

## Space-time localization with electromagnetic fields

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**ABSTRACT.** The necessity to reconcile the laws of mechanics and electromagnetism has led to question our representation of space and time and to the development of relativity theory. Present most accurate determinations of positions in space and time rely on the exchange of electromagnetic signals and strongly depend on the relativistic conception of space-time. Despite the quantum nature of the electromagnetic fields and atomic clocks used for localization, space-time positions are yet usually treated as classical parameters. As discussed here, relativistic localization may be described by quantum observables representing space-time positions, and defined on the exchanged quantum fields. These quantum localization observables differ from their classical analogs by their non vanishing commutator, which involves spin observables. Alternatively, quantum positions may be defined as commuting but complex observables, which do not commute with their complex conjugates. There result new relations of space-time to algebra and geometry, and in particular modifications of relativistic frame transformations and covariance rules.

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### 1 Introduction

Although, since the very beginning, time and space have played an essential role in the development of physical theories, their present status still suffers from serious ambiguities. In either theoretical or experimental practice, and according to the domain of physics, different notions of

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time or space are actually used. Such a situation cannot remain indefinitely innocuous, especially as new fields are being prospected, which lie at the borderland between general relativity, quantum theory or metrology. Moreover, the very high and ever increasing accuracy reached in measurements of space and time becomes incompatible with a fuzzy status.

The existence of different notions of time and space was first clearly recognized and explicitly stated by Newton in his *Principia* [1]:

”I. Absolute, true, and mathematical time, of itself, and from its own nature, flows equally without relation to anything external, and by another name is called *duration*. Relative, apparent and common time, is some sensible and external measure of duration by the means of motion: such are measures of *hours*, of *days*, of *months*, etc. which are commonly used instead of true time.

II. Absolute space, in its own nature, without relation to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute space, which our senses determine by its position to bodies; and which is commonly taken for immovable space.”

The first notion of time, which Newton called absolute and mathematical, allowed him to write mathematical equations for the laws of mechanics and gravitation. Used as a curvilinear coordinate on a planet's trajectory, it gave the necessary tool to deal with infinitesimals. The second notion, which he called common and sensible, allowed him to relate the motions of different physical systems. Physical time enters through Kepler's area law as a measure of inertial motions, which planetary motions can be compared with. The mathematical representations of time and space as real parameters, clearly privileged by Newton, still underlies the differential formalism of modern physical theories. On another hand, the idea of a universal arena for all motions, represented by observable time and space built from real clocks, lies at the basis of modern coordination systems and metrology.

The introduction of electromagnetic fields as intrinsic physical entities, with the recognition of their universal velocity of propagation [2], raised the question of the consistency of the physical laws for electromagnetism and mechanics. This led to a questioning of their constitutive basis, represented by space and time. In his drastic solution to this compatibility problem, Einstein emphasized the primary role played by

time, and especially by physical time delivered by clocks [3]:

"If we wish to describe the *motion* of a material point, we give the values of its co-ordinates as functions of time. Now we must bear carefully in mind that a mathematical description of this kind has no physical meaning unless we are quite clear as to what we understand by "time". We must keep in mind that all judgments where time plays a role are always judgments about *simultaneous events*. When, for instance, I say: "This train has arrived here at 7 o'clock", this roughly means: "The passage of the small hand of my watch on the 7 and the arrival of the train are simultaneous events."

"... this definition is no longer sufficient when it is the matter of temporally relating series of events which occur at places far from my watch."

"... So we see that we cannot attach any *absolute* signification to the concept of simultaneity, but that two events which, viewed from a system of co-ordinates, are simultaneous, can no longer be looked upon as simultaneous events when envisaged from a system which is in motion relatively to that system."

The breakthrough of relativity theory was made possible by insisting on the necessary observable character of time and space. Introducing and privileging the notion of event, Einstein argued that space and time had to be reconstructed from physical observables. This can be performed using some primitive physical systems, like clocks and light signals, to deliver and disseminate time observables, and then some primitive procedures, like clock synchronization and event localization, to coordinate events both in time and in space. This conception of a space-time related to observables delivered by physical systems and to a set of consistent procedures to coordinate their values led to theoretical predictions and applications, globally known as relativistic effects. The relativistic conception of space and time and its associated constructive procedures lie at the heart of modern metrology [4] and high precision coordination systems, such as the Global Positioning System [5, 6].

Although the relativistic conception was first developed in a classical context, its modern applications must face the quantum nature of physical observables. It is then clear that physical time and space belong to the quantum world. Indeed, in modern metrology, time and space units are defined from atoms and hence are deeply rooted in quantum theory. The time delivered by an atomic clock is intrinsically the phase of a

quantum oscillator. Electromagnetic signals used in synchronization and localization procedures are quantum fields, so that the time references they carry must be defined as quantum observables. But this raises a new compatibility problem, that is the existence of a description of space and time which relies on quantum and relativistic observables. The difficulty in obtaining a description satisfying both demands has been first clearly stated by Schrödinger, and revives the fundamental distinction made by Newton [7]:

”But in quantum mechanics, ... time is not treated as an observable, there is no operator which could be used to find its “statistics”. It is a parameter the value of which is supposed to be exactly known: it is in fact the old good time of Newton and quantum mechanics does not worry about the existence of the old good clock which she would need to know the value of this parameter  $t$ .

... But it seems to me doubtless that we will *have to* give up this too classical notion of time, and not only because of relativity. This notion of time is a serious lack of coherence in quantum mechanics (or in its usual interpretation), without mentioning the postulates of relativity. For the knowledge of the variable  $t$  is obtained in the same manner as that of any other variable, by observing a physical system, namely a clock.  $t$  is therefore an observable and must be treated as an observable; time must in general have a “statistics” and not a “value”. The exceptional role of time is thus not justified.”

As underlined by Schrödinger, the formalism of quantum mechanics provides time position a different status as that of space positions, so that it is not compatible with relativistic requirements. As is well known, this compatibility is restored in quantum field theory, but at the price of having both time and space lose their observable character. In standard quantum field theory, both space and time positions are represented as classical real parameters. Even if space positions may be given a representation in terms of quantum operators conjugate to momentum [8, 9], it is commonly admitted that time cannot be given a similar description as an operator conjugate to energy [10, 11]. This entