one can read in every textbooks of electromagnetism that we can describe propagation of potential waves in either the Coulomb or the Lorenz gauge because in any case the propagation of the electromagnetic waves remains unchanged... We will show that the Coulomb constraint cannot describe propagation at finite speed but instantaneous propagation in a coaxial cable.

What does it mean experimentally that a quantity propagates instantaneously? Imagine the following experiment. Let's take a coaxial cable. One can relate it to a function generator which delivers for instance a scalar potential pulse of whatever shape : square, triangular... Experimentally, the scalar potential seems to propagate instantaneously in a short coaxial cable of one meter long. We are in the so-called quasi-static limit where quantities are time-dependent but do not propagate (with the help of the analogy, one understand that we should use the Coulomb constraint). Experimentally, the scalar potential does not propagate instantaneously in a long coaxial cable of one hundred meter long because we are able to detect with an oscilloscope a time delay introduced by the finite speed propagation of a pulse of scalar potential between the entry and the exit of the cable. This last experimental fact is in contradiction with the assertion that we can use Coulomb gauge to describe propagation because the scalar potential is solution of a Laplace equation in this gauge that is, it must propagate instantaneously [1]. From the anal-

ogy, one concludes that we should use the Lorenz constraint to describe propagation and not the Coulomb constraint. Of course, one also uses Coulomb constraint in the time-independent case. A close look to the range of validity of the so-called quasi-stationary approximation (c_L is infinite) permits to understand that there is no contradiction with the above statement concerning the fact that the potentials and the fields can or can not propagate depending on the problem...

To conclude this part, whatever the potentials are undetermined or not, the so-called gauges conditions seem to be physical constraints which would tell us if the velocity of light is a relevant parameter or not (that is finite or not) and not mathematical conditions to fix the potentials. Indeed, depending on what type of phenomena you are studying, some imply that the velocity of light is finite and some other do not. More precisely, is there a consistent galilean electromagnetism (c_L is infinite) coexisting with a relativistic electromagnetism (c_L is finite) ? This question was addressed and answered for the fields by Lévy-Leblond & Le Bellac [10] and it was revisited recently by Holland & Brown [11]. Our paper extends these last works for the gauge conditions.

4 The mathematical proof of the non-equivalence between the gauge conditions

As a matter of fact, Lévy-Leblond & Le Bellac have shown that the full set of Maxwell equations has two well defined galilean limits which they called the magnetic limit (used for example in Ohmic conductors and in Magnetohydrodynamics ([12, 13]) : also called the magneto-quasistatic approximation) and the electric limit (used for example in dielectrics and in Electrohydrodynamics ([14, 13, 15]): also called the electro-quasistatic approximation). The two limits are obtained by taking the velocity of light as infinite. Contrary to mechanics which allows only one galilean limit, the two limits of electromagnetism come from the fact that $c_L = \sqrt{1/\mu_0\epsilon_0}$ can tend to infinity if either μ_0 or ϵ_0 tends to zero separately. For example, the magnetic limit is the result of keeping μ_0 constant during the process while ϵ_0 tends to zero.

Moreover, Lévy-Leblond & Le Bellac have derived the galilean transformations for the potentials [10]. In the magnetic limit ($\mathbf{u} \ll c_L$ and $V \ll c_L \cdot |\mathbf{A}|$), they read : $\mathbf{A}^* = \mathbf{A}$ and $V^* = V - \mathbf{u} \cdot \mathbf{A}$ whereas in the electric limit ($\mathbf{u} \ll c_L$ and $V \gg c_L \cdot |\mathbf{A}|$): $\mathbf{A}^* = \mathbf{A} - \mathbf{u}/c_L^2 V$ and $V^* = V$. Now, if we apply the limiting process used by these authors ($\mathbf{u} \ll c_L$ and $V \ll c_L \cdot |\mathbf{A}|$ or $V \gg c_L \cdot |\mathbf{A}|$) to the Lorenz gauge

