The "fine structure" of Special Relativity and the Thomas precession

Yves Pierseaux

Inter-university Institute for High Energy
Université Libre de Bruxelles (ULB)
e-mail: ypiersea@ulb.ac.be

RÉSUMÉ. La relativité restreinte (RR) standard est essentiellement un mixte entre la cinématique d'Einstein et la théorie des groupes de Poincaré. Le sous-groupe des transformations unimodulaires (boosts scalaires) implique que l'invariant fondamental de Poincaré n'est pas le quadri-intervalle mais le quadri-volume. Ce dernier définit non seulement des unités de mesure, compatibles avec l'invariance de la vitesse de la lumière, mais aussi une différentielle scalaire *exacte*. Le quadri-intervalle d'Einstein-Minkowski nécessite une définition non-euclidienne de la distance spatio-temporelle et l'introduction d'une différentielle *non-exacte*, l'élément de temps propre.

Les boost *scalaires* de Poincaré forment un sous-groupe du groupe général (avec deux rotations spatiales). Ce n'est pas le cas des boosts *vectoriels* d'Einstein étroitement liés à la définition du temps propre et du *système propre*. Les deux rotations spatiales de Poincaré n'apportent pas de physique nouvelle alors que la précession spatiale introduite par Thomas en 1926 pour une succession de boosts non-parallèles corrige (facteur $\frac{1}{2}$) la valeur calculée classiquement du moment magnétique propre de l'électron. Si la relativité de Poincaré est achevée en 1908, c'est Thomas qui apporte en 1926 la touche finale à la cinématique d'Einstein-Minkowski avec une définition correcte et complète (*transport parallèle*) du système propre. Nous montrons, à la suite de l'énergie propre (Einstein), de la masse propre (Planck), du temps propre (Minkowski), que **le moment magnétique propre** (et aussi le spin 1/2) de *l'électron ponctuel* est inscrit dans la **RR d'Einstein-Thomas**, *clairement séparée de celle de Poincaré* (électron classique). Le spin 1/2 est ainsi *déduit* du groupe d'Einstein-Thomas. L'équation de Dirac (*premier ordre* par rapport au temps) est invariante au sens d'Einstein-Thomas tandis que celle de Klein-Gordon (*second ordre* par rapport au temps) est covariante au sens de Poincaré. Nous montrons également que la précession du système propre implique une structure projective et lobatchevskienne de l'espace tridimensionnel dans la RR d'Einstein-Thomas dans lequel l'électron n'est *jamais accéléré* dans son système propre et n'émet donc pas de radiation.

The standard Special Relativity (SR) is essentially a mixture between Einstein's kinematics and Poincaré's theory of groups. The subgroup of unimodular transformations (scalar boosts) implies that Poincaré's fundamental invariant is not the four-interval but the four-volume, which defines not only the units of measure, compatible with the invariance of light speed but also an *exact-scalar* differential. Minkowski's four-intervall supposes a non-Euclidean definition of the space-time distance and the introduction of an *non-exact* differential, the element of the proper time. Poincaré's *scalar* boosts form a subgroup of the general group (with two space rotations). This is not the case for vector Einstein's boosts, connected with the concepts of proper time and *proper system*. Poincaré's two space rotations don't bring new physics whereas Thomas' space rotation, that completes Einstein's composition of vector boosts, corrects (factor $\frac{1}{2}$) the value of the magnetic moment of the electron.

If Poincaré's SR is completed in 1908, Thomas completed only in 1926 Einstein-Minkowski's kinematics by his correct and complete definition of the proper system (*parallel transport*). We show that it is not only the proper energy (Einstein), the proper mass (Planck), the proper time (Einstein-Minkowski), but also the **proper magnetic moment** (and also the spin $\frac{1}{2}$) of the pointlike electron which is inscribed in **Einstein- Thomas' SR**, *clearly separated of Poincaré's one* (where the electron has a finite volume). The electron's spin $\frac{1}{2}$ is **deduced** from Einstein-Thomas' group. Dirac's equation (*first order* with respect to the time) is invariant in Einstein's sense while the Klein-Gordon's equation (*second order* with respect to the time) is invariant in the sense of Poincaré. In this respect we show that Thomas' precession implies a projective and Lobatchevkian structure of the tridimensional space in Einstein-Thomas' SR, in which the electron (without structure) is *never accelerated* and therefore doesn't emit radiation.

1 Introduction : Einstein's SR, Poincaré's SR and the Thomas precession

The two famous papers, that of Einstein ("Elektrodynamik bewegter Körper") and that of Poincaré ("La dynamique de l'electron") contain not only two approaches of SR but two different *theories* of SR. The polemical questions of historical priorities prevent us from seeing very interesting points for physics. In order to avoid these questions of priorities, we consider that the two theories are simultaneous (1905) and independent events . The title ” The ”fine structure” of SR” means therefore that, if the two theories are very close, they are not merged. This ”spectral” analogy has been chosen because the existence of a ”fine structure” of SR is connected with a question of atomic physics: The Thomas precession of the spin of the electron. Pais tells us Einstein’s impressions in1925:

> Twenty years after his seminal 1905 paper on SR, **Einstein heard something about the Lorentz group that greatly surprised him**. In 1925 Uehlenbeck and Goudsmit had discovered the spin of the electron and thereby explained the occurrence of the alkali doublets, but for a brief period it appeared that the magnitude of the doublet splitting did not come out correctly. Then Thomas supplied the missing factor, 2, now known as the Thomas factor. Uhlenbeck told me that he did not understand a word of Thomas work when it first came out. I remember that, when I first heard about it, it seemed unbelievable that a relativistic effect could give a factor of 2 instead of something of order v/c. **Even the cognoscenti of the SR (Einstein included!) were quite surprised**. At the heart of the Thomas precession lies the fact that a LT with velocity $\mathbf{v}_1$ followed by a second LT in a different direction $\mathbf{v}_2$ does not lead to the same inertial frame as one single Lorentz transformation (LT) with the velocity $\mathbf{v}_1+\mathbf{v}_2$ (*It took Pauli a few weeks before he grasped Thomas point*). [18. Pais]

Thomas’ precession is entirely based on Einstein’s kinematics. It is therefore very strange that Einstein was greatly surprised by the Thomas’ rotation. The Lorentz group had been indeed defined by Poincaré in 1905 with his two space Euclidean rotations. Unfortunately Poincaré was no longer there in 1926 to give his own impressions. We immediately underline that Thomas’ deduction of only one rotation from the composition of two **v**-LT (Lorentz Transformations), that conducts to a very simple correction (Thomas’ factor), is *logically* very difficult (”Uhlenbeck didn’t understand a word...” and ”it took Pauli a few weeks...”). Many physicists think that the essence of the Thomas’ precession is only situated in the non-commutativity of the composition of two **v-**LT but there is, in fact, another fundamental root of the Thomas reasoning in the definition of "the *proper system"*. Goldstein writes about the Thomas rotation:

> Consider a particle moving in the laboratory system with a velocity v that is not constant. Since the system in which the particle is at rest is accelerated with respect to the laboratory, the two systems should not be connected by a LT. **We can circumvent this difficulty** [the acceleration] by a frequently used stratagem (elevated by some to the status of an additional postulate of relativity). **We imagine an infinity of inertial systems moving uniformly relative to the laboratory system,** one of which instantaneously matches the velocity of the particle. The particle is thus instantaneously at rest in an inertial system.[12. Goldstein]

Thomas' precession is directly connected with the acceleration and the status of acceleration is a crux problem in SR [13. Jaeckel M. and Reynaud S.] Goldstein speaks about a "stratagem" or even an "additional postulate" to the theory of relativity about the way to circumvent the problem of acceleration. It is impossible to understand, in the least details, the two relativistic roots of Thomas' discovery without distinguishing Poincaré's principles and Einstein's principles of SR because the **status of acceleration** ("a second derivation with respect to the time") is not the same in Poincaré's and in Einstein's theory.

2 The mixture SR: Poincaré's groups and Einstein's kinematics

The "standard" or the "common" SR is essentially a mixture between *Einstein's principles of kinematics* and *Poincaré's theory of groups.*

The process of splitting off the standard mixture is not "antirelativistic". On the contrary, Poincaré's SR confirms all the formulas of the standard SR. This is the reason why we adopt (by analogy of course) the concept of the "fine structure" of SR because the observation of the fine structure doesn't delete the definition of the spectral lines. But the doublet "Poincaré's SR- Einstein's SR" can conduct to non-standard results (see conclusion), invisible in the mixed state.

2.1 Poincaré's mathematical group of the Lorentz transformation

When one thinks about the "Poincaré's group and relativity", one thinks first about what Wigner has called (perhaps by compensation)

”Poincaré’s group” (the non-homogeneous group) . That group is not a good starting point because it is the only one that is not present in Poincaré’s work. If the LT are of course in Lorentz work, the group’s properties (the most general and the most restricted) are not in Lorentz work but in Poincaré’s work. The group defined and called the Lorentz group by Poincaré is the following (for any scale factor *l*, Poincaré poses in his paragraph 1: $c = 1$ and $k = \frac{1}{\sqrt{1-\varepsilon^2}}$)

> It is noteworthy that the LT form a group. For that, if we put
>
> $$x' = kl(x + \varepsilon t) \quad y' = ly \quad z' = lz \quad t' = kl(t + \varepsilon x)$$
>
> and
>
> $$x'' = k'l'(x' + \varepsilon' t') \quad y'' = l'y' \quad z'' = l'z' \quad t'' = k'l'(t' + \varepsilon' x')$$
>
> with
>
> $$k^{-2} = 1 - \varepsilon^2 \quad k'^{-2} = 1 - \varepsilon'^2$$
>
> we find
>
> $$x'' = k''l''(x + \varepsilon'' t) \quad y'' = l''y \quad z'' = l''z \quad t'' = k''l''(t + \varepsilon'' x)$$
>
> with
>
> $$\varepsilon'' = \frac{\varepsilon + \varepsilon'}{1 + \varepsilon\varepsilon'} \quad l'' = ll' \quad k'' = kk'(1 + \varepsilon\varepsilon') = \frac{1}{\sqrt{1 - \varepsilon''^2}} \quad (1)$$
>
> [26. Poincaré H. 1905, paragraph 4]

These ”$(\varepsilon, l) - LT$” define ”the continuous and homogenous Lorentz group of Poincaré”. At this stage of the reasoning, these ”$(\varepsilon, l) - LT$” are only the mathematical transformations (1) that let invariant the Maxwell equations in their most general form, with notably the second order wave equation, written by Poincaré with the d’Alembertian,

$$\Box' = l^{-2}\Box \quad (2)$$

in his first paragraph.

Poincaré considers successively *two* **decompositions** of his Lorentz group. The first is the following:

> Any transformation of this group can always be resolved into a transformation of the form
>
> $$x' = lx \quad y' = ly \quad z' = lz \quad t' = lt$$
>
> and a linear transformation that doesn't change the quadratic form:
>
> $$x^2 + y^2 + z^2 - t^2 \tag{3}$$
>
> [26. Poincaré H. 1905, idem]

Poincaré envisages the invariance of the quadratic form by any linear transformations $(\varepsilon, l) - LT$. He doesn't say anything about the sign of this quadratic form. He doesn't consider the null value of the quadratic form and consequently he doesn't associate the quadratic form with the propagation of light in order to define an null interval like Einstein or a metric like Minkowski (4.1). The propagation of light waves is described by the (covariant) second order wave equation (2) and *only* by the second order wave equation. Physical meaning of the quadratic form (3) is not discussed by Poincaré. Many physicists think that the "mathematician"Poincaré developed only the mathematical group structure leaving the fundamental physical interpretation to the "physicist" Einstein.

2.2 Poincaré's physical subgroup of scalar ε-LT and Euclidean space rotations

The problem is that Poincaré doesn't introduce physics (*the principle of relativity*) in his first decomposition (3) but in his **second decomposition** of his Lorentz' group (just after the first in the text):

> The group can also be generated in another way. Any transformation of the group may be regarded as comprising a transformation having the form {1}

$$x' = kl(x + \varepsilon t) \quad y' = ly \quad z' = lz \quad t' = kl(t + \varepsilon x)$$

preceded and followed by an appropriate rotation.

For our purposes, however, we have to consider only certain of the transformations in this group.

We must regard l as being a function of ε, the function being chosen **so that this partial group P is itself a group**. Let the system be rotated through 180° about the y-axis; then the resulting transformation must belong to P. This operation is equivalent to changing the sign of x, x’, z and z’; hence we have {2}

$$x' = kl(x - \varepsilon t) \quad y = ly \quad z' = lz \quad t' = kl(t - \varepsilon x)$$

Thus l is unchanged when ε is replaced by -ε. Next, if P is a group, the substitution inverse to {1}, that is {3}

$$x' = kl^{-1}(x - \varepsilon t) \quad y' = l^{-1}y \quad z' = l^{-1}z \quad t' = kl^{-1}(t - \varepsilon x)$$

must likewise belong to P; it must be identical to {2}
so that

$$l = l^{-1}$$

Consequently we have:

$$l = 1$$

[26. Poincaré H. 1905, idem]

Poincaré’s relativistic physics begins here. In other words Poincaré’s demonstration of his principle of relativity is in this second decomposition of the group. Let us sum up Poincaré’s demonstration of the principle of relativity:

1) The second Poincaré decomposition with **two** Euclidean rotations, R_3 and R'_3 can be written:

$$R'_3 \circ (\varepsilon, l) - LT \ \circ R_3 \tag{4}$$

2) "*For our purposes,... we must regard l as being a function of* ε": If the scale factor depends only on the velocity $l(\varepsilon)$ then the $(l(\varepsilon), \varepsilon)-LT$ must form a **subgroup**.

3) In order to have a subgroup (the principle of relativity), we must have:

$$l(\varepsilon) = 1 \tag{5}$$

and therefore we have "$\varepsilon - LT$" :

$$x' = k(x + \varepsilon t) \quad y' = y \quad z' = z \quad t' = k(t + \varepsilon x) \tag{6}$$

The principle of relativity and the structure of subgroup of $\varepsilon - LT$ *are exactly the same thing in Poincaré's logic.*

Poincaré's demonstration of the principle of relativity is that the Lorentz mathematical (the scale factor l) group forms a Lorentz physical (the mathematical scale factor l can only depend on the velocity ε) subgroup if and only[1] if $l(\varepsilon) = 1$.

If we return to the property of associativity (1) with $l = 1$ of the $\varepsilon - LT$ it is now clear the velocity ε is a *relative velocity.* Poincaré's ether is a completely relativistic space-time medium[2]. If we return now to the second decomposition (4), with two space rotations, we have with (5), $l = 1$:

$$R_3' \circ \varepsilon - LT \ \circ R_3 \tag{4'}$$

The velocity (the direction of the movement) is always defined along the x-axis by Poincaré (6). This is obviously not restrictive because, thanks to the two Euclidean space rotations, R_3 and R_3', the general case where the two velocities are not parallel can be reduced to (6). So if the relative velocity of K' is not aligned on the x-axis of K, it is very easy to make two space rotations in order to find again the same

[1]Poincaré gives up, in his fundamental 1905 paper, the discussion of the $\varepsilon-$dependance of the scale factor l (1904 Poincaré's letter to Lorentz).

[2]Let us notice that at this stage the absolute space is already mathematically and physically impossible. In which frame is the ether at rest (K, K', K")? [28. Poincaré H. (1907), La relativit de l'espace]. If one considers two inertial frames, the ether is *by definition*, in Poincaré's own words , at rest in one of the two frames.

orientation for the two systems K and K’. With these two Euclidean rotations, we obviously find again the same *orientation* when we make *one* revolution of 2π for each rotation (see conclusion).

Poincaré’s Euclidean space rotations introduce no new physics because they allowed to reduce the study of *any movement* (any vector velocity) in Euclidean R_3 space in three dimensions into a space in one dimension. The first rotation lines up the x-axis of K’ with the x axis of K, then Poincaré’s $\varepsilon - LT$ in this direction transforms K’ to a frame which is at rest relative to K and a final rotation lines up the coordinate y’ and z’ of the frame K’ with that ones y, z of K’. So instead of (6), it is not restrictive to consider the two-dimensional space-time *subgroup* $\varepsilon - LT$ (7):

$$x' = k(x + \varepsilon t) \quad t' = k(t + \varepsilon x) \tag{7}$$

By definition relative velocity ε is a **scalar** in Poincaré’s scalar boost, $\varepsilon - LT$.

Poincaré shows also that his two-dimensional space-time scalar $\varepsilon - LT$ is analogous ($t \rightarrow it$) to an Euclidean rotation R_2 (”around a fixed origin” in Poincaré’s own words, paragraph 9 [26. Poincaré H. 1905]) in the complex plan.

$$R'_3 \circ R_2(t \rightarrow it) \circ R_3 \tag{8}$$

It is very important to notice that if Poincaré didn’t introduce a metric (a definition of units of measure with the velocity of light) in his first decomposition, he no longer introduces the definition of unit of length in his second decomposition. We will see (3.1) that Poincaré absolutely needs a physical phenomena that occurs in one dimension (along the direction of the motion) in order to define his metric (units of measure).

What is the geometrical meaning of Poincaré’s demonstration of the principle of relativity?

$$l = l(\varepsilon) \Longrightarrow l(\varepsilon) = 1$$

Poincaré's demonstration consists of demonstrating that the **determinant** of the $(l(\varepsilon), \varepsilon) - LT)$ is equal to one.

$$D = \begin{vmatrix} lk & lk\varepsilon \\ lk\varepsilon & lk \end{vmatrix} = k^2(1-\varepsilon^2)l^2 = l^2 = 1 \tag{9}$$

Given that $(l(\varepsilon), \varepsilon) - LT$ are linear affine (homogeneous) transformations[3], then in order to form a subgroup, they must be **unimodular** affine transformations. If the affine invariant is the harmonic ratio between three lined up points, the affine **unimodular invariant** is the **area**, defined by three non-aligned points. We shall show (3.3) that Poincaré's metric and Poincaré's relativistic differential equations are entirely based on this invariant and not on the other invariant, the quadratic form (3) .

2.3 Einstein's vector v-LT and Lobatchevskian kinematics

The standard mixture SR is based on Poincaré's group theory and Einstein's kinematics. But what is the most essential element in Einstein's kinematics? It is of course the addition of velocities:

> In the system k moving along the x-axis of the system K with velocity **v**, let a point move [with velocity w] (φ is then to be looked upon the angle between the velocities **v** and **w**). After a simple calculation we obtain
>
> $$u = \sqrt{\frac{v^2 + w^2 + vw\cos\varphi - \frac{vw\sin\varphi}{c^2}}{1 + \frac{vw\cos\varphi}{c^2}}} \tag{10}$$
>
> It is worthy of remark that **v** and **w** enter into the expression of the resultant velocity **u** in a **symmetrical** manner.
>
> If **w** has the direction of the x-axis, we get
>
> $$u = \frac{v+w}{1+\frac{vw}{c^2}} \tag{11}$$

[3]The inhomogeneous group of Poincaré-Wigner is also an affine transformation. On the other hand, a projective linear transformation supposes that only the homogeneous coordinates are valid.

> For the case $v = c$ [in LT!!!], we have
>
> $$u = \frac{c + w}{1 + \frac{w}{c}} = c \tag{12}$$
>
> If in addition to the system K and k, we introduce another system k' moving parallel to k From which we see that such parallel transformations-necessarily forms a group.[7.Einstein A.1905, paragraph 5]

Einstein writes the general law of composition of the velocities (for non-parallel velocities). He considers a *"point"* with the velocity **w** and applies a $\mathbf{v} - LT$. Einstein's boost $\mathbf{v} - LT$ is therefore fundamentally a **vector** boost.

Poincaré's supporters often claim that Einstein didn't know anything about the concept of group. This is not true because Einstein shows the structure of group for parallel transformations. The genuine problem is that he introduces a **vector** definition of the boost (paragraph 2, deletion of the ether) and he doesn't say anything about the group properties of the general case of the composition of two **vector** $\mathbf{v} - LT$ in different directions. Nowhere Einstein introduces space rotations (neither in this text, nor in his second fundamental text, see in the introduction: "Twenty years after his seminal 1905 paper on SR, Einstein heard something about the Lorentz group that greatly surprised him). **Einstein's $\mathbf{v} - \mathbf{LT}$ don't form a group** because the composition of two non-parallel **v**-LT don't give a $\mathbf{v} - LT$ but a **v**-LT with one space rotation .

The historical situation of the two SR is diametrically opposed. Indeed Poincaré's demonstration of the group structure of LT is nearly finished in 1905 (without the 1908 definition of units of measure) while Einstein's problem of the group structure of the $\mathbf{v} - LT$ - coupled with the nature of the geometrical representation of the law of composition of non parallel velocity **v** - is characterized by a very long historical development.

Einstein's vector $\mathbf{v} - LT$ defined a 3-velocity space often call **3-kinematics space.** What is the geometrical character of this vector kinematics space? That question has been completely solved by Sommerfeld, Varicak and Borel.Sommerfeld shows in 1908 that Einstein's above formula of vectorial addition of velocities is a formula of trigonometry

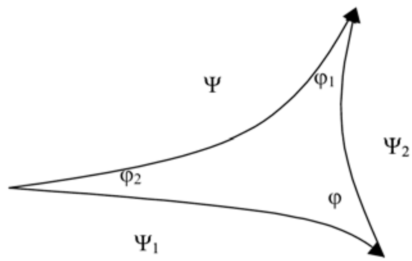

Figure 1: Hyperbolic triangle of rapidities

on a sphere of imaginary radius (c^2 = -1 or $c = i$) [34.Sommerfeld A.]. If Sommerfeld's representation is mathematically correct, it is not a satisfactory physical solution because the 3-kinematics space cannot be an imaginary space (the light lines "$c = i$" are isotropic lines in a mathematical sense) but a real space. The satisfactory solution for the geometrical character of the 3-kinematics space (the light lines are Minkowski's isotropic lines) has been found by Varicak (1909). Einstein's formula (10) is a formula (13) of the trigonometry of *Lobatchevski [40. Varicak V.]*:

$$\cosh\Psi = \cosh\Psi_1 \cosh\Psi_2 - \sinh\Psi_1 \sinh\Psi_2 \cos\varphi \qquad (13)$$

that is written with Minkowski's hyperbolic angles (called rapidity by Robb [32. Robb A.A.]): with $\frac{u}{c} = th\Psi, \frac{v}{c} = th\Psi_1, \frac{w}{c} = th\Psi_2$.In most of the standard books, we can only find the formula: $th(\Psi_1 + \Psi_2) = \frac{th\Psi_1 + th\Psi_2}{1 + th\Psi_1 th\Psi_2}$ with $\Psi = \Psi_1 + \Psi_2$ (11'). The standard interpretation is almost always reduced to the parallel translation (this last equation 11' appears then only as another notation for the particular case 11 of addition of parallel velocities). But that's not the fundamental discovery of Varicak. He demonstrates that this last equation (11') is a **vectorial** addition (14) in **Lobatchevski's 3-kinematic space** :

$$\overrightarrow{\Psi} = \overrightarrow{\Psi_1} + \overrightarrow{\Psi_2} \qquad (14)$$

In Lobatchevski 3-space of velocities (see also Fock V.[11. Fock V.]), the **vector addition** hold good. The sum of the three angles of the

hyperbolic triangle (figure 1) is lesser than π and the hyperbolic defect ϵ is:

$$\epsilon = \pi - (\varphi + \varphi_1 + \varphi_2) \tag{15}$$

Varicak shows also that the aberration of light is the Lobatchevki angle of parallelism (4.5) but we focus now the attention on the fact that the additive property extend to vectorial addition (14) [2. Barrett J.F.], because we shall demonstrate (4.5) the relation between the hyperbolic defect and the Thomas rotation.

We can think that one of the greatest specialist of the Lobatchevski geometry in the world (Poincaré) had not realized that he worked in his SR with a Lobatchevski space. It would be not only not very reasonable but it would be also completely wrong. In Poincaré's SR, the kinematic 3-space is by definition (second decomposition of the Lorentz group, 2.2) Euclidean thanks to the two space rotations R_3 and R'_3. *The problem of the composition of vector $v - LT$, followed by only one space rotation is Einstein's problem ("at his great surprise...", see introduction), not Poincaré's problem.*

Pauli notes (note 111) the origin of Varicak's determination of Einstein's 3-kinematic space:

> This connection with the Bolyai-Lobatchevsky geometry can be briefly described in the following way: if one interprets dx_1, dx_2, dx_3, dx_4 as homogeneous coordinates in a three-dimensional projective space, then the invariance of the equation amounts to introducing a Cayley system of measurement, base on a real conic section. The rest follows from the well-known arguments by Klein.[19. Pauli W., note 111]

So Pauli shows clearly that Varicak's fundamental discovery is based on the homogeneous coordinates of $\mathbf{v} - LT$ that is a projective point of view. We can generally be sure that Klein's point of view is not Poincaré's point of view and that is particularly true in SR [14. Klein F.].

2.4 Einstein's symmetrical ("v=c") addition of velocity vectors

Let us return now to Einstein's quotation [7.Einstein A.1905, paragraph 5] about the motion of a "point" at the beginning of (2.3): "For the very important case "$v = c$" , we have "$u = c$" ". The young Einstein doesn't hesitate to put "$v - c$" in $\mathbf{v} - LT$!!! (12). The only meaning of the relativistic ether in Poincaré's logic is that the light is a wave and only a wave (there is no point that travels with velocity c) while in Einstein's logic the light is a particle with the velocity $v = c$. Einstein in 1905 put everywhere $v = c$ in his equations. But our problem is not the history but the mathematical physics. It is in principle strictly forbidden to make $v = c$ in $v - LT$ because the γ factor becomes infinite. So we can think that it is an Einstein's youthful misdemeanour and that he would have to make $w = c$ in order to obtain $u = c$. But precisely, Einstein writes just before making $v = c$ that "$\mathbf{v}$ and $\mathbf{w}$ enter into the expression of the resultant velocity $\mathbf{u}$ in a **symmetrical** manner". L*ogically* if one makes $\mathbf{w} = \mathbf{c}$ and one takes into account Einstein's demonstration of commutativity for $\varphi = 0$, that means that the order of the composition of two parallel boosts, "$w = c$ and v" or "w and $v = c$" doesn't have any importance.

Nothing prevent us to make $v = w = c$ in order to find $u = c$. If we compose $x = ct$ with $v = c$ we find the result:

$$\xi = \gamma(x - vt) \qquad \tau = \gamma(t - \frac{v}{c^2}x) \qquad \xi = \infty.0 \qquad \tau = \infty.0$$

So let us note that there is no contradiction in order to find $\xi = c\tau$ on condition that we choose the units ($t = 1$ and $x = c$)

$$\xi = \infty.0 = c \qquad \tau = \infty.0 = 1 \tag{16}$$

So the point $t = 1$ and $x = c$ becomes the point $\tau = 1$ and $\xi = c$. We shall demonstrate that young Einstein's apparent mistake $v = c$ in $\mathbf{v} - LT$ is in perfect harmony with his definition of units ("$\mathbf{c} - LT$" is defined by (16)), with Minkowski's identification of the light lines with the isotropic lines and with Pauli's note 111 about Varicak's homogeneous coordinates (4.1).

3 Poincaré's scalar relativistic kinematics

Poincaré's second decomposition (4') is not compatible with Einstein's kinematics. So logically the question of the existence of a relativistic Poincaré's kinematics compatible with his subgroup is posed.

3.1 Poincaré’s two principles: principle of relativity and principle of longitudinal contraction of length

The principle of relativity (”the impossibility of demonstrating the absolute movement”) is by definition the structure of subgroup of scalar $\varepsilon - LT$. This is Poincaré’s first principle (first part of his 1905 work). What is Poincaré’s second principle ? The answer is in the second part of his 1905 paper:

> So the Lorentz hypothesis is the only one that is **compatible** with the impossibility of demonstrating the absolute motion ; if we admit this impossibility, we must admit that moving electrons are contracted in such a way to become revolution ellipsoids whose two axis remain constant. [26. Poincaré H. 1905, paragraph 7]

Poincaré’s two principles are (1900-1911):

1) principle of relativity

2) principle of longitudinal contraction of length

According to Poincaré, the Lorentz hypothesis is not only *compatible* with the principle of relativity but is *the only one* compatible with the principle of relativity (unlike e.g., Langevin’s and Abraham’s hypotheses) in others words with $l = 1$).

3.2 Poincaré’s use of ε-LT and the purely longitudinal (scalar) contraction

The historical difference, with respect to Einstein, is that Poincaré has never developed explicitly his method of using of $\varepsilon-$LT on a basic example (a deformable rod). Our problem is not an historical problem. In the same way that Einstein’s two principles are compatible, we must understand, on the level of mathematical physics, why and how Poincaré’s two principles are compatible.

