The relativistic motion of charged particles in an electromagnetic field

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RÉSUMÉ. Le mouvement relativiste des particules chargées dans un champ électromagnétique est réexaminé. Deux résultats nouveaux : (i) l'hélice euclidienne tri-dimensionnelle est généralisée aux courbes gauches quadri-dimensionnelles, et le rôle de la troisième courbure analysé; (ii) le mouvement de la particule chargée confirme l'approche de Franklin-Trocheris-Takeno, et évite les difficultés liées à la covariance des équations de Maxwell, et au recours *a priori* au cylindre de lumière en Astrophysique.

ABSTRACT. The relativistic motion of charged particles in an electromagnetic field is revisited. Two new results: (1) the Euclidean 3D helix has been generalized to the 4D spacetime, and the role of the third curvature analyzed; (2) the natural description of the rotational motion is exhibited and corresponds to the Franklin-Trocheris-Takeno type, thus avoiding difficulties with the covariance of Maxwell's equations in rotating frames, or the *a priori* introduction of light-cylinders in Astrophysics.

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C'est avec plaisir que je dédie cet article à O. Costa de Beauregard, que j'ai eu le privilège de rencontrer dès 1950, dans le séminaire Louis de Broglie; il combattait alors les anti-relativistes en compagnie du Général A. Metz. Nos chemins se sont croisés depuis, dans la rédaction d'un ouvrage collectif sur les grandes théories de la Physique moderne, et dans l'utilisation du tenseur d'énergie asymétrique (du type

Tetrode), notamment dans le cadre de la théorie d'Einstein-Cartan. Avec la vigueur qui le caractérise, il a mené beaucoup d'autres combats plus importants pour construire une théorie quantique relativiste des champs, clarifier la notion d'irréversibilité, ou délimiter la place de la rétrodiction en Physique.

Je me souviens également que nous étions les deux seuls membres du Collège A à nous joindre aux étudiants pour soutenir le maintien de la spécialité "Physique Théorique" lors de la réunion de constitution de Paris VI en 1968.

Son attitude calme en toute circonstance ne l'empêche pas de s'engager pour défendre ses amis; je l'ai remarqué au moins deux fois.

1 Introduction

The relativistic counterpart of the guiding center approximation of the motion of charged particles in an electromagnetic field has been first considered by Hellwig [3] and further elaborated by Vandervoort [17], motivated by the presence of high-energy particles, or of a strong magnetic field with a crossed or nearly crossed electric field. Their approach consists mainly in finding exact solutions for the trajectory of a charged particle in a constant e.m. field, and then perturbing it in the lab frame, under suitable adiabaticity conditions; the helical trajectory occurs as a basic component of the solution of the equations of motion of the particle; it is interpreted as the usual superposition of a gyration about the guiding center and a motion of this center, possibly modified by further "drift" motions. These considerations are adopted by Northrop [13] and subsequent authors; a conventional relativistic Lagrangian formulation may be found in Morozov and Solov'ev [12]. On the other hand, Synge [14] gives an exhaustive geometrical study of helices in Minkowski spacetime M_4, defined as twisted timelike world-lines for which all the Frenet-Serret curvatures are constant, and relates them to the motion of particles in different algebraic classes of constant e.m. fields.

Two points deserve consideration, namely the definition of rotation, and that of a helix.

The Galilean rotation can only be valid at small velocities, for otherwise, the linear relation $\vec{v} = \vec{\omega} \times \vec{r}$ would lead to speeds greater than the light velocity c. An alternative approach avoiding this difficulty has been proposed as early as 1922 by Franklin [2], and rediscovered by Trocheris [16] and Takeno [15]. It will therefore be referred to as the FTT approach; its physical meaning is discussed in [7], and its applications to models of pulsars and to the covariance of Maxwell's equations are worked out in [8, 9] and [10] respectively. Essentially, in the Galilean approach, the speed $v = \omega r$ and the angular velocity ω are both linearly additive; in the conventional relativistic approach, the speed $v = \omega r$ obeys the relativistic law of composition of velocities but, as a consequence, ω is not additive any more, whereas in the FTT approach, the linear additivity of ω and the relativistic law of composition of velocities are both preserved, thanks to the relation $v = c \tanh(\omega r/c)$.

Helices in Euclidean three-space may be defined geometrically by requiring the existence of a constant vector $\mathbf{k}$ such that the unit tangent $\mathbf{u}$ has constant projection on it ($\mathbf{k}\cdot\mathbf{u}$ =const.). This is in particular true if the curvature and torsion are both nonzero and constant, as in the usual helix, but includes more general trajectories: it suffices that the ratio of torsion to curvature be constant (this will be recovered in section 2 below). We study the natural generalization of this geometric definition to world-lines in M_4, which does not seem to have been considered in the literature, and study its relation to Synge's classification. We will see in particular that the latter overlooks the importance of the relativistic law of composition of velocities.

In this paper, we therefore obtain the conditions for the relativistic motion of charged particles to be helical in the generalized sense. In the case of a constant e.m. field, we show that (i) rotational motion corresponds to the FTT map if one requires that velocities be added according to the relativistic law and (ii) for the world-line to be a generalized helix, the third curvature (essentially related to the electric field) must vanish. Further properties of the general helical motion will be discussed in a forthcoming paper.

This paper is organized as follows.

Section 2 gives the geometric definition of helices in M_4 and discusses its properties.

Section 3 recalls background information on the electromagnetic field, and section 4 gives the expression of the curvatures of world-lines of particles in a general field F_{ab}; combined with the results of section 2, this gives the general conditions for the motion to be helical.

The results are specialized to constant fields in section 5; the properties of the Frenet-Serret frame according to the reduced form of the field is discussed, in the spirit of Synge [14].

Section 6 shows how the correct expression for the 4-velocity of charged particles leads naturally to the FTT map when the relativistic law of composition of velocities is taken into account.

2 Generalized helix in M_4

Let $\mathbf{u} = u^a\partial_a$, with $u^a = dx^a/ds$, be the time-like unit vector tangent to the world-line (L) of a particle in the space-time M_4 endowed with the metric

$$\eta_{ab} = \delta_{ab} - 2\delta_a^0\delta_b^0.$$

(Latin indices run from 0 to 3 and bold-face letters denote 4-vectors). The associated Frenet-Serret tetrad $\{\mathbf{u}, \mathbf{n}, \mathbf{b}, \mathbf{c}\}$ is determined by

$$\dot{\mathbf{u}} = a\mathbf{n}, \quad \dot{\mathbf{n}} = a\mathbf{u} + \tau\mathbf{b}, \quad \dot{\mathbf{b}} = -\tau\mathbf{n} + \sigma\mathbf{c}, \quad \dot{\mathbf{c}} = -\sigma\mathbf{b}, \tag{1}$$

where the dot denotes d/ds; a, τ and σ denote the curvature, torsion, and third curvature of (L) respectively. Of course, these 4-vectors are mutually orthogonal, and

$$u^a u_a = -1, \quad n^a n_a = b^a b_a = c^a c_a = 1.$$

Definition. A time-like world line is a (generalized) *helix* if there is a vector field k^a and a non-zero constant λ such that

$$\dot{k}^a = 0 \ \text{ and } \ k^a u_a + \lambda = 0.$$

For such a curve, let us express k^a in the Frenet-Serret frame:

$$k^a = \lambda u^a + \mu n^a + \nu b^a + \rho c^a.$$

Differentiating with respect to s, we find that (L) is a helix if and only if

$$\mu a = 0, \quad \dot{\mu} = \lambda a + \nu\tau, \quad \dot{\nu} = -\mu\tau + \rho\sigma, \quad \dot{\rho} = -\nu\sigma.$$

This system may be simplified.

If $a\tau\sigma \neq 0$, we let

$$\theta = -a/\tau$$

and find

$$\mu = 0, \quad \nu = \lambda\theta, \quad \rho = \lambda\dot{\theta}/\sigma,$$

and

$$\frac{d}{ds}(\sigma^{-1}\frac{d\theta}{ds}) + \sigma\theta = 0. \tag{2}$$

Conversely, if (2) holds, then (L) is a helix with

$$k^a = \lambda(u^a + \theta b^a + \frac{\dot{\theta}}{\sigma}c^a).$$

Note that condition (2) may be written

$$\ddot{\theta} + \sigma^2\theta - \dot{\theta}\frac{\dot{\sigma}}{\sigma} = 0. \tag{3}$$

If $\sigma = 0$ but $a\tau \neq 0$, we now find

$$\mu = 0, \quad \nu = \lambda\theta, \quad \lambda\dot{\theta} = 0.$$

Conversely, if $\dot{\theta} = 0$, then (L) is a helix with

$$k^a = \lambda(u^a + \theta b^a).$$

One can say more in this case: if $0 < \theta^2 < 1$, one verifies that

$$h^a = u^a - \theta b^a$$

is constant ($\dot{h}^a = 0$). If we decompose u^a as

$$u^a = v^a + \alpha h^a, \tag{4}$$

with $v^a h_a = 0$, we find, multiplying this equation by h_a, that $\alpha = 1/(1-\theta^2)$. Using equations (1), we find

$$\ddot{v}^a = \frac{\dot{a}}{a}\dot{v}^a + a^2(1-\theta^{-2})v^a. \tag{5}$$

If we further assume $\dot{a} = 0$, we find

$$\mathbf{v} = \Lambda(\mathbf{e}_1 \cos\omega s + \mathbf{e}_2 \sin\omega s),$$

where $\omega^2 = \tau^2 - a^2$. It follows, letting $\Gamma^2 = -\alpha^2 h^a h_a = \tau^2/(\tau^2 - a^2)$, that

$$\mathbf{u} = \Gamma \mathbf{e}_0 + \Lambda(\mathbf{e}_1 \cos \omega s + \mathbf{e}_2 \sin \omega s). \tag{6}$$

where the $(\mathbf{e}_j)$ form an orthonormal 3-frame. The generalized helix reduces to the timelike world-line tangent to the above $\mathbf{u}$.

The discussion may be summarized as follows:

- If $a\tau\sigma \neq 0$, then $\theta = -a/\tau$ obeys (3).
- If $\sigma = 0$ and $a\tau \neq 0$, then the ratio of curvature to torsion is constant and u^a obeys (4-5).
- If $\sigma = \dot{a} = 0$, and $\theta^2 < 1$, then the world-line is determined by (6).

3 Electromagnetic field

Let us recall a few facts, in a form suitable for our purposes. The e.m. field is described by a two-form

$$F_{ab} = \eta_{abmn} t^m B^n + t_a E_b - t_b E_a,$$

where $t^a = \delta^a_0$, and E^a, B^a are respectively the electric and magnetic fields, both orthogonal to t^a. At each point of M_4, they determine two algebraic invariants

$$I = \frac{1}{2} F_{ab} F^{ab} = B^2 - E^2 \quad \text{and} \quad J = \frac{1}{4} F_{ab} \overset{*}{F}{}^{ab} = \vec{B} \cdot \vec{E}.$$

Under a Lorentz map relating two frames S' and S, S' having the velocity $\vec{\beta}$ with respect to S, we have

$$\vec{E}' = \gamma(\vec{E} + \vec{\beta} \times \vec{B}), \quad \vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E}).$$

Two cases may be distinguished.

1. $\vec{E} \cdot \vec{B} = 0$. This condition, unlike the requirement that $\vec{E}$ (or $\vec{B}$) alone vanishes) is preserved under Lorentz transformations.

2. $\vec{E} \cdot \vec{B} \neq 0$. There exists [11] a Lorentz map $L^{a'}_b : S \to S'$ with a relative velocity $\vec{\beta}$ orthogonal to both $\vec{E}$ and $\vec{B}$, such that $\vec{E}'$ and $\vec{B}'$ are collinear. In fact, the conditions

$$\vec{E}' \times \vec{B}' = 0 \quad \text{and} \quad \vec{\beta} = \beta \frac{\vec{E} \times \vec{B}}{|\vec{E} \times \vec{B}|}$$

lead to

$$\beta^2 - |\vec{E} \times \vec{B}|^{-1}(B^2 + E^2)\beta + 1 = 0$$

with $0 \leq \beta < 1$. We then get

$$\beta = \frac{1}{2}|\vec{E} \times \vec{B}|^{-1}\{B^2 + E^2 - \sqrt{\delta}\}, \tag{7}$$

where $\delta = I^2 + 4J^2$.

4 Charged particles in an e.-m. field

The world-line (L) of a particle of mass m and charge Q in an e.-m. field F_{ab} is determined by

$$\dot{u}^a = kF^a{}_b\, u^b, \tag{8}$$

where $k = Q/mc$. Combining with equations (1), the curvatures of (L) are

$$a = n^a F_{ab} u^b, \qquad \tau = ka^{-1}(b^a \dot{F}_{ab} u^b + b^a F_{ab} n^b) \tag{9}$$

$$\sigma = k(a\tau)^{-1}\{c^a \ddot{F}_{ab} u^b + \dot{a} c^a F_{ab} n^b + 2ac^a \dot{F}_{ab} n^b + a\tau c^a F_{ab} b^b\}. \tag{10}$$

(L) is a generalized helix if (a, τ, σ) satisfy (3). The angular velocity corresponding to u^a is

$$\omega^a = \eta^{abcd} u_b \nabla_c u_d. \tag{11}$$

Remark. The motion is *uniformly accelerated* [5] when $\ddot{u}^a = a^2 u^a$, and *uniformly suraccelerated* [6] when $\dddot{u}^a = 3a\dot{a}u^a$;

5 Case $\dot{F}_{ab} = 0$

In this case, the three curvatures may be computed from (9-10):

$$a = kn^a F_{ab} u^b, \quad \tau = kb^a F_{ab} n^b, \quad \sigma = kc^a F_{ab} b^b,$$

and one verifies, using (1) and (8), that

$$\dot{a} = \dot{\tau} = \dot{\sigma} = 0.$$

Define

$$(V_A{}^a)_{A=0,1,2,3} = [u^a, n^a, b^a, c^a].$$

Equations (1) and (8) may be expressed in terms of ${V_A}^a$ in the form

$$\dot{V}_A{}^a = S_A{}^B V_B{}^a \quad \text{and} \quad \dot{V}_A{}^a = kF^a{}_b V_A{}^b,$$

where

$$(S_A{}^B) = \begin{bmatrix} 0 & a & 0 & 0 \\ a & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix}.$$

The matrices S and $(kF)^T$ are conjugate (through V) and therefore have the same characteristic equation, which may be written

$$\lambda^4 + (\tau^2 + \sigma^2 - a^2)\lambda^2 - a^2\sigma^2 = 0 \quad \text{or} \quad \lambda^4 + k^2 I\lambda^2 - k^4 J^2 = 0.$$

There are four eigenvalues: $\pm\chi$ and $\pm i\omega$, where

$$\chi^2 = \frac{1}{2}(a^2 - \tau^2 - \sigma^2 + \Delta) \quad \text{and} \quad \omega^2 = -\frac{1}{2}(a^2 - \tau^2 - \sigma^2 - \Delta),$$

with

$$\Delta^2 = (a^2 - \tau^2 - \sigma^2)^2 + 4a^2\sigma^2 = k^4[I^2 + 4J^2] = k^4\delta.$$

It follows that $V_A(s)$ is a linear combination of solutions of the form $V_A(s) = V_A(0)\exp(\lambda s)$.

We summarize the procedure followed by Synge to find $\mathbf{u}$ in the general case, for the benefit of the reader. First, $\mathbf{u}$ has the form

$$\mathbf{u} = \mathbf{v}_0 \cosh\chi s + \mathbf{v}_1 \cos\omega s + \mathbf{v}_2 \sin\omega s + \mathbf{v}_3 \sinh\chi s.$$

Since $\mathbf{u}^2 = -1$ for all s, the constant vectors $(\mathbf{v}_k)$ should satisfy

$$\frac{1}{2}(\mathbf{v}_0^2 + \mathbf{v}_1^2 + \mathbf{v}_2^2 - \mathbf{v}_3^2) = -1,$$
$$\mathbf{v}_1^2 - \mathbf{v}_2^2 = 0, \qquad \mathbf{v}_0^2 + \mathbf{v}_3^2 = 0,$$
$$\mathbf{v}_0 \cdot \mathbf{v}_3 = \mathbf{v}_1 \cdot \mathbf{v}_2 = 0,$$
$$\mathbf{v}_j \cdot \mathbf{v}_k = 0 \ \text{ if } \ j = 0, 3 \ \text{ and } \ k = 1, 2.$$