which, we know, is Lorentz invariant (c_L is finite), we find that the Lorenz gauge resumes to the Coulomb gauge in the magnetic limit and that the Lorenz gauge remains the same in the electric limit. The Lorenz (Coulomb) gauge is now covariant with respect to the "electric" ("magnetic") transformations of the potentials. The Coulomb gauge is the only possible constraint that we can apply when we deal with Ohmic conductors or in Magnetohydrodynamics that is within the range of the magnetic limit. The Coulomb gauge cannot apply in the electric limit as well as in relativistic electromagnetism which was not recognized before. The important point is that the Coulomb gauge is obtained mathematically by a limiting process from the Lorenz gauge and is not independent of the Lorenz gauge. We clearly state that it is hence forbidden to plunge the Coulomb gauge which is galilean into the full set of Maxwell equations which are relativistic contrary to what is stated in almost all the textbooks. The Lorenz gauge describes both relativistic electromagnetism and galilean electromagnetism within the electric limit and it cannot apply in the magnetic limit.

Once again, the analogy can help us to grasp the underlying physics. If a flow is said to be incompressible, the velocity of sound is considered to be infinite. More precisely, the compressibility of the fluid tends to zero while the density is kept constant. Moreover, we characterized usually media where waves propagate by using the concept of impedance which for an acoustic wave is $Z_s = \rho_0 c_s$ and for a light wave is $Z_L = \mu_0 c_L$. Hence, μ_0 is the analogue of ρ_0 . Now, we can remark easily that the magnetic limit is the analogue of an incompressible flow while there is no mechanical counterpart for the electric limit. One understands why the Coulomb gauge is the only gauge which does apply in Ohmic conductors within the magnetic limit which are analogous to Newtonian fluids in incompressible flow [9]. Recently, Brown & Holland [16] have shown that the Schroedinger equation which is a galilean equation was only coherent with the use of the magnetic limit which explains why we use the Coulomb gauge with this equation when dealing with an electron in a vector potential.

One century ago, H.A. Lorentz noticed that the electromagnetic field remains invariant ($\mathbf{E}' = \mathbf{E}$ and $\mathbf{B}' = \mathbf{B}$) under the so-called gauge transformations [17] : $\mathbf{A}' = \mathbf{A} + \nabla f$ and $V' = V - \partial_t f$ where $f(x, t)$ is the gauge function. Hence, this indetermination is believed to be an essential symmetry of Classical Electromagnetism [17]. We showed that the Coulomb and Lorenz gauges were not equivalent because they must be

interpreted as physical constraints that is continuity equations. So, to make a gauge choice is not related to the fact of fixing a special couple of potentials. Gauge conditions are completely uncorrelated to the supposed indetermination of the potentials. The gauge choice must be taken with respect to the type of electromagnetism we study that is relativistic or not by taking care also of the type of galilean limit. What is the meaning of gauge transformations ? We believed that it is only a structural feature (that is linearity) of the definitions of the potentials from the fields. The potentials of Classical Electromagnetism do have a physical meaning as recalled in the introduction. If we defined the fields from the potentials and not the contrary, the gauge transformations loose their sense.

As a conclusion, we propose to reject gauge transformations. Gauge invariance is preserved but in a weaker sense : the potentials are defined up to a constant. The proposed rejection of gauge transformations is not new in the literature : it was foreseen by L. de Broglie in the application of the principle of inertia of energy in relativity [18]. More recently, A. Van Oosten proposed a non-gauge-invariant theory of electromagnetism based on the Fermi Lagrangian which is a valid alternative to the standard approach as it makes the same experimental predictions [19].

5 The Nature of Light

Now, we can have a closer look to the way the propagation of light is described usually. One can find for example in *The Classical Theory of Fields* by Landau & Lifshitz [20] the following description. Thanks to gauge invariance, one can take the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ and the assumption that the scalar potential is zero in order to describe light propagation. As a matter of fact, one obtains : $\mathbf{E} = -\partial_t \mathbf{A} - \nabla V = -\partial_t \mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$. The vector potential (so the fields) is solution of a propagation equation. From our point of view, this derivation is misleading because the Coulomb gauge is not Lorentz invariant and we advocated in this paper that it can not describe propagation (c_L is infinite). Indeed, if we apply a Helmholtz decomposition to the Lorenz gauge (c_L is finite), one finds : $\nabla \cdot \mathbf{A}_{longitudinal} + 1/c_L^2 \partial_t V = 0$ and independently : $\nabla \cdot \mathbf{A}_{transverse} = 0$. The magnetic field is : $\mathbf{B} = \nabla \times \mathbf{A}_{transverse}$ with : $\nabla \times \mathbf{A}_{longitudinal} = 0$. The electric field writes : $\mathbf{E} = -\partial_t \mathbf{A} - \nabla V = -\partial_t \mathbf{A}_{transverse}$ with : $\partial_t \mathbf{A}_{longitudinal} + \nabla V = 0$.