3.2.1 The first $\varepsilon-$LT and the definition of simultaneity

Poincaré’s compatibility implies a specific use of LT. Poincaré writes in his fundamental work:

> In accordance with Lorentz Hypothesis, moving electrons are deformed in such a manner that the real electron becomes an ellipsoid, while the ideal electron at rest is always a sphere of radius r (...) The LT replaces thus a moving real electron by a motionless ideal electron.[26. Poincaré H. 1905, paragraph 6]

We adopt Poincaré's (K, K', K" ε, k, t, t') and Einstein's (K, k, k', v, γ, t, τ) *respective notations* in the following of this paper. In order to illustrate this, we can use the following diagrams:

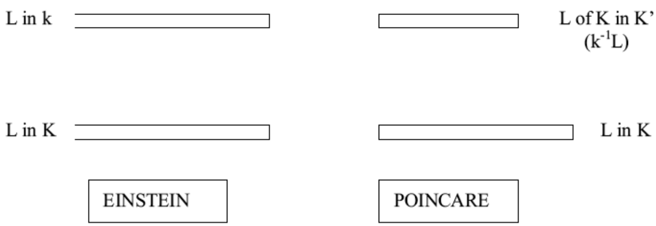

Figure 2: Units of length before LT

In Einstein's SR , the rods are *by principle* (see 4.1) identical (figure 2). The contraction of the moving rod $\gamma^{-1}L$ is the result of a comparison of measurements (figure 3) from one system K (k) to the other k (K) with the well known use of LT. The calculation with Einstein's use of LT is easy and can be found in any standard book on SR. In Poincaré's SR, the moving rod K' is by principle contracted (figure 2) $k^{-1}L$ (sometimes called "real contraction"). By the use of LT (figure 3) the length of the rod in K' (for observers in K') is equal to L.

We can obviously reverse the role of K and K' (Poincaré's subgroup, 2.2). *Poincaré's longitudinal contraction is completely reciprocal.* That is the essential difference with respect to the non-relativistic point of view of Lorentz. According to Poincaré's kinematics, the "real" differences are compensated by a "good use" of LT. According to Einstein's kinematics, the identical processes "appear" to be different by another "good use" of LT. Poincaré's calculation with the LT is also very easy. Suppose the ether is chosen by definition at rest in K. The real length of the rod placed in the moving (ε) system K' is thus $k^{-1}L$. The first LT $x' = k(x - \varepsilon t)$

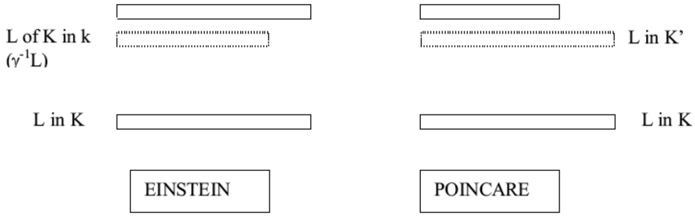

Figure 3: Units of length after LT

"replaces" (in Poincaré's words), *in the same time t*, the (real) contracted length $k^{-1}L$ of the moving rod in a motionless rod L. The rod of length $k^{-1}kL$ by the use of LT is at rest in K'. In Einstein's use of LT the reciprocal contraction of rigid rods is deduced from the definition of simultaneity. In Poincaré's use of LT the simultaneity is deduced from the reciprocal contraction of deformable rods. This is a very important result: *In Poincaré's relativistic kinematics the simultaneity is deduced from the use of the first LT on a contracted moving rod.* The simultaneity is therefore completely relative in Poincaré's kinematics. Then there is no problem with the relativistic character of Poincaré's finite electron.

> But in the Lorentz hypothesis, also, the agreement between the formulas does not occur just by itself; it is obtained together with a possible explanation of the compression of the electron under the assumption that the deformed and compressed electron is subject to constant external pressure, the work done by which is proportional to the variation of volume of this electron.[26. Poincaré H. 1905, introduction]

Poincaré's finite electron is not compatible with the quantum electron but it is perfectly compatible with a relativistic definition of the simultaneity. Poincaré's pressure[4] is also an invariant but we don't want to

[4] Poincaré speaks about the electron as a hole in ether. Poincaré's pressure is not a pressure of ether (Langevin) but a negative pressure of "classical" vacuum. The relativistic mechanics of continuous medium is the starting point of Poincaré's SR and the final point of Einstein's SR. Laue[15. Laue M.] in particular rediscovers in 1911 (and Fermi ten years later) Poincaré's pressure but in a purely static sense while, in Poincaré's text, this pressure has an explicit dynamical sense (with an implicit kinematics sense we develop here).

discuss this question here. We underline only here that Poincaré's pressure supposes a force. Then, Poincaré's use of $\varepsilon - LT$ supposes that between two states of velocity the electron undergoes an acceleration (see 4.2).

3.2.2 The second ε−LT and the definition of duration (elongated light ellipsoids)

The difficult problem for Poincaré is clearly to deduce the (real) dilation of time from the (real) contraction of length, by taking in account the constancy of the velocity of light. Poincaré poses $c = 1$ (paragraph 1) but he doesn't raise (like Einstein) the constant in the Maxwell equations to a status of a principle. In Poincaré's kinematics c is only the constant in the Maxwell equations, which is covariant in virtue of the first principle (covariance of the Maxwell equations, in particular (2) the second order relative to the time wave equations, that defined the electromagnetic medium).

The synchronization method by exchange of signals of light is developed by Poincaré in 1900 in a paper on the reaction principle in the Lorentz theory [24. Poincaré H. 1900]. Poincaré explains that when Lorentz's local time $t' = x + vx/c^2$ is used in the moving system K' relative to the ether K, the observers of K remark no difference (*to first order*) between the *forward* travel time and the *backward* travel time of the light. For the second order Poincaré envisages already in 1900 that the hypothesis of Lorentz is necessary. There is no calculation in Poincaré's 1904 philosophical talk about the general principle of physics, in Saint Louis, in which he repeats simply the analysis of his 1900 paper. Poincaré's calculation (to second order) is published in 1908 [5]:

> A body that is spherical when in repose will thus assume the form of a flattened ellipsoid of revolution when it is in motion. But the observer will always believe it to be spherical, because he had himself undergoes an analogous deformation, as well as all the objects that serve him as points of reference. On the contrary, the surfaces of the waves of light, which have remained exactly spherical, will appear to him as *elongated ellipsoids*. **What will happen then?** Imagine an

[5]The elongated light ellipsoids are however in his cours à la Sorbonne in 1906[27. Poincaré H. (1906)]

> observer and a source involved together in the transposition. The wave surfaces emanating for the source will be spheres, having as centre the successive positions of the source. *The distance of this centre from the present position of the source will be proportional to the time elapsed since the emission - that is to say, to the radius of the sphere.* But for our observer, on account of the contraction, all these spheres will appear as elongated ellipsoids. This time the compensation is exact, and this is explained by Michelson’s experiments. [29. Poincaré H. (1908)]

This is **Poincaré’s exact synchronization** [21. Pierseaux Y.] which takes (in all his texts) into account his fundamental second hypothesis:

1) Spherical waves are solutions of the covariant Maxwell (second order with respect to the time) wave equation for the two systems K and K’.

2) In one of the two systems (say K’), the metres (unit of length) are longitudinally contracted relative to the other (Poincaré’s second principle)

Poincaré’s deduction is very simple:

* The spherical waves become elongated ellipsoidal waves in K’ (the system of the observer and the source involved in the translation).

* The elongation of the ellipsoids is proportional to the *(forth and back)* time elapsed since the emission.

The contraction of units of length, coupled with Poincaré’s covariance of the speed of light, gives the dilation of time (back and forth travel: the variation k of the duration is inversely proportional to the variation k^{-1} of length (i.e. Poincaré’s light waves mean that the unit of time changes inversely with respect to unit of length). Therefore *fundamental connection between the unit of space, the unit of time and the unit of light speed* is:

$$\Delta x \Delta t = k^{-1} \Delta x k \Delta t = \Delta x' \Delta t' \tag{17}$$

The physical invariant is the space-time area $\Delta x \Delta t$ (the ”fourvolume” in four-dimensional space-time) corresponds to the second decomposition (2-2), the subgroup of scalar boost with unimodular determinant

($D = 1$). **The fundamental Poincaré's kinematics invariant compatible with Poincaré's second decomposition of the group (4') is the four-volume.** Poincaré elongated ellipsoidal light waves clarifies not only his definition of units but bring also two other very important clarifications:

a) Poincaré's exact synchronization shows that the time for the moving observers is not Lorentz' local time $t' = t + \varepsilon x$ but Poincaré's local time, given by the second $\varepsilon - LT$: $t' = k(t + \varepsilon x)$. If we have two systems in uniform translation with respect to each other, we can define t in one of the two systems and therefore t' depends on t (and inversely). Poincaré's duality "local time-true time" is completely relativistic" (he never uses these concepts in his 1905 work).

b) Poincaré's transformation of successive spherical light waves into successive ellipsoidal light waves (Poincaré's dilation of space or better: Poincaré's *expansion* of space, see conclusion) immediately induces the relativistic *Doppler-Fizeau* formulas in connection with Poincaré's interpretation of the relativistic aberration.

3.3 Poincaré's exact differential of the four-volume and the finite units of measure

We showed that the physics is introduced by Poincaré by the unimodular transformation (2.2, second decomposition ($D = 1$) and not by the quadratic form (2.1, first decomposition). The demonstration of the invariance of action (Poincaré's paragraph 2 and 3) is the central piece in Poincaré's deduction of relativistic mechanics ("La Mécanique Nouvelle"). This invariance (see "La relativité restreinte d'Einstein-Planck avec entropie invariante et la relativité restreinte de Poincaré avec action invariante", [21. Pierseaux Y., (3)]) is directly deduced by Poincaré from the invariant of the electromagnetic field $l^4(E'^2 - H'^2) = E^2 - H^2$ *and from the invariance of the fourvolume.* Poincaré writes in paragraph 2:

> We have firstly $dt'dV' = l^4 dtdV$ since $x', y', z', t'$are related to x, y, z, t by linear expressions whose **determinant** is l^4"
>
> $$J = \int_{-\infty}^{+\infty} dtdV \frac{1}{2}(E^2 - H^2) \qquad (18)$$
>
> the result

$$J = J' \tag{19}$$

> However for this equation to be valid, the limits of integration must be the same ...[26. Poincaré H. 1905, paragraph 3]

We will discuss Poincaré's limits of integration in (4.2.3). The fundamental differential in Poincaré's paper on relativistic mechanics is given by the four-volume. Taking into account the result of the paragraph 4 (subgroup, $l = 1$) the fundamental differential is:

$$dt'dV' = l^4 dtdV \tag{20}$$

Poincaré's reduction from four-dimensional space-time (6) to two-dimensional space-time (7) is directly based on the compatibility with purely longitudinal contraction and the principle of relativity ($l = 1$):

$$dt'dV' = dtdV \quad \Longleftrightarrow \quad dt'dx' = dtdx \tag{20'}$$

Poincaré's fundamental relativistic reduction $4 \rightarrow 2$ (space-time) or $3 \rightarrow 1$ (space) is entirely based on the fact that (20) define scalar exact differentials (see, a contrario, 4.2.2). Now it is clear that, without a metric (*finite* units of time and space), Poincaré's SR would be only a mathematical theory. The dtdV is an *exact* differential and there is according to Poincaré no problem to integrate, *if the limits are the same.* Unlike Minkowski and Planck [21. Pierseaux Y., (3)], who integer between fixed limits t_1 and t_2 (between two events, 4.2.2), Poincaré doesn't integrate between fixed finite limits:

> However for this equation to be valid, the limits of integration must be the same. Hitherto we have assumed that t ranged from t_0 to t_1 and x, y, z from -∞ *to* $+\infty$. The limits of integration would then be altered by the LT; but there is no bar to assuming that t_0 = - ∞, t_1 = ∞; and the limits for J and J' are the same. [26. Poincaré H. 1905, paragraph 3]

Poincaré's SR is first a field theory and therefore he integers on the electromagnetic field between t_0 = - ∞, t_1 = ∞. With the electromagnetic Lagrangian, $L = \int_{-\infty}^{+\infty} dV \frac{1}{2}(E^2 - H^2)$, Poincaré obtains the relation $L = k^{-1}L'$. Then he finds the exact dilated differential $dt = kdt'$ compatible with the invariance of action. $\int_{-\infty}^{+\infty} Ldt = \int_{-\infty}^{+\infty} L'dt'$. The differential dilation is introduced exactly on the same way as the purely longitudinal contraction $dV = k^{-1}dV'$ according to the **exact scalar** differential of the fourvolume $dVdt = dV'dt'$. Poincaré finally obtains the **finite** Lagrangien of the **finite** electron which is (in his notation: $m = c = 1$) equal to k^{-1} (in 1906 Planck's language, $\gamma^{-1}mc^2$). That confirms dynamically the purely relativistic character of Poincaré's electron (3.2.1). But that also confirms kinematically the purely relativistic character of Poincaré's definition of units of measure: *the fundamental unit of length is naturally given in Poincaré's SR by the classical radius of the electron* ($\frac{e^2}{mc^2}$). The connection between *the infinitesimal,* $dt'dx' = dtdx$, and *the finite,* $\Delta x \Delta t = \Delta x' \Delta t'$, in Poincaré's SR is therefore clarified.

3.4 Poincaré's space-time area invariant and wave light velocity

Poincaré's invariant ($\Delta x \Delta t = k^{-1}\Delta x k \Delta t = \Delta x' \Delta t' = 1$) seems not to be a geometrical credible invariant because the light velocity seems not to be present. In order to show that the two space-time area is directly connected with the velocity of light, i.e. with Poincaré's covariance of light velocity, let us develop the underlying geometry of the unimodular ε-LT (7): $x' = k(x + \varepsilon t) \quad t' = k(t + \varepsilon x)$. Like Poincaré, we poses $c = 1$. We represent the units of measure in K: $\Delta x = \Delta t = 1$. The unimodular invariant (**space-time area**) is based on (at least) **three** points . We represent the two light wave lines in dashed lines because they represent the velocity 1 of a wave and no of a point ($\varepsilon < 1$) on the figure 4:

The points $(0, 0)$, $(1, 0)$, $(0, 1)$ *and* $(1, 1)$[6] become respectively $(0, 0)$, $(k, k\varepsilon)$, $(k\varepsilon, k)$ *and* $(k(\varepsilon + 1)$, $k(\varepsilon + 1)$. The square and the rhomb, constructed on the **two** light lines, have the same space-time area. The direct (forth) light line $x = t$ and the inverse light line (back) $x = -t$ are orthogonal. The invariant space-time area is given by the half product of the two diagonals respectively for the square $\frac{1}{2}\sqrt{2}\sqrt{2} = 1$ and for the rhomb ($\frac{1}{2}$ ($\sqrt{2}k(1 + \varepsilon) * \sqrt{2}k(1 - \varepsilon)) = k^2(1 - \varepsilon^2) = 1$).

[6] It is not a point (a particle) v=c. It is a point of the space-time medium in which an optical perturbation (solution of a second order with respect to the time wave equation) propagates.