One can then choose, modulo transformations preserving the canonical planes $(\mathbf{v}_0, \mathbf{v}_3)$ and $(\mathbf{v}_1, \mathbf{v}_2)$, an orthonormal tetrad $(\mathbf{e}_k)$ such that

$$\mathbf{v}_0 = \Gamma\mathbf{e}_0, \quad \mathbf{v}_1 = \Lambda\mathbf{e}_1, \quad \mathbf{v}_2 = \Lambda\mathbf{e}_2, \quad \mathbf{v}_3 = \Gamma\mathbf{e}_3,$$

with

$$\Gamma^2 - \Lambda^2 = 1.$$

By suitable transformations preserving the canonical planes, one gets

$$\mathbf{u} = \Gamma(\mathbf{e}_0 \cosh w + \mathbf{e}_3 \sinh w) + \Lambda(\mathbf{e}_1 \cos\varphi + \mathbf{e}_2 \sin\varphi),$$
$$\dot{w} = \chi, \qquad \dot{\varphi} = \omega.$$

We then obtain

$$\begin{aligned}
\mathbf{n} &= \frac{\Gamma\chi}{a}(\mathbf{e}_0 \sinh w + \mathbf{e}_3 \cosh w) + \frac{\Lambda\omega}{a}(-\mathbf{e}_1 \sin\varphi + \mathbf{e}_2 \cos\varphi),\\
\mathbf{b} &= -\Lambda(\mathbf{e}_0 \cosh w + \mathbf{e}_3 \sinh w) - \Gamma(\mathbf{e}_1 \cos\varphi + \mathbf{e}_2 \sin\varphi),\\
\mathbf{c} &= \frac{\Lambda\omega}{a}(\mathbf{e}_0 \sinh w + \mathbf{e}_3 \cosh w) + \frac{\Gamma\chi}{a}(\mathbf{e}_1 \sin\varphi + \mathbf{e}_2 \cos\varphi),
\end{aligned}$$

and

$$a^2 = \Gamma^2\chi^2 + \Lambda^2\omega^2, \quad \tau^2 = \frac{\Gamma^2\Lambda^2}{a^2}(\chi^2 + \omega^2)^2 \quad \sigma^2 = \frac{\omega^2\chi^2}{a^2}.$$

In terms of $\vec{B}$ and $\vec{E}$, one should consider two cases:

1. $\vec{B} \cdot \vec{E} \neq 0$. By a suitable transformation (see section 3), one may adopt a frame in which $E^a = E\delta^a_3$ and $B^a = B\delta^a_3$, and therefore $\Delta = k^4(B^2 + E^2)^2$, so that $\chi^2 = k^2E^2$ and $\omega^2 = k^2B^2$.

2. $\vec{B} \cdot \vec{E} = 0$. It follows that $\sigma^2 = 0$; there are three sub-cases of interest:

   - $E = 0$, and therefore $\chi = 0$, $\omega^2 = k^2B^2$, and the world-line (L) can be shown to be tangent to

     $$\mathbf{u} = \Gamma\mathbf{e}_0 + \Lambda(\mathbf{e}_1 \cos\omega s + \mathbf{e}_2 \sin\omega s),$$

     and is a timelike helix which is confined to a three-dimensional subspace of M_4. We find $a^2 = \Lambda^2\omega^2$ and $\tau^2 = \Gamma^2\omega^2$.
   - $B = 0$, and therefore $\omega = 0$, $\chi^2 = k^2E^2$, and the world-line (L) is tangent to

     $$\mathbf{u} = \mathbf{e}_0 \cosh\chi s + \mathbf{e}_3 \sinh\chi s;$$

     we find $a^2 = \chi^2$ and $\tau = 0$.