Indeed, concerning the nature of light, one can wonder if light should still be considered as a transverse wave. As a matter of fact, the po-

tentials do have a physical meaning in Classical Electromagnetism as recalled in the introduction. Moreover, we gave a physical interpretation of the gauge and particularly of the Lorenz gauge which implies by Fourier transform : $V = c_L A_x$ where x is the direction of propagation of a plane light wave in vacuum ($k = \omega/c_L$). Hence, if we can say - as usual - that the longitudinal electric and magnetic fields cancel ($E_x = -\partial_t(A_x \cos(kx - \omega t)) - \partial_x(V \cos(kx - \omega t)) = 0$ and $B_x = (\nabla \times \mathbf{A})_x = 0$ because $V = c_L A_x$), the last equation shows that, under the Lorenz constraint, the vector potential has a non-zero longitudinal component which is a gradient. As pointed out by B. Leaf, the time-like and longitudinal potential components constitute a Lorentz-covariant null vector which is not amenable to boson quantization as the transverse components [21]. So from the point of view of the potentials from which the fields derive, light is neither a transverse or a longitudinal wave : it is a composite wave...

Once again, one can understand the longitudinal propagation for light with the sound analogy. By Fourier transformation of the continuity equation for the fluid, one obtains : $\delta p/\rho_0 = c_S \delta u_x$. One recalls that the propagation of sound waves is vorticity-free ($B_x = 0$) and that one gets the propagation equations by combination of the continuity equation with the linearized Navier-Stokes equation ($E_x = 0$) [8] : $\partial_t \delta \mathbf{u} + \nabla(\delta p/\rho_0) = 0$. The longitudinal propagation for light is not in contradiction with polarization experiments which do show that light can not be a pure longitudinal wave but that the electric field is transverse despite the fact that the vector potential has a longitudinal component...

The longitudinal propagation of the potentials is also a feature of electromagnetic waves in a coaxial cable with the difference that the longitudinal vector potential is not a gradient in this case [7, 13]. The unconvinced reader could argue that all the results regarding light can be derived without any reference to the potentials. Formerly, it is right but there is an implicit statement when we use the full set of Maxwell equations to derive light propagation that is we consider the velocity of light as finite. That's why we advocated in this paper that it is equivalent to use the Lorenz gauge.

6 Conclusions

In conclusion, one can understand that the gauges express electromagnetic continuity from a physical point of view based on an analogy with hydrodynamics. From this analogy, we concluded that the Lorenz gauge

is more fundamental, in general, than the Coulomb gauge which is an approximation for the stationary case and for the time-dependent case when one neglects the propagation of electromagnetic waves and more generally relativistic phenomena within the magnetic limit. From the pedagogical point of view, the analogy facilitates the use and understanding of the vectorial operators and allows to find solutions of electromagnetic problem much more readily in terms of hydrodynamics equivalent [9].

The author is fully aware that the conclusions of this paper are controversial as they defy old-established opinions about the non-physical character of the potentials as well as the so-called gauge conditions. Anyway, it is the author's belief that Electromagnetism cannot continue to be transmit to young generations without understanding the fundamentals of this discipline and in particular of the potentials which are the primary quantity in relativity and quantum field theory. Let us remind James Clerk Maxwell's own words : [the vector potential] is the mathematical quantity which can be considered as the fundamental quantity of the electromagnetic theory ([4], Vol.2, p. 187). It is funny to notice that Maxwell used also the following expressions : electrotonic state, electrokinetic momentum or electromagnetic momentum to designate the vector potential...

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