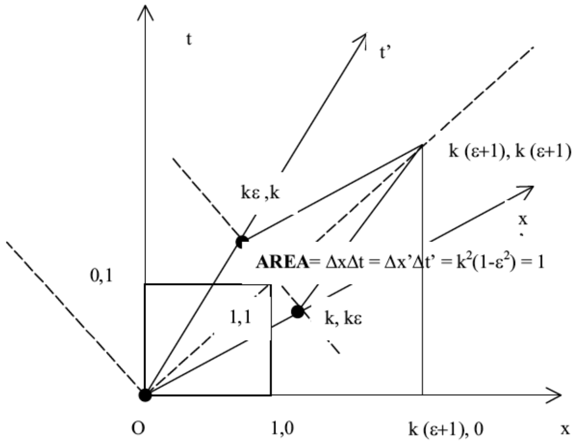

Figure 4: Poincaré’s area invariant

The non-null distance on two Poincaré’s two light lines is therefore the same thing as the invariant area that defines the units of measure. We put the emphasis on the fact that $k^2(1-\varepsilon^2) = 1$ defines an *area* $(\Delta x \Delta t = \Delta x' \Delta t')$ equals to one $(D = 1,\ 9)$ and not a distance equals to one (see 4.1). Poincaré’s unimodular ε-LT (7) are in perfect harmony with Poincaré’s Euclidean-complex (analogy with Euclidean rotation in R_2, equation 8) representation: the norm of a non-null complex number is always not null. Without any contradiction we can define a metric (a physical definition of units) without identifying the light lines to isotropic lines. The (non null) distance on the successive three points on the direct light line are: $\sqrt{2}$, $\frac{\sqrt{2}}{2}k(1+\varepsilon)$, $\sqrt{2}k(1+\varepsilon)$. The (non null) distance on the two inverse light lines are: $\sqrt{2}$, $\sqrt{2}k(1-\varepsilon)$. Poincaré’s invariant is defined with the **two** light lines, i.e. the ”two-ways (back and forth) speed of light ”. (see mean light speed, ”Light elongated ellipsoidal waves”[7]).

In Poincaré’s unimodular *affine* space-time, there is *no metric* in the sense of a definition of a space-time distance between **two points**. But there is obviously (Poincaré’s two space rotations) an Euclidean metric in R_3 (3-space or 3-kinematic space) for K and K’. So with Poincaré’s definition of units by the four-volume ”$k^{-1}k$” we have therefore in K’:

[7] The homothetic light ellipsoids of Poincaré is exactly the physical representation of that structure (the light lines are affine unimodular lines but not isotropic lines).

$\Delta x' = \Delta y' = \Delta z' = 1$. Poincaré's second hypothesis [31 Reignier J.] is not a supplementary hypothesis (we rejoin on this precise point the analysis of Jean Reignier in Peyresc Congress) because *it is directly induced by the structure of the unimodular subgroup.* That definition of units is very important because there is no longer *objective* reason for the physicists, henceforth, to prefer Einstein's kinematics to Poincaré's kinematics. Let us return to the connection with Poincaré's definition of units and Poincaré's covariance of light speed. Let us remark that if we consider only one light wave in figure 4 and therefore the isocele right triangles with respectively the basis in K$(1,0)$ and $k(1+\varepsilon)$, we have $xt = \frac{1}{2}(x^2+t^2)$ or $x^2+t^2-2xt = x'^2+t'^2-2x't' = 0$. (For the other light wave, we replace - by +). In order to make easier the comparison with Einstein-Minkowski's invariant (4.1.3), we can also introduce the velocity of light $c \neq 1$:

$$(x-ct)^2 = 0 \tag{21}$$

So that is the definitive enlightenment of the question of ether in Poincaré's SR: the relativistic space-time medium is defined by affine unimodular ε-LT. Poincaré's ether is not "metaphysics" because it is no more "metaphysics" to do affine geometry than to do projective geometry (see 4.1.1).

4 Einstein-Minkowski-Thomas' vector relativistic kinematics

Einstein showed in 1905 that his two fundamental principles (principle of relativity and principle of speed light invariance) are compatible.

4.1 Einstein's space-time invariant and Minkowski's isotropic light lines

The statement of Einstein's principles are in the paragraph 2 and the demonstration of the compatibility is in the paragraph 3:

> At the time $t = \tau = 0$, when the origin of the two coordinates (K and k) is common to the two systems, let a spherical wave be emitted therefrom, and be propagated -with the velocity c in system K. If x, y, z be a point just attained by this wave, then $x^2+y^2+z^2 = c^2t^2$. Transforming this equation with our equations of transformation we obtain after a simple calculation $\xi^2+\eta^2+\zeta^2 = c^2\tau^2$. The wave under consideration is therefore no less a spherical wave with velocity

> of propagation c when viewed in the moving system. This shows that our two fundamental principles are **compatible**. [7.Einstein A.1905, paragraph 3]

Unlike Poincaré (3), Einstein directly connects the quadratic form to the light propagation. He doesn't say in which system, K or k, the source is at rest. The light speed depends neither on the speed of the source (at rest in K or in k) nor on the speed of the moving system (respectively k and K). Let us also remark that Einstein's spherical "waves" are not spherical waves in the classical sense not only because the ether is deleted but also because they are not defined in Poincaré's sense as a solution of the second order with respect to the time wave equation (2). Einstein spherical "waves" are *defined* by Einstein's null interval between **two** events ($s = 0$, see 4.2):

$$x^2 + y^2 + z^2 - c^2t^2 = \xi^2 + \eta^2 + \zeta^2 - c^2\tau^2 = 0 \tag{22}$$

In order to make easier the comparison with Poincaré, let us consider Einstein's light propagation in only one space dimension. Einstein's spherical "waves", $x^2 - c^2t^2 = \xi^2 - c^2\tau^2 = 0$, are therefore by definition **Minkowski's isotropic lines** with null-distance between **two** world-points. So the difference of squares means not only the equations of the cone light lines but also the null-distance between two different points on each light line.

In Einstein-Minkowski's kinematics the units of measure are defined on **one light line** with the distance between **two** space-time points (figure 5).

The non-Euclidean distance between the **two** points $(0, 0)$ and $(1, 1)$ respectively in K and k, is null. Minkowski's metric (scale) hyperbolas defines the hyperbolic distance (**difference of squares**) between the **two** points $(0, 0)$ and $(1, 0)$ by $\gamma^2v^2 - \gamma^2 = 1$ (idem for $(0, 0)$ and $(0, 1)$). The correspondent expression, $k^2\varepsilon^2 - k^2 = 1$, in Poincaré's SR determines not a non-Euclidean invariant *distance* but an unimodular invariant *area (*equations 9 or 17*)*. In Poincaré's SR, the metric is not determined by the scale hyperbolas but by the ellipsoidal elongated light waves. Minkowski's only one isotropic light line defines Einstein's identical units of measure (in order to make easier the comparison with

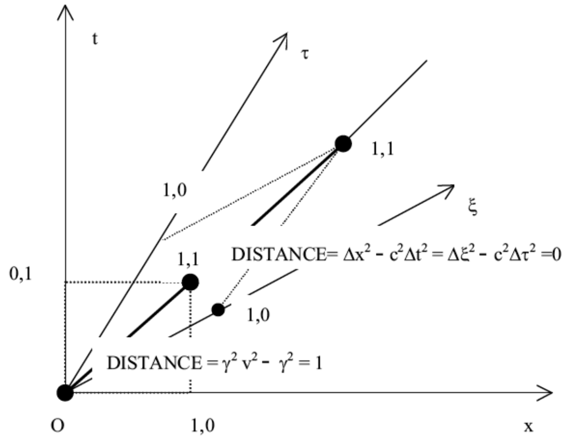

Figure 5: Einstein-Minkowski's distance invariant

Poincaré's kinematics, we can pose $c = 1$):

$$\frac{\Delta x}{\Delta t} = \frac{\Delta \xi}{\Delta \tau} = 1 \Longleftrightarrow \Delta x = \Delta \xi = 1 \text{ and } \Delta t = \Delta \tau = 1$$

(idem for the other isotropic line). In other word Einstein's identical light velocity *within* each system implies Einstein's definition of *identical* units. That definition of identical units is already obvious in Einstein's 1905 text:

> **In accordance with the principle of relativity** " the length L of the rod in the moving system" - **must be identical** to " the length L of the stationary rod.". The length to be discovered [by v-LT], we will call " the length of the (moving) rod in the stationary system". This we shall determine on the basis of our two principles, and we shall find that it differs from L. [7.Einstein A.1905, paragraph 2]

Einstein's statement of the principle of relativity implies clearly that the space-time units are a priori identical *within* each inertial system K and k.

4.1.1 Einstein's identical units and the projective homogeneous coordinates

Max Born, one of rare physicists who has understood the crucial role of this principle of identity[8], thought however that Einstein's principle was in contradiction with Einstein's use of **v**-LT (the units are actually respectively contracted-dilated by Einstein's use of $v - LT$, see 3.1).

There is no contradiction because Einstein's identical units result from the use of $v - LT$ on Einstein's light point "$v = c = 1$"(Einstein's light particle) on one and only one isotropic line. The point $x = 1$, $t = 1$ becomes $\xi = 1, \tau = 1$ by any $v - LT$.

We can also reverse that deduction if we define "$v = c$" $- LT$ for the point $x = 1$ and $t = 1$ at the limit as $\xi = \infty.0 = 1$, $\tau = \infty.0 = 1$ (16).

That $c - LT$ (system with $m = 0$) transforms any point (x, t) into a point $(1, 1)$ on the light isotropic line.

So the law of addition of velocities (10 or 11) becomes perfectly **symmetric** (see Einstein's quotation, 2.3) between v and w when the audacious young Einstein (12) put by symmetry "$v = c$" in $v - LT$ ("$c - LT$"). So in order to precise which kind of geometrical representation refers to Einstein's use ($v \le c$) of the *linear fractional law* of relativistic addition, let us return to (11) with $c = 1$:

$$u = \frac{v + w}{1 + vw} \tag{11"}$$

In Poincaré's *affine* unimodular LT, we have strictly $v < c$. At any point of the right line $v < c$ that becomes $u < c$ it corresponds a point of the plan x, t that becomes x', t'. Except for $v = c$ because on the **affine** *right line of velocities*, we have an **open** interval $]-1, 1[$. So that linear fractional relation is not defined for $v = 1, u = 1$ because we find that the point (of the space-time medium) $x = t = 1$ becomes $x' = t' = o.\infty$.

If we consider now that *the linear fractional* relation is defined on a **projective** *line right of velocities* on a **closed** interval $[-1, 1]$, then,

[8] A fixed rod that is at rest in the system S and is of length 1 cm, will, of course, also have the length 1 cm, when it is at rest in the system S', provided that the remaining physical conditions are the same in S' as in S. Exactly the same would be postulated of the clocks. We may call this tacit assumption of Einstein's theory the "principle of the physical identity of the units of measure".[6. Born M.]

at the velocity transformation from $v = 1$ to $u = 1$, it corresponds the coordinates transformations from $x = t = 1$ to $\xi = \tau = 1$.

At linear projective linear fractional transformation for velocity $v = c = 1$, $1 \leftrightarrows 1$, corresponds the transformation for the homogeneous space-time coordinate $(1,1) \leftrightarrows (1,1)$. The Lorentz coordinates (x,t), (ξ,τ) are the **homogeneous** coordinates of the velocity. In projective transformation there is no difference between a finite (velocities v, w or rapidities Ψ_1, Ψ_2) and an infinite point ($v = c$, $w = c$ or $\Psi_1 = \infty$, $\Psi_2 = \infty$). We see now that addition of velocities (11) and addition of rapidities (11') are completely identical. So the *homogeneous coordinates* of the velocity $v = c = 1$ are $x = t = 1$ $\xi = \tau = 1$, i.e. *Einstein's identical units.* Einstein's principle of identity is valid if and only if the space-time coordinates in ($v \leq c$)-LT are the homogeneous coordinates of the velocity $[\ -c,\ c\ ]$. So let us return now to the four-space-time. We have entirely clarify Pauli's note 111 (2.3) "if one interprets dx_1, dx_2, dx_3, dx_4 as **homogeneous** coordinates in a three-dimensional projective space [Lobatchevkian 3-kinematics space], then the invariance of the equation amounts to introducing a Cayley system of measurement [Einstein's identical units of measure], base on a real conic section [Minkowski's isotropic lines]". So in Einstein's kinematics, clearly separated from Poincaré's one, we not only *can* make "$v = c$" in LT, but we **absolutely must make** "$v = c$" in $v{-}LT$ in order to obtain the famous Minkowski's metric (entirely based on Minkowski[9]'s isotropic light lines).

4.1.2 Einstein's invariance of the one-way-speed of light and Einstein's photon

So logically we must now show, in the framework of Einstein-Minkowski's **vector** kinematics, that it is not only the scalar substitution $v = c$ that characterizes Einstein's SR but the vector substitution "$\mathbf{v} = \mathbf{c}$"($\mathbf{c}{-}LT$). Let us examine in details Einstein's synchronization. Poincaré and Einstein use the same method of distant clocks synchronization (with light signals[10]). Both authors speak about a convention (an assumption). It

[9] We showed that Minkowski's 1908 "mystische Formel" (in his own words) "$c = i$" corresponds to Einstein's "$v = c$".

[10] There is a metaphysical version of SR without velocity of light, without synchronisation of the clocks (without Poincaré and Einstein) and therefore without physical definition of units of measure (Ignatowski, 1911).

is reasonable to think that Poincaré knew what he said when he insisted in 1911 on the fact that the conventions were not the same [11].

1-(Einstein's "stationary time of a stationary system K")

> But it is not possible without further assumption to compare, in respect to time, an event at A with an event at B. We have so far defined only an "A time" and a "B time". We have not defined a common "time" for A and B, for the latter cannot be defined at all unless to establish by definition that the time required by light to travel from A to B equals the time it requires to travel from B to A. Let a **light ray** start at the "A time t_A" from A towards B, let it at the "B time" t_B be reflected at B in the direction of A, and arrive again at A at the A time t'_A. In accordance with definition the two clocks synchronize if:
>
> $$t_B - t_A = t'_A - t_B$$

.2- (Einstein's "stationary time of a stationary system k")

> It is essential to have time defined by means of stationary clocks in stationary system.[7.Einstein A.1905, paragraph 1]

The repetition of the concept stationary is essential because in his §3, Einstein notices about his second system k (ξ, η, ζ, τ):

> To do this [deduce LT] we have to express in equations that τ is nothing else than the set of data of clocks at rest in system k, which have been synchronized according to the rule given in paragraph 1.[7.Einstein A.1905, paragraph 3]

$$\tau_B - \tau_A = \tau'_A - \tau_B$$

[11] Poincaré's writes a short time before his death: "Today some physicists want to adopt a new convention. This is not that they have to do it; they consider that this convention is easier, that's all; and those who have another opinion may legitimately keep the old assumption in order not to disturb their old habits".[30. Poincaré H. 1912)]

Einstein's synchronization (without the length's contraction) of identical clocks *within* his second system k is exactly the same as Einstein's synchronization within his first system because the speed of light is identical. Moreover in Einstein's definition **"forward travel time ≡ backward travel time"** *within* any inertial system (K or k), we have $\overrightarrow{c} = \overleftarrow{c}$, the fundamental invariant is the **one-way-speed of light**. Einstein's identical units are defined with **only one light isotropic** line while Poincaré's units are defined with by the invariance of the two-way-speed of light. The invariance is defined by Einstein with a direction and a sense on this direction:

$$\overrightarrow{c} = \overleftarrow{c}$$

So the velocity of light wave in Poincaré's SR is fundamentally a scalar (a constant that appears in a second order wave equation) whereas in Einstein-Minkowski's SR, the velocity of a **light ray** (Einstein's 1905 preferred concept with this one of light quantum) is a fundamentally a vector. The crux difference between an *electromagnetic wave and a photon* is therefore revealed by the existence of a "fine structure" of SR: The electromagnetic wave (2) supposes a two-ways relativistic invariance of light speed while the photon supposes the one-way relativistic invariance of light speed.

4.1.3 Einstein's "difference of squares" versus Poincaré's "square of differences"

Let us now demonstrate that the two metrics are *algebraically incompatible.*

Einstein writes in 1922:

> Let be a light ray that propagates through the empty space from one point to another point of K. If x represent the distance between the two points, the propagation of the light must satisfy this equation
>
> $$x = ct$$
>
> If one put this equation at the square one can also write
>
> $$x^2 = c^2t^2 \qquad \text{or} \qquad x^2 - c^2t^2 = 0$$

For Einstein’s pseudospherical ”waves” (actually vector rays in any direction), we have in three space dimensions (22):

$$r^2 - c^2t^2 = 0 \tag{22}$$

(the same for k)[10. Einstein A., first conference]

Einstein put at the square the two members of the equations while Poincaré put also the equation at the square *but after a change of member*. And he obtains (21):

$$(x \pm ct)^2 = x^2 + c^2t^2 \pm 2cxt = 0$$

For Poincaré’s genuine spherical waves (with the spherical classical symmetry, $1/r$), we have:

$$(r \pm ct)^2 = r^2 + c^2t^2 \pm 2crt = 0 \tag{23}$$

The ”fine structure” of SR is now geometrically demonstrated in last resort on this elementary algebraic calculation : **the difference of the squares[12] is not the same that the square of the difference**. This algebraic discrepancy is situated in the *core* of the standard mixture SR. ”Give me a fulcrum and I lift the world up”, said Archimedes... Here it is only question of the splitting of the *standard pseudoEuclidean mixture* into its two components: ”Poincaré’s affine-unimodular metric” and ”Minkowski’s projective-hyperbolic metric”.

4.2 Einstein-Minkowski’s invariant proper time (eigenzeit)

Poincaré’s relativistic *scalar* kinematics is based on Poincaré’s convention of synchronization which is defined with longitudinal contraction. Einstein’s exact synchronization is obtained without the contraction. The contraction is deduced from the simultaneity by the ”$(v \neq c) - LT$ (paragraph 2). That suggests that the definition of the (unit of) time is more fundamental than the definition of space in **vector** Einstein-Minkowski’s kinematics.

[12]The difference of squares is profoundly connected with non-Euclidean structure of space-time and with the quantum structure of the light.

4.2.1 Einstein's identical clocks and Minkowski's invariance of the element of proper time

Let us return to Einstein's previous quotation in which we have replace "the rods" by "the clocks".

> [In accordance with the principle of relativity " the duration T of the clock in the moving system" - must be identical to " the duration T of the stationary clock." (The duration to be discovered [by v-LT] we will call " the duration of the (moving) clock in the stationary system". This we shall determine on the basis of our two principles, and we shall find that it differs from T.[7.Einstein A.1905, " paragraph 2"]

In others words this extraordinary Einstein's statement of the relativity principle means that it is not only the Maxwell equations that are invariant (by $\mathbf{v}-LT$) but also the units of length and time (by $\mathbf{c}-LT$). This is an important result because Einstein's invariant units or Einstein's identical units are exactly the same concept. Einstein's principle of identity is particularly clear in Einstein's concept of identical atomic clocks (clocks with identical rhythm τ or frequency ν): Einstein writes in his second fundamental paper on SR in 1907:

> Since the oscillatory process that corresponds to a spectral line is to be considered as a intra-atomic process, whose frequency n is determined by the **ion alone**, we can consider such an ion as a clock of definite frequency υ_0; this frequency is given, for example, by the light emitted by **identically** constituted ions at rest with respect to the observer.[8. Einstein A.1907]

The concept of identical atom is a quantum concept [13] But why Einstein's definition of **invariant unit of time** τ become more fundamental than the unit of space in Einstein-Minkowski's SR? Minkowski

[13] Einstein's identical clocks suggest that Einstein's clocks are quantum clocks because the concept of identical is a quantum concept.Weiskkopf [43.Weisskopf V.] writes:"the main idea of quantum theory, I said, there is idea of identity. Understanding the idea of identity, there is the understanding the concept of quantum state established by Bohr in the first period of his scientific activity.

introduces the **Element of "Eigenzeit"** (proper time) $d\tau$ as a fundamental *invariant* because it is directly connected with the fundamental invariant ds [16. Minkowski H.1908]:

$$ds = cd\tau \tag{24}$$

directly obtained from the differential of Einstein's quadratic form $s = 0$ (22)

$$ds^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2) = c^2d\tau^2 - (d\xi^2 + d\eta^2 + d\zeta^2)$$

with $d\xi^2 + d\eta^2 + d\zeta^2 = 0$: the element of proper time, $d\tau = \gamma^{-1}dt$, is taken always at the same place in the proper system of the particle. So Einstein defines an *invariant* stationary time in stationary system at the same place (in other words a duration at rest attached to a body) but in his 1905 paper, the three system K, k and k' are always in uniform translation. Minkowski extends (in principle) this definition to any motions. In other words Einstein defines τ and $s = 0$ while Minkowski defines $s \neq 0$, ds and $d\tau$. But what is the main characteristics of the element of Eigenzeit?

4.2.2 Minkowski's element of proper time is a non-exact differential

Sommerfeld writes in 1921:

> As Minkowski once remarked to me, the element of proper time $d\tau$ is not a complete (exact) differential."Thus if we connect two world-points O and P by two different world-lines 1 and 2, then

$$\int_1 d\tau \neq \int_2 d\tau \tag{25}$$

[9. Einstein A.]

The main characteristic of the **Element** of Eigenzeit is that it is a **non-exact differential**. What does it mean "non-exact differential"? That means that the proper time τ measured from K, with $d\tau = \gamma^{-1}dt$, depends on the followed path, or in Minkowski's own word, depends on the result of the integral (24$\Longleftrightarrow$25) on different world-lines (path 1 and path 2 between finite limits t_1 and t_2 of integration). This is directly connected to the famous "paradox" of the twins.

In Poincaré' SR the times t ("true time") and t' ("local time") that are given by the $\varepsilon-$LT are completely relativistic and the exact differentials dt and dt' are related by the four-volume to the differential $dV = dxdydz$. The differential four-volume used by Poincaré is a scalar complete (exact) differential [14].

The question of different paths of integration has no meaning in Poincaré's scalar SR (3.3). Minkowski's non-exact differential (24) *makes* "Poincaré's reduction from four-dimensional space-time (6) to two-dimensional space-time" *impossible.*

4.2.3 The proper time and the proper system

Let us examine now in details the relation between the proper time and the proper system and therefore Goldstein's quotation in the introduction.

According to Goldstein the "stratagem" consists of reducing any accelerated motion into a infinite series of "inertial systems moving uniformly relative to the laboratory system". Nothing prevents Poincaré to make an ε-LT (with his *two rotations* in Euclidean space) at each instant, t or t', in order to find the formulas of transformation of the components of the acceleration **a** or **a'**. These general formulas of relativistic transformation of the acceleration for an electron (on any trajectory) are in Poincaré's paper (paragraph 7).

What is the difference between the two relativistic kinematics (with or without Element of Eigenzeit)? The stratagem at least supposes three systems respectively noted K, k, k', k'' ... by Einstein and K, K', K'', K'''... by Poincaré. The relations between $K-k$, $K-k'$, $K-k''$ in

[14] It is anyway impossible to imagine that the greatest specialist of all the times of the differential equations didn't know that he worked with non-exact differential in his own SR. Nowhere Poincaré introduces the famous paradox of the twins and nowhere he introduces the non-exact differential ds or dt. There is an antinomy with the fourvolume differential.

Einstein's convention are exactly the same as the relations between $K-K'$, $K-K''$, $K-K'''$ in Poincaré's convention. In the two cases K represents the laboratory system.

But that's not true for the connection between respectively $k-k'-k''-k'''$...(Einstein-Minkowski) and $K'-K''-K'''$(Poincaré).

The proper time $d\tau$ is an invariant for the series $k-k'-k''-k'''$. In other words the successive uniform states of motion are related for each element of time $d\tau$ by a $dv-LT$, because precisely $d\tau$ is an invariant (4.2.1). The $dv-LT$ are defined without the $\gamma - factors$: the $\gamma - factors$ that appear only in (the integration of) the successive relations $K-k$, $K-k'$, $K-k''$...in virtue of the definition of the non-exact element of proper time. In Poincaré's relativistic point of view, the successive "capital K", $K'-K''-K'''$...are *not in uniform translation* with respect to each other. We discover now the reason of keeping the original notations respectively, t', t", t"'..., dt', dt", dt"'... in Poincaré's text and τ and $d\tau$ in Einstein-Minkowski's text.

With Einstein-Minkowski's definition of proper time, the series of "little k", $k-k'-k"-k'''$, defines the **proper system** of the particle. In other words, in order to pass from k to k', k' to k", k" to k"' there is no acceleration. If it was not the case it would be impossible to relate that series with Lorentz transformations (see Goldstein's quotation), $dv-LT$, and the **Element** $d\tau$ would not be an invariant. The composition of two vector $\mathbf{v}-LT$ *and* $\mathbf{dv}-LT$ would be impossible. That extraordinary "stratagem" in Einstein-Minkowski's SR, in order to reduce the acceleration, is not a Minkowski's invention but an Einstein's invention. Einstein's 1905 launching of the boost is indeed defined by Einstein without acceleration: the acceleration must be infinitely "slow" in such a way that it has no effect on the clocks and the rods that remain identical after the launching of the boost (the second system k). We rediscover Einstein's concept of stationary system (4.1) "It is essential to have time defined by means of stationary clocks in stationary system". We have, on one hand, Poincaré's series with acceleration $K'-K''-K'''$... and, on the other hand, Einstein-Minkowski's eigenseries $k-k'-k"-k'''$, without acceleration (equation 25).

Einstein's theory, clearly separated from Poincaré's theory is a **steady-state** theory. If one introduces an acceleration, we have an effect on the rods and thus a force and thus a work of the force and thus a pressure. We showed that Einstein's thermodynamical **adiabatic hy-**

pothesis is in fact the same concept as Einstein's quantum identical clocks [21. Pierseaux Y., (3)]. Any classical pendulum undergoes, in Poincaré's series K', K'', K''', a mechanical deformation[15].

4.2.4 The insufficient Einstein-Minkowski's definition of the proper system (*eigensystem*)

Minkowski's proper time is defined for any world-line and therefore for any trajectory in 3-space. The proper time is defined by Minkowski in **3-space** and the proper system is only defined by Minkowski in 1-space. Why? Here is Thomas' crucial point. The succession of proper systems $k(v'), k'(v"), k''(v'')$...are related by $d\mathbf{v} - LT$ if and only if the velocity $\mathbf{v}$ points always in the same direction, and therefore in one-dimension. If the orientation (not the intensity) of the velocity changes, we rediscover the fact that the composition of two non-parallel successive vector boost is not a vector boost because there appears from K a space rotation. The logical necessity of this extension is neither in Minkowski's texts nor in Einstein's texts. And it is not astonishing that Einstein was very astonished by the relativistic discovery of Thomas.

In other words, if the proper time consists of transforming the classical accelerated trajectory into a succession of stationary state in the proper system (i.e, "transforming the classical accelerated trajectory of a material point into **a succession of events**" [21. Pierseaux Y.]), we

[15]Synge writes: "The dependance of the integral on the path of integration in space-time is sometimes called the clock paradox (i.e. Moller). But it presents no paradoxe here. Indeed, the fact that ds is not an exact differential constitutes the essentiel difference between relativity and Newtonian physics."What is the difference between Newton's invariant (the absolute classical time) and Einstein-Minkowski's invariant (the proper time)? If we consider that the proper time (at the same place and at rest in the proper system) of our wrist is given by our watch at rest relative to our wrist, it is a mistake. The time is given by the place of the moving little hand of the watch. This is therefore not a proper time because the little hand is moving relative to the watch or the wrist. If the clocks are considered physically as instruments of measure of time, all the classical clocks are the same as the little hand of the watch. The time is defined by a motion and if we want a time "at rest and always at the same place", we must give up all the clocks that give the time with a displacement in space and therefore all the classical harmonic oscillators. The Bohr atom or the quantum oscillator gives proper time (frequency) without change in space but with change of energy. The **"eigenvalues"** of the observable **"eigenzeit"** must be given by the eigenfrequency associated with the changes of eigenstate of energy (quantum harmonic oscillator) and not by a velocity associated to the changes of state in space (classical harmonic oscillator).

can only transform, with Minkowski's definition, an accelerated motion along a right line into a succession of parallel $d\mathbf{v} - LT$. In addition to this first problem if one adopts Minkowski definition of the proper system "just by declaring that its origin is moving with the accelerated particle"(see 4.4), the series $k - k' - k" - k'''$... related to each other by $d\mathbf{v} - LT$ is no longer a proper system because obviously the **orientation** of the system k changes at each element of proper time.

In fact Einstein-Minkowski's definition of the proper system is only valid for uniform accelerated translations. It becomes insufficient when the acceleration correspond to a change of *orientation* and not to a change of intensity. In other words, if the acceleration is reduced in the scalar sense (second derivative with respect to the time), what happens about the acceleration in the vector (orientation) sense?

Here is the critics of Thomas: *"How should we orient the x, y, z axes?"* (4.4). The gap in Einstein-Minkowski's definition is not situated in Minkowski's definition of proper time but in Minkowski's definition of *proper system.* The proper system is defined in 3-space. If the definition of the proper time is fundamental in Einstein's SR, we would have to be able to find the relationship between the proper time and the proper (system's) *orientation* in 3-space.