- $I = B^2 - E^2 = 0$. One can adopt a frame in which F_{ab} has only two non-vanishing components $F_{20} = F_{12} = E = B$. One can then proceed as in the general case, to find the corresponding u^a, taking it in the form

$$\mathbf{u} = \mathbf{v}_0 + s\mathbf{v}_1 + s^2\mathbf{v}_2 + s^3\mathbf{v}_3,$$

and such that $u^a u_a = -1$ for all s. One finds

$$\mathbf{u} = (1 + \frac{1}{2}s^2a^2)\mathbf{e}_0 + \frac{1}{2}s^2a^2\mathbf{e}_1 + sa\mathbf{e}_2.$$

A direct computation shows that $a^2 - \tau^2 = 0$.

6 Significance of Λ and Γ

Let us first examine the case when $\sigma = 0$ and $(S_A{}^B)$ has one double zero eigenvalue, and two purely imaginary ones. One then has

$$\mathbf{u} = \Gamma\mathbf{e}_0 + \Lambda(\mathbf{e}_1 \cos\varphi + \mathbf{e}_2 \sin\varphi)$$

with $\varphi = \omega s$,

$$\omega = \sqrt{\tau^2 - a^2}, \quad \Lambda^2 = \frac{a^2}{\tau^2 - a^2}, \quad \Gamma^2 = \frac{\tau^2}{\tau^2 - a^2}.$$

Synge writes in this case: "It is in fact the history of a particle which moves in a circle of radius $\Lambda\omega^{-1}$ [taking $c = 1$] with proper angular velocity ω." This result follows by integration of $dx^a = u^a ds$.

However, one can see that the result depends on the law of composition of velocities connecting $\beta(r)$ and $\beta(r + dr)$. In fact, $\beta = \Lambda/\Gamma$ and $\Gamma = 1/\sqrt{1 - \beta^2}$, hence

$$d\beta = \frac{d\Lambda}{\Gamma^3} = \frac{d\Lambda}{(1 + \Lambda^2)^{3/2}} = \Gamma^{-2}\frac{d\Lambda}{\sqrt{1 + \Lambda^2}}.$$

Now, $d\beta$ may be taken to be the velocity increment according to either the Galilean or the relativistic law of composition of velocities.

- In the first case,

$$d\beta = \frac{\omega\, dr}{c},$$

and integrating

$$\frac{d\Lambda}{(1+\Lambda^2)^{3/2}} = \frac{\omega\, dr}{c},$$

we obtain

$$\Lambda = \frac{\omega r/c}{\sqrt{1-\omega^2 r^2/c^2}} \quad \text{and} \quad \Gamma = \frac{1}{\sqrt{1-\omega^2 r^2/c^2}},$$

which are the values which follow from the instantaneous Lorentz transformation (ILT) with velocity $\omega r/c$. As such, when one composes two ILTs corresponding to ω_1 and ω_2, one does not recover the additive law for angular velocities. Furthermore, the relation $\beta = \Lambda/\Gamma = \omega r/c$ introduces the unwanted particle velocity greater than c. Also, Maxwell's equations are then not covariant in rotating frames (see [1, 4]).

- In the second case,

$$\frac{d\beta}{1-\beta^2} = \frac{\omega\, dr}{c} = \frac{d\Lambda}{\sqrt{1+\Lambda^2}},$$

which leads to

$$\Lambda = \sinh\frac{\omega r}{c}, \quad \beta = \tanh\frac{\omega r}{c},$$

that is, to the Franklin-Trocheris-Takeno (FTT) map. It is clear that velocities remain less than c under this map [8, 9], and one can see that the covariance of Maxwell's equations in rotating frames is restored [10]. One can also check that the value of ω is consistent with equation (11).

Note that the above discussion of Γ and Λ remains valid in a constant e.m. field whenever one can introduce a canonical decomposition of spacetime into two canonical planes $(\mathbf{e}_0, \mathbf{e}_3)$ and $(\mathbf{e}_1, \mathbf{e}_2)$.

7 Conclusion

In this part of the study of the relativistic motion of charged particles in a e.m. field, we have sorted out the main geometric features, with special reference to the case of a constant field. The Larmor rotation has been related, not to a Galilean motion, but to a relativistic one of

FTT type. The Euclidean 3D helix has been properly generalized to a Minkowskian one; one recovers the characterization of the former when the third curvature vanishes. The case of non-constant e.m. field and further physical applications will be presented elsewhere.

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