4.3 Einstein's "slowly accelerated" motion of a point electron that doesn't emit radiation (eigenstate)

So before examining the decisive Thomas' contribution to vector SR, it is necessary to specify the role and the representation of the electron in Einstein's SR. The last paragraph of 1905 Einstein's paper, entitled "Dynamics of the *slowly accelerated* electron" leads to the deduction of relativistic kinetic energy and also to the fundamental equation of the relativistic dynamic (Planck, 1906).

> Let there be in motion in an electromagnetic field a **pointlike (Punkförmige)** particle with the charge e, called an electron, for the law of motion of which we assume as follows : If the electron is at rest at a given epoch, the motion of the electron ensues in the next **element of time ("Zeit-teilchen")** according to the equations:
>
> $$ma_x = eE_x \qquad ma_y = eE_y \qquad ma_z = eE_z \qquad (26)$$

where x, y, z denote the coordinates of the electron, and m the mass of the electron, as long as its motion is slow. Now, secondly, let the velocity of the electron at a given epoch be v. We seek the law of motion of the electron in the immediately ensuing element of time. It is then clear that at the given movement the electron is at rest relatively to a system of co-ordinates which is in parallel motion with velocity v along the axis X of K. From the above assumption, in combination with the principle of relativity, it is clear that in the immediately **ensuing element of time** the electron, viewed from the system k, moves in accordance with:

$$ma'_x = eE'_x \qquad ma'_y = eE'_y \qquad ma'_z = eE'_z \qquad (26')$$

in which the symbols a'_x, a'_y, a'_z, E'_x, E'_y, E'_z refer to the system k.[7.Einstein A.1905, paragraph 10]

The similarity with Einstein's launching of the boost (paragraph 3) is very clear (4.2.3). According to Einstein, the point electron moves without acceleration from element of time to element of time (Einstein's Zeitteilchen is the immediate forerunner of Minkowski's Element of Eigenzeit) from one proper system to the other. We have the series $k - k' - k" - k'''$. Einstein's identical electron within each identical (26, 26') stationary system means that not only the charge e is an invariant but also the *proper mass* m of the electron is an invariant. Einstein's parallel motion of a point electron in an electric field is fundamental because it is the only rigorous method to deduce the relativistic dynamics from relativistic kinematics of the material point in the framework of Einstein's SR. Einstein tries to find the dynamical equation of the electron by a succession of **v**-LT in the *same direction*. He doesn't succeed [16] but he deduces the relativistic equation for uniform slowly accelerated movement. Indeed, with the help of the transformation of the electromagnetic field Einstein obtains:

[16]One year later Planck finds from this equations the fundamental relativistic equation (one year after Poincaré) "by making a simple rotation of the system of coordinates". Einstein adopts in 1907 Planck's deduction. The vector velocity **v** is no longer aligned on Ox. The orientation of the axis of the proper system of the electron are aligned on the velocity of the electron. So Planck finds the classical relativistic dynamical equation in the improper system. But he gives up the definition of proper system.

$$a_x = \frac{e}{m}\frac{1}{\gamma^3}E_x \qquad a_y = \frac{e}{m}\frac{1}{\gamma}(E_y - \frac{v}{c}H_z) \qquad a_z = \frac{e}{m}\frac{1}{\gamma}(E_z + \frac{v}{c}H_y)$$

And therefore:

$$m\gamma^3 a_x = F_x \qquad m\gamma a_y = F_y \qquad m\gamma a_z = F_z$$

What is the status of the acceleration in Einstein’s parallel motion from one proper system to another proper system? On one side there is no acceleration (the series $k - k' - k" - k'''$): the uniform accelerated movement is transformed in a series of $dv - LT$. On the other side there is an acceleration (a and a’, 26 and 26’) of the electron at each ”Zeitteilchen” relative to the laboratory system (Einstein’s series $K - k$, $K - k'$, $K - k"$, is the same as Poincaré’s series, see 4.2.3). In his adiabatic SR (or better: ”adiabatic and isentropic” SR [21. Pierseaux Y.]), Einstein is not very interested by the calculation of the second derivatives, he prefers to integrate on the velocity:

> We will now determine the kinetic energy of the electron. If an electron moves from rest at the origin of coordinates of the system K along the axis of K under the action of an electrostatic force F_x, it is clear that the energy withdrawn from the electrostatic field has the value $\int eE_x dx$. **As the electron is to be slowly accelerated, and consequently may not give off any energy in the form of radiation**, the energy withdrawn from the electrostatic field must be put down as equal to the energy of motion of the electron. Bearing in mind that during the whole process of motion which we are considering the first of the above equations applies, we therefore obtain[7.Einstein A.1905, paragraph 10]

$$W = m\int_0^v \gamma^3 v dv = mc^2(\gamma - 1)$$

So Einstein defines the *proper energy* and the improper energy. But the fundamental point is here that Einstein’s electron **never accelerates** relative to its proper system and therefore **it doesn’t emit any**

radiation. Einstein's electron is, in Einstein's own words, in a stationary state ("eigenstates", a succession of stationary systems).This is a very important fact for the classical electromagnetic theory of emission. But Einstein only considers the proper systems $k - k' - k" - k'''$ always in the same direction. One more time, Einstein defined vector **v**-LT but his calculations are limited to the case of parallel motion. Exactly like Minkowski, Einstein's definition of proper system is limited to the rectilinear uniform accelerated system ($dv - LT$); the proper system is not defined if the system of the electron changes of direction ($\mathbf{dv} - LT$). We have on one hand Poincaré's series with acceleration $K' - K'' - K'''$... and on the other hand Einstein-Minkowski's eigenseries $k - k' - k" - k'''$, without acceleration. That's not all. Einstein's electron that doesn't emit radiation is a **pointlike (Punktförmige) electron** "Let there be in motion in an electromagnetic field an pointlike particle with the charge e, called an electron..."

4.3.1 Poincaré's classical electron (finite volume)

Poincaré discusses all the models of the electron in his 1905 paper except of course this one of Einstein . That was of course impossible for historical reasons because the two theories of electron ("Dynamique de l'électron" and "Electrodynamik bewegter Körper") are quasi-simultaneous 1905 events. But the interesting point for mathematical physics is that Einstein's representation of the electron is not a classical representation but it is a quantum representation.

So "the fine structure" of SR is visible not only for the light (Poincaré's waves and Einstein's photons) but also about the electron. According to Poincaré, the only relativistic (compatible with the principle of relativity $l = 1$) model for the finite electron is the model of the longitudinally deformable electron based on the Lorentz hypothesis. For Poincaré's relativistic deformable electron, we have a relation that has the same form as the four-volume (V is the volume of the electron and e its density of charge:

$$k\rho k^{-1}V = k\rho' k^{-1}V'$$

So we have the relativistic conservation of the charge of the finite electron:

$$e' = e$$

4.3.2 Einstein’s ”quantum” point electron (Minkowski’s ”substantial” point electron)

In his second fundamental 1907 work on SR [8. Einstein A.1907], Einstein defined ”the electron” as ”a material point” (and reciprocally). For the ”Punktförmige” Einstein’s electron, we have:

$$V = V' = 0 \tag{27}$$

So we have an identical ”quantum” of charge attached[17]

$$e \equiv e \tag{28}$$

We put of course quantum between inverted commas because, in Dirac’s own words as long as we don’t have a rigorous theory of magnetic monopole we no longer have a theory of quantization of the charge. In standard presentation of SR (the mixed state), the essential role of the electron in the two papers (”Elektrodynamik der bewegter Körper”[7.Einstein A.1905] and ”Dynamics of the electron”[26. Poincaré H. 1905] is very often completely forgotten. One speaks about material point without connection with this extraordinary intuition (thermodynamic origin, see [20. Pierseaux Y.]) of Einstein about the elementary (point) character of the electron. As we shall see Einstein’s ”Punktförmige electron without structure” (quantum electron) is particularly important in the discussion of the Thomas rotation and the spin.One can think that the point electron with a quantum of charge is not necessary in Einstein’s SR but it would be a serious mistake. If we introduce a finite electron, we must consider a distribution of charge and therefore a pressure that balances the electrostatic force and we find again Poincaré’s SR. Let us note that Minkowski [16. Minkowski H.1908] defined in 1908 two kinds of electron: On one hand, Einstein’s electron or the material point (in Minkowski’s language, the ”substantial” point) and, on the other hand, the purely phenomenological ”Lorentzian electron” (sic). Minkowski never quotes Poincaré in 1908 and he tries to show that his kinematics is completely compatible with Lorentz’s dynamic (with two kinds of electron !). We find again the mixture...

[17] Einstein often repeats that e is a stranger in electromagnetic theory.

4.4 Thomas' complete vector definition of the proper system

Einstein's and Minkowki's definition of the proper system are limited to accelerated systems in the same direction ($dv - LT$). There is therefore an acceleration by change of orientation ($\mathbf{dv} - LT$). The main difficulty in Thomas' 1927 paper is not mathematical but *logical* [36. Thomas L.H. (1927)]. Tomonaga's analysis is particularly clear:

> If the electron is moving with constant velocity, then such a coordinate system can easily be obtained from the laboratory coordinate system by a LT. However if electron has acceleration, there is a complication. We can indeed consider coordinate system in which the electron is motionless. Specifically, this is a coordinate system which has the electron at the origin and is moving together with the electron. However we cannot uniquely determine the coordinate system **just by declaring that its origin is moving with the electron**. **How should we orient the x, y, z axes?** In addition to the condition that the origin is moving with the electron, we must add the condition that the x, y, z axes are always moving in translation, i.e. not rotating. This **"parallel motion"** is obvious when the electron motion is maintaining a constant velocity, but when the electron has acceleration, it is not that obvious. This is the point that Thomas realized. Therefore Thomas first discussed what it meant by the **parallel translation** of the coordinate axes. He concluded that the parallel translation of the axes means that the axes at each instant are parallel to the axes at an infinitesimally small time before that instant. The coordinate system whose origin is moving with the electron, in this sense translating parallel to itself, may be called **the proper coordinate system of the electron**. Thomas derived that this coordinate system, seen from the laboratory system, is not translating parallel to itself but is accompanied by rotation if the electron has acceleration. [37. Tomonaga S.]

How should we orient the axes? How a point electron whose velocity changes at each time of direction could be at rest in a proper inertial system? The only solution is that **the electron must have a device that is pointing permanently in the same direction in**

such a way that the proper system never changes of direction (the precession occurs naturally in the improper system)[18]. All the text of Thomas' relativistic deduction [36. Thomas L.H. (1927)] consists of defining the proper system with the *parallel transport.* The current idea according to which the only ingredient of the Thomas precession is the non-commutativity of two non-parallel **v**-LT is not sufficient (that is already in Sommerfeld's paper in 1909[34.Sommerfeld A.]). Thomas' definition precisely transforms the non-oriented systems into parallel proper systems. *There are two ingredients in Thomas's completion of Einstein-Minkowski's SR: the parallel transported proper systems and the composition of non-parallel vector* $\mathbf{v}{-}LT$ *and* $\mathbf{dv}{-}LT$

Now what is the nucleus of Thomas' deduction? Thomas absolutely needs three systems K, k, k' in order to extend group Einstein's demonstration, limited to parallel systems (2.3). Without affecting the general character of our considerations, we can consider a point in uniform rotation. In this way we extract the essence of Thomas reasoning that is the problem of change of **orientation of velocity** ($a_n = \frac{v^2}{R}$) and not change of intensity ($a_t = 0$) of the velocity. In order to compose two boosts $\mathbf{v} - LT$ *and* $\mathbf{dv} - LT$, Thomas must define $\mathbf{dv} - LT$ as a boost and the only way is the parallel transport: "The parallel translation of the axes means that the axes of k and k' at each [element of time] are parallel to the axes at an infinitesimally small [element of] time before that instant". Let us consider the figure 6:

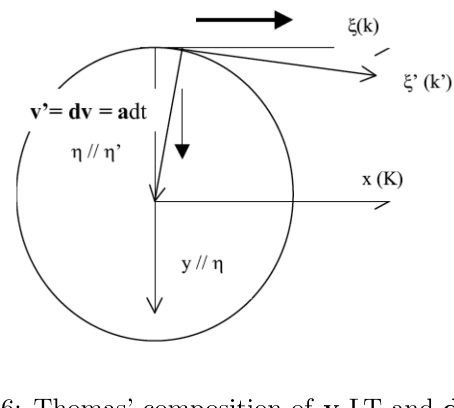

Figure 6: Thomas' composition of **v**-LT and **dv**-LT

[18]If one alignes simply the direction of the system on this one of the velocity we obtain the classical equation of the relativistic dynamic.

At $t = 0$ the proper system k is moving in the x // ξ direction with velocity $\mathbf{v}$. At the next element of time dt the proper system k' of the electron is moving perpendicular to ξ' in the $\eta' // \eta$ direction with velocity $\mathbf{v}' = d\mathbf{v} = \mathbf{a}dt$. The instantaneous direction of the velocity v' is orthogonal to v ($\mathbf{v} \perp \mathbf{a}$). There is no tangent component of the acceleration (no second derivative with respect to the time). So k is parallel moving relative to K and k' is parallel moving relative to k We compose the two successive orthogonal boosts $\mathbf{v} - LT$ and $\mathbf{dv} - LT$ into the boost $(\mathbf{v}' = \mathbf{v} + d\mathbf{v}) - LT$.

first boost $\mathbf{v} - LT$

$$\begin{aligned} \xi &= \gamma_v(x - vt) \\ \tau &= \gamma_v(t - \frac{v}{c^2}x) \\ \eta &= y \end{aligned} \tag{29}$$

second (infinitesimal) boost $dv - LT$

$$\begin{aligned} \xi' &= \xi \\ \tau' &= \gamma_{v'}(\tau - \frac{v}{c^2}\xi) \\ \eta' &= \gamma_{v'}(\tau - \frac{v}{c^2}\xi) \end{aligned} \tag{30}$$

The following figure 7 represents the situation after the time dt [1. Aharoni]:

The line from the origin O of K to the origin O' of k' makes an angle θ in K and θ' in k'. So k is in parallel movement relative to K and k" is in parallel movement relative to k and however k' is not parallel to K. We have $k \;//\; K$ and $k' \;//\; k \quad \not\Rightarrow \quad k' \;//\; k$. That's impossible in affine and therefore also Euclidean geometry.

Let us note that for Galilean transformation we would have

$$\tan\theta_{Galilée} \quad \tilde{} \quad \theta_{Galilée} = \quad \tan\theta'_{Galilée} \quad \tilde{} \quad \theta'_{Galilée} = \frac{v'}{v}$$

and therefore the transitivity of the parallelism ($k \;//\; K$ and $k' \;//\; k \quad \Rightarrow \quad k' \;//\; k$). This is directly connected with the addition of vector in three dimensions Euclidean space. In Einstein's SR the velocities don't add as vectors in Euclidean space but as vectors in Lobatchevskian

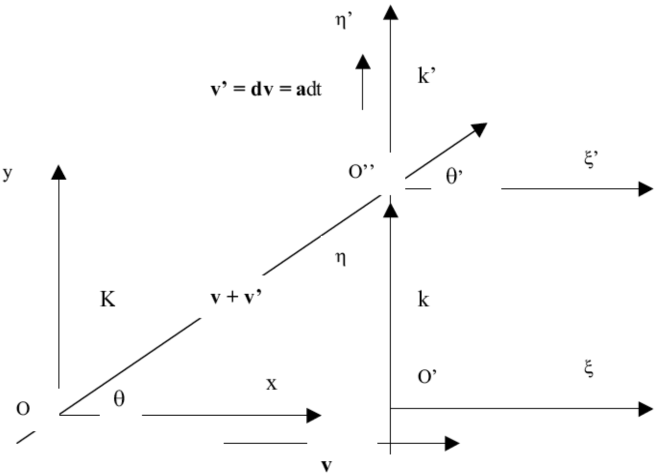

Figure 7: Non-transitivity of parallelism

kinematics space (see hyperbolic triangle). So the relativistic calculation gives with (29 and 30, see Aharoni) by applying the $v - LT$:

$$d\theta = \theta' - \theta = \gamma_{v'} \frac{v'}{v} - \frac{v'}{\gamma_v v} \tag{31}$$

We obtain θ (in K) $\neq \theta'$(in k') and $d\theta$ represents therefore the (Lobatchevskian) *deviation* from parallelism.

We see also immediately that if one composes $(\mathbf{v} = \mathbf{c}) \perp \mathbf{v}'$ (a star light ray **c** orthogonal to the direction of the Earth velocity $\mathbf{v}'$ with respect to the Sun), we obtain, with $\gamma_v^{-1} = \gamma_c^{-1} = 0$, and the angle of aberration α ($\tan\alpha = \gamma_{v'} \frac{v'}{c}$). Varicak demonstrated in 1910 that aberration angle is the *Lobatchevskian angle of parallelism* [41. Varicak V.]. We rediscover here that the correct definition of the star aberration, in Einstein-Varicak's SR, is based, on the composition of two **vector** velocities **v'** and **c** (v' is not the relative velocity Earth-star).

Thomas considers, because dv is very small, that $\gamma_{v'} \simeq 1$. We remark here that, in virtue of the definition of the *invariant* proper time $d\tau$ by d**v**-LT, (4.2.3), we must have in (30): $\gamma_{v'} = 1$ (the proper time is an operator that transforms the continuous trajectory (unnumerable infinity) into a numerable infinity of (countable) events (states), see note 21).

$$d\theta = \frac{dv}{v}(1 - \frac{1}{\gamma_v}) \tag{31'}$$

So this the Thomas angle (the space rotation). We don't have yet the Thomas precession, in other words the variation *in the time* of the orientation:

$$\frac{d\theta}{dt} = \frac{dv}{vdt}(1 - \frac{1}{\gamma_v}) = \frac{a}{v}(1 - \frac{1}{\gamma_v})$$

The expression$\frac{d\theta}{dt}$ represents the variation of the orientation of the proper system with respect to K *in the time of K.* But precisely the orientation of the proper system doesn't vary because it is parallel transported. So the question is "in which duration *the deviation of parallelism* (The Thomas angle) must be defined ?" The very subtle Thomas' deduction is based on the deep connection between the non-integrable change of orientation and the non-integrable proper time on a closed path. That deep connection between **proper time** and **proper orientation** is very clear in Aharoni's analysis of the Thomas precession.

> From the foregoing it follows that when a frame of reference P is carried along a curved path parallel to itself, on return to some original position O it is no longer parallel to its original orientation. The **orientation is not integrable** since the deviation from the original orientation will depend on the shape of the orbit and the velocity along it. In this respect the orientation behaves similarly to the time indication of a clock which is taken round a closed path and then compared which was left in the original position. (Aharoni, Special Theory of Relativity)[1. Aharoni]

The Element of eigenorientation $d\theta$ and the Element of eigentime $d\tau$ are invariant ($\gamma_{v'} = 1$) . By integration along different paths, we find different values for θ and τ with respect to K. The Thomas angular velocity or the Thomas precession defines a very fundamental connection between the two non-exact differentials, the proper time and the proper space orientation:

$$d\theta = \omega d\tau \tag{32}$$

The expression $\frac{d\theta}{d\tau}$ represents the deviation of parallelism with respect to K *in the proper time of k (or k').*

So we find the intensity of the Thomas precession:

$$\omega_{\text{Thomas}} = \frac{d\theta}{d\tau} = \gamma_v \frac{d\theta}{dt} = \frac{a}{v}(\gamma_v - 1)$$

The angular velocity is a vector oriented along Oz

$$\boldsymbol{\omega}_{\text{Thomas}} = \frac{v \times a}{v^2}(\gamma_v - 1) \tag{33}$$

In first approximation, we find the well known expression for the Thomas precession for v $\lll c$

$$\boldsymbol{\omega}_{\text{Thomas}} = \frac{v \times a}{2c^2} \tag{34}$$

Minkowski's diagram space-time of world lines has no general meaning without Thomas definition of proper system (31'). We only can define rectilinear world lines and the hyperbolas world line that represent the uniformly accelerated translation (and also the definition of the identical units of measure). In order to give a general sense to Minkowski's curve trajectory in 3-space, we must attach to the proper time a precession (34) of the orientation of the parallel transported proper system.

4.5 Curved space in Einstein-Minkowski-Thomas' SR

Thomas' extension of the definition of the proper system for non-parallel boost supposes the parallel transport. What is the geometrical meaning of Thomas' parallel transport?

> The notion of parallel transport offers a good approach to understanding the characteristic properties of curved spaces. Given two points in a space and a vector at one of them, one may construct that vector at the other point that is parallel to the first. To this end, one connects the two points by a geodesic and then transports the original vector along the geodesic, seeing to it that the angle between the straight line and the vector remains constant in the course of the transfer. **In a flat space the end result depends not on the path** of transfer but solely of the original vector. A single direction is parallel at each point of space to that originally given elsewhere. **In a curved space, the result of parallel transfer depends of both the original vector and the path of transport** chosen (Bergman, "The riddle of Gravitation").[3. Bergman]

So Thomas' decisive contribution to Einstein-Minkowski's SR consists of showing the connection between the proper time and the orientation in space. The orientation is not integrable since the deviation from the original orientation will depend on the shape of the orbit and the velocity along it. Without any calculation, we can conclude that the space in Einstein-Minkowski-Thomas SR is a **curved space** (i.e., a non-Euclidean space). The change of the orientation is a result of the curvature of the space in which the electron is moving. We can now specify what kind of non-Euclidean space. Sommerfeld's non-commutativity for the composition of two velocities at right angles, in two possible ways, can be illustrated by the following figure 8 [34.Sommerfeld A.]:

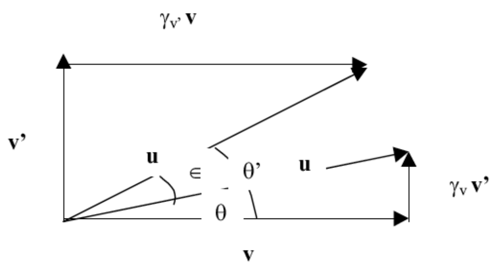

Figure 8: Sommerfeld's non-commutativity

$$tg\theta = \frac{v'}{v\gamma_v} \qquad tg\theta' = \gamma_{v'}\frac{v'}{v} \tag{35}$$

In Sommerfeld's representation of the kinematics space, the triangles are not congruent. If we use Varicak's representation, the only one compatible with Minkowski's definition of isotropic lines, we have congruent triangles (figure 9)[39. Varicak V.]:

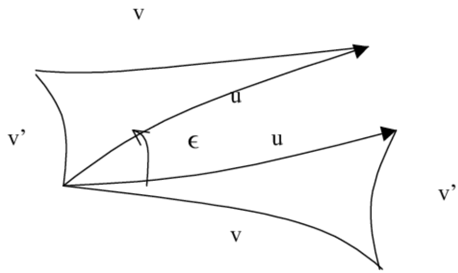

Figure 9: Varicak's hyperbolic representation

The factor γ occurs only in the Euclidean representation of the velocities; in non-Euclidean representation, it is implicit in the geometry [2. Barrett J.F.].

$$\epsilon = \theta' - \theta \tag{36}$$

Varicak demonstrates in 1910 that the Sommerfeld angle, $\epsilon = \pi - (\varphi + \varphi_1 + \varphi_2)$, is the *hyperbolic defect* in hyperbolic triangle (see figure 1). But in Varicak's 1910 paper the hyperbolic defect doesn't have any physical interpretation. The non-commutativity is based on the order of the composition of $\mathbf{v}$ and $\mathbf{v}'$ (first $\mathbf{v}$ and then $\mathbf{v}'$ or inversely). It we now realize that **hyperbolic triangle** represents **vector addition** in kinematic 3-space (figure 10), when we compose $\mathbf{v}$ and $\mathbf{v}'$ into $\mathbf{v} + \mathbf{v}'$, we have a space rotation and therefore *the hyperbolic defect is the Thomas angle.*

Anyway, by the equations (31) and (35) we deduce immediately the relationship between the hyperbolic defect ϵ and the Thomas angle $d\theta$ (see 30)? We have obviously (33):

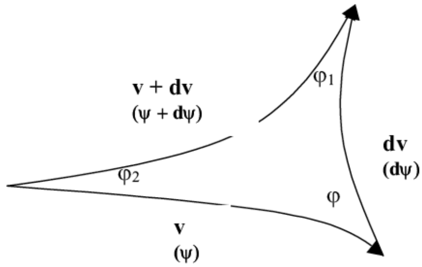

Figure 10: Thomas hyperbolic vector composition

$$d\theta = \epsilon \tag{37}$$

The Thomas angle ***is*** *the hyperbolic defect (with the proper system,* $\gamma_{v'} = 1$, we have the Thomas precession ω)

So we know that the *hyperbolic defect* ϵ is proportional to the *area of the triangle (in fondamental area units of the Lobatchevskian "sphere" whose negative radius of curvature is* $c = 1$ and area is 4π*).* A succession of triangles forms a polygonal closed path that gives by integration the (non-restrictive) circular path considered by Thomas. After one revolution on the closed path, when we return to the starting point, the orientation of the proper system (intrinsically parallel) will have **revolved of an angle proportional to the area of the "circle"** (see conclusion).

Borel (who recognized Varicak's priority) writes: "In classic kinematics, kinematics space is an Euclidean space. The principle of relativity corresponds to the hypothesis that kinematics space is a space **with negative constant curvature**, the space of Lobatchevski and Bolyai."[4. Borel E.] In classical mechanics, the 3-kinematics space and the 3-space itself are both Euclidean. With Varicak (1910), we see that kinematics 3-space is Lobatcheskian (curvature radius $c = 1$), with Thomas (1926), i.e. the integration of $d\theta$ along a closed path (parallel transport) *in space*, we see that the *3-space itself is Lobatchevskian.* In the standard mixture SR we have a Euclidean 3-space and a Lobatchevskian kinematic 3-space. That's obviously completely incoherent. If we split off the mixture into its two components, the space in Einstein-Thomas SR is *really* Lobachteskian exactly on the same way that the space in Einstein's GR is *really* Riemanian. **Let us summary**

the fine structure between Poincaré’s scalar SR and Einstein-Thomas’ vector SR:

Poincaré’s subgroup $\Rightarrow$ metric based on the unimodular invariant four-volume $\Rightarrow$ exact differential, $dt \ and \ dt'$, connected by the scalar invariant $dtdV = dtdV$.

Minkowski’s metric based on the nonEuclidean invariant the four-intervall $\Rightarrow$ non-exact differential of element of proper time, $ds = cd\tau$, connected with Thomas’orientation dθ of proper system.

But we must now answer the haunting question ”in order to apply the parallel transport to the material point of Einstein’s SR, ”**the electron must have a device that is pointing permanently in the same direction in such a way that the proper system never changes of direction”**.

In other word, Einstein-Thomas’ SR is not yet a complete theory because it is impossible to apply on the material point (the electron), the parallel transport which is absolutely necessary for a rigorous definition of the proper system. Moreover the separation of the mixture (Einstein’s kinematics and Poincaré’s groups) means that Einstein’s vector kinematics lost Poincaré’s guarantee of the structure of group (with two space rotations). We must find why and how only one space rotation combined with the parallel transport (the device...) can lead to a structure of group (conclusion).

5 Thomas’ principle of correspondence between the proper magnetic moment of the classical electron and the spin of the quantum electron

Let us now sum up Thomas’ contribution to atomic physics [35. Thomas L.H. 1926]. What is the situation in 1925? The purely theoretical quantum description of the spin is established by Pauli is in harmony with the experimental measurements (a difference of energy for alkali doublets) of Uelhenbeck and Goudsmit. The image of self-rotation classical electron was rejected by Pauli ”the classically indescribable two-valuedness”. The non-relativistic classical theory was unable to explain the experimental results: there is no magnetic field and thus no interaction with the supposed proper magnetic moment of the electron and therefore no fine structure in the spectrum.

5.1 Kronig’s self rotating classical electron and the magnetic field

So the first intervention of SR is this one of Kronig:

> How do we use relativity here? Let me explain. The electrons in an atom are in an electric field and are therefore influenced by that electric field, but the self-rotating magnetic moment of the electron will not be directly affected by the electric field. However, if an electron is moving, then a magnetic field which did not exist in the laboratory frame appears for the moving electron ["the usual expression is seen by the electron"] through a Lorentz transformation. According to Einstein [as well Poincaré], this field is:[37. Tomonaga S.]

$$\mathbf{H} = \frac{1}{c}\frac{\mathbf{E}\times\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}$$

The magnetic field is a fundamental relativistic effect for little velocities $v << c$. The electron sees a magnetic field which is perpendicular to the plan of the orbit. We can deduce the Larmor frequency with the gyromagnetic experimental ratio g_0

$$\boldsymbol{\omega}_{Larmor} = g_0 \frac{e}{2mc}\mathbf{H} \tag{38}$$

And so thanks to this first intervention of SR, we have a fine structure for alkalis doublets:

$$\Delta W = \hbar\nu_{Larmor} = g_0 \frac{e\hbar}{2mc} H \tag{39}$$

The experimental measure of the gyromagnetic ratio for the electron is 2. Kronig put g_0 equals 2 (with an angular momentum of self-rotation of $\frac{1}{2}$h)you find a difference twice as large as the one which is actually observed for the anomalous Zeeman effect. So with the first intervention of SR (the mixed state), we have a structure fine but not the good (experimental) value for the energy of the fine structure. So Kronig's calculation was completely rejected by Pauli. We note that Kronig doesn't define the proper system. That's precisely Thomas's critic.

5.2 Thomas' factor and the proper magnetic moment of the electron

The second intervention of SR [35. Thomas L.H. 1926] consists of defining correctly the proper system (Thomas):

> Speaking in terms of classical theory what Kronig , Uehlenbeck and Goudsmit has done was to think that the angular velocity of the precession is itself the angular velocity in the laboratory system. However Thomas argued as follows. This angular velocity is indeed the angular velocity as seen from the proper coordinate system of the electron, but if you view the motion from the laboratory system, the proper coordinate system is itself rotating with an angular velocity (the Thomas precession)[37. Tomonaga S.].

Thomas obviously had in his mind the Bohr model where the electron is in uniform rotation in an electric field (see 4.3, Einstein’s last paragraph 10) and he introduces (with 38 and 34) a corrected angular velocity in the laboratory system:

$$\boldsymbol{\omega}_{Lab} = \boldsymbol{\omega}_{Larmor} + \boldsymbol{\omega}_{\text{Thomas}} = g_0 \frac{e}{2mc}\mathbf{H} + \frac{1}{2c^2}\mathbf{a} \times \mathbf{v} \tag{40}$$

So we recall that the acceleration is the acceleration of the electron with respect to the laboratory system (the nucleus system), the electron has a device that points always in the same orientation (the electron is in its successive proper systems, k, k', k'', k'''..., in parallel non-accelerated motion).

$$\mathbf{a} = -\frac{e}{m}\mathbf{E} \tag{41}$$

That’s exactly Einstein’s formula for uniform parallel translation (26, 26’) but that is now valid for uniform circular moving. Thomas obtains:

$$\boldsymbol{\omega}_{Lab} = (g_0 - 1)\frac{e}{mc}\mathbf{H} \tag{42}$$

If we substitute $g_0 = 2$ we obtain **one half** of the previous result (the famous Thomas factor $\frac{1}{2}$) and the discrepancy (39) with experiment disappears. Tomonaga writes:

> When this was clarified, even Pauli (after a few weeks) decided not to categorically oppose the idea of self-rotating electron, and in spite of its remaining problem.[37. Tomonaga S.]

We underline that the Thomas relativistic effect (factor $\frac{1}{2}$) is obtained with the formula for small ($\frac{c}{137}$) velocities $v << c$.

In 1925 Pauli had given a quantum ("classically indescribable") description of the spin. But the spin (the proper kinetic moment of a charged particle) must be connected to the magnetic moment of the electron. Nobody knows a quantum theory of magnetism (see conclusion). Thomas himself thinks that his relativistic calculation is purely classical. He uses indeed, on one side, Einstein's composition of vector **v**-LT and Minkowski's proper time and, on the other side, Poincaré's representation of the electron. It is always very interesting to read the original texts. Let us examine therefore the 1926 Thomas paper:

> The precession of the spin axis so calculated is its precession in a system of coordinates (2) in which **the center of the electron** is momentarily at rest. [35. Thomas L.H. 1926]

With the extended (with respect to Minkowski) definition of the proper system, Thomas defines a correspondence (42) exactly in the Bohr meaning ($\Longleftrightarrow$) that can be can be sum up in the following way:

spin of quantum electron $\Longleftrightarrow$ magnetic moment of classical electron

But what is the remaining problem according to Pauli? The remaining problem is that the magnetic moment has a classical definition and that the spin has a quantum definition.

quantum electron $\neq$ **classical electron**

The situation is very curious because, on one hand, we have the quantum electron with spin (without internal structure or finite volume) and, on the other hand, we have a classical electron with a proper magnetic moment. The two calculations coincide experimentally but this **quantum-classic coincidence** is based on a **correspondence principle** in Bohr's meaning. It is now time to conclude.

6 Conclusion: The spin of the electron in Einstein-Thomas' Special Relativity

On one side of Thomas' principle of correspondence we have a classical electron and on the other side we have a quantum electron. That's not very satisfying and the Dirac equation doesn't solve this problem (the good magnetic moment appears as a classical limit) .

The "fine structure" of SR solves the problem: all you need is to replace, in Thomas' deduction, Poincaré's classical electron by Einstein's quantum electron (4.3.1 and 4.3.2). The discrepancy completely disappears. This operation is not only sufficient but it is *necessary*. The only solution to complete Einstein-Thomas' SR, in order to have a correct definition of the proper system, is that "**the electron must have a device that is pointing permanently in the same direction in such a way that the proper system never changes of direction".**

That device can only be the proper magnetic moment.
In Einstein-Thomas's SR Einstein's quantum electron must have a proper magnetic moment and therefore a spin.

It is another proper magnitude to a long list: proper energy (Einstein), proper mass (Planck), proper time (Minkowski), proper length (Born) and finally proper magnetic moment (Thomas). If the first intervention of SR corresponds to the introduction of magnetic field on the electron as a relativistic effect for $v << c$, the second intervention (Thomas) corresponds also to the necessary introduction of a proper magnetic moment of the electron as a relativistic effect for $v << c$. So if we have the magnetic moment of the electron in Einstein-Thomas' SR, it seems impossible, at the first sight, to deduce the famous spin $\frac{1}{2}$ from Einstein-Thomas' SR clearly separated from Poincaré's SR . Let us remind however that the fundamental characteristic of a vector state associated to the spin $\frac{1}{2}$ is that it is necessary to operate two revolutions 4π to find again the initial state. In the Lobatchevskian 3-space of Einstein-Thomas SR, the change of orientation of the magnetic moment (of the system) is proportional to the enclosed polygonal area (4-5) along which the parallel transport occurs[4. Borel E.]. Here the polygonal area is, at the limit, a circle whose radius is $c = 1$ (4.5). After one revolution 2π, the area equals to π and therefore the spin undergoes a change of direction of π (minus one for the vector state).

$$\theta_{one \text{ revolution } 2\pi} = \pi \tag{43}$$

After two revolutions 4π, the enclosed area is 2π and therefore we **find again the same initial** vector state for the spin.

$$\theta_{two\ \mathrm{revolutions}\ 4\pi} = 2\pi \tag{44}$$

So remember that Einstein's vector kinematics, with the separation of the mixture, lost the guarantee of Poincaré's structure of group. We obtain (at last!), thanks to the introduction of the spin, **the structure of group** and therefore Einstein-Thomas SR is now complete.

The "fine structure" of SR is clearly established: in Poincaré's SR we find again the same orientation of the system with one revolution of 2π for each rotation. In Einstein-Thomas' SR, we find again the same orientation of the system (given by the spin) for only one rotation but with two revolutions 4π (along the path of integration on Lobatchevkian sphere with solid angle 4π,4.5).

We note that the "fine structure" conducts (but much more directly because the spin is deduced from Einstein-Thomas' group) to the same result as Rindler-Penrose's idea about the similarity of the group structure on the Riemann spheres for the transformation of the spin $\frac{1}{2}$ and for the transformation of the relativistic aberration (here Penrose rediscovers Varicak [31bis. Rindler and Penrose]).

We showed that Thomas rotation is fundamentally connected with the proper system without acceleration, in the tangential sense, i. e. *without the second derivative with respect to the time.* Thomas' rotation is only connected with a purely vector change of orientation recorded in the fixed system K. Dirac's equation that gives the spin of the electron is precisely an equation of the first order with respect to the time (We point out that we showed that Einstein's concept of independent events in SR is directly connected with *Einstein's definition of probability* by inversion of Boltzmann's principle [21. Pierseaux Y.]).

But that's not all. We have another very interesting application of the Lobatchevski's 3-space to the quantum mechanics. Let us return to Einstein's point electron that doesn't emit any radiation. The explanation is very clear: with respect to its proper system the electron never accelerates in Lobatchevski 3-space. The space is itself (constantly) curved.. We have so a very simple geometrical relativistic explanation of a famous riddle of a quantum concept. **We know henceforth why an electron that moves, in Bohr model, around the nucleus doesn't emit radiation**: The uniform circular movement becomes, thanks to the Thomas precession and its associated device (spin), **a geodesic path** in Lobatchevski 3-space of Einstein-Thomas' SR (any proper system follows by definition a geodesic path).

We showed that the main border (classical-quantum) of the present physics passes between the two SR (PIRT 2000, [22. Pierseaux Y.]). So the two components of the standard mixture SR can be called with reference to the representation of light "quantum SR" (**QSR**) and classical or "wavy SR" (**WSR**). The question of the connection of the doublet "WSR and QSR" respectively with QED, QM, GR and so on... is an open question .The proper magnetic moment of the electron corresponds to a spin that is incompatible with Poincaré's (relativistic) classical electron whose radius is $\frac{e^2}{mc^2}$): the classical electron's surface reaches a velocity higher than that of light. Only Einstein's point electron is compatible with the spin of the electron. Nevertheless Poincaré's WSR remains very important for another theoretical equation. It is indeed also possible to show that if **the first order (with respect to time) Dirac's equation** becomes a formula of QSR, **the second order (with respect to time) Klein-Gordon's equation** becomes a formula of WSR (the fundamental meaning of the quadratic form that doesn't define the light wave propagation in Poincaré' SR (3) because the light wave propagation is defined by the second order (with respect to time) wave equation (2). In order to answer a question of Jacques Robert about the acceleration we want to specify that we only focus the attention in this paper on the consequence of Einstein-Thomas' SR about the **eigenstate (eigensystem)** of the electron (without emission of radiation) in the Bohr model. We didn't discuss the changes of state with emission of radiation. In the framework of Einstein-Thomas' SR, the photonic mode of emission for bound electron in atom can only be defined by change of energy while, in the framework of Poincaré's SR, the classical wavy mode of emission for free electrons can only be defined by acceleration (see synchrotron radiation in particles accelerators). That question, that will be discussed in another paper, is connected with the discrepancy, discovered by Louis de Broglie, between the relativistic variation of clock frequency and wave frequency. The "fine structure" shows that it is implicit in Einstein's definition of photon (4-1-2) and we will show that *de Broglie's concept* leads to a quantum definition of time with "only one atom at rest in space" (*cold atom*, Dehmelt NP 1989). Without that identification between "proper atomic frequency" and "proper time" (quantum oscillator, see note 20), we must define time classically, like Poincaré, by the propagation of light waves in space and reciprocally elongated distance by dilated time (or dilated velocity). In other words, it is easy to show, from Poincaré's deduction of relativistic Doppler and aberration formulas (3.2.2), that in front of the atomist *Einstein-Thomas's SR* with the spin of electron, we have a cosmologist P*oincaré-Hubble's SR* with the expansion of space.

So we are therefore also in the interface between the general relativity (GR) and the two extended SR. Minkowski's metric is based on the "difference of squares" (4.1.1) while Poincaré's metric is based on the "square of the difference" (3.4). It is well known that Einstein reintroduces in 1921 a space-time ether exactly in the sense of Poincaré. Which is the valid local SR in Einstein's GR. Poincaré's WSR, Einstein's QSR or the standard mixture? That's very important for the quantization of gravitation. In addition, what is the relation between the magnetic hyperbolic space in Einstein-Thomas' QSR and Riemannian gravitational space in GR? The existence of the doublet "QSR and WSR", directly resulting from the splitting of the standard pseudo-Euclidean mixture SR, is based in last resort on an irreducible opposition between a Euclidean and a non-Euclidean conception. The great geometrician F. Klein regretted that Minkowski, after his famous 1908 conference, "never again referred to a manifold as both four-dimensional and non-Euclidean" and later hid from view his "innermost mathematical especially invariant-theoretical thoughts on the theory of relativity" [42. Walter S.].Klein knew that Minkowski's definition of isotropic light line with real conics at the infinity was based on a *projective* point of view and not on Poincaré's *affine* point of view[14. Klein F.]. In most of the standard book on SR we find the mixture affine-hyperbolic. That is also true for Pauli's book on the theory of relativity. But in Pauli's book we can also find a very strange contradiction. We have first note 111:

> This connection with the Bolyai-Lobatchevsky geometry can be briefly described in the following way: if one interprets dx_1, dx_2, dx_3, dx_4as **homogeneous coordinates in a three-dimensional projective space**, then the invariance of the equation amounts to introducing a Cayley system of measurement, base on a real conic section. The rest follows from the well-known arguments by Klein.[19. Pauli W., note 111]

And secondly, as Serge Reynaud once remarked to me in Peyresc, we have Pauli's judgement on the 6 groups of transformations (A: orthogonal transformations, B and B': affine transformations, D: Einstein's point transformations for GR, E: Weyl's transformations). All the groups according to Pauli are very important for the physics. Except Group C:

> C: The projective group of **projective linear fractional transformations**. This was mainly used by mathematicians in earlier investigations in non-Euclidean geometry. For physics it is of minor importance.[19. Pauli W., p 24]

Pauli's "note 111" and Pauli's "judgement on C" are obviously in contradiction. Minkowski's metric, clearly separated from Poincaré's metric, is based on a projective point of view for the homogeneous Lorentz transformation that conducts to a Cayley system of measurement of units (4.1.2). The coordinates of a point are the homogeneous coordinates of the velocity (2.4, 4.1.1). With the Möbius (linear fractional) transformations, we also have the **Möbius strip** (by 43 and 44). So the most important theory for physics, thanks to the existence of a "fine structure" of SR, could become the projective QSR (without Kaluza's fifth dimension that Einstein didn't like!). It is true for the definition of a time operator (note 111 and 31 $\rightarrow$ 31$'$) but it is also true for another enigma in theoretical physics. Indeed, Pauli was a rationalist who liked the symbols (see his dialogue with Karl Gustav Jung): The number of the room where he died was 137. That's obviously a pure coincidence of events but if the "fine structure" of SR is theoretically really interesting, it could have something to say about the fine structure... constant. Last but not least: A good physical theory must not only clarify the old unsolved questions but it must be also able to predict new phenomena. Is that the case for QSR? The young Albert wrote to his young wife Mileva in 1901:

> Drude's theory is a kinetic theory of thermal and electrical processes completely in the mind of the kinetic theory of gas. If only there was no this damned magnetism about which we don't know what we can do. [April 1901, letter n°27] [20. Pierseaux Y.]

At this epoch we showed that the Democritean Albert was a follower of Boltzmann and saw atoms (quanta of matter) everywhere even for the (quanta of) light and for the (quanta of) charge. The only problem was that there is no atom of (quanta of) magnetism, i.e. a magnetic monopole. Twenty years after the miraculous year 1905, Einstein was very surprised by the purely special relativistic reasoning of Thomas. He would have been still more surprised to discover that behind the Thomas precession, in his proper theory, there was an atom of magnetism. Now let us indeed think about the magnetic moment of Einstein's pointlike electron in QSR. If it is possible to attach a magnetic dipole on the classical finite electron (Thomas's principle of correspondence), **it is obviously impossible to attach a dipole to a point**. So QSR is a quantum theory of magnetism (with quantization of the charge) that predicts, in agreement with Georges Lochak's talk in Peyresc, directly *the existence of* **magnetic monopole**.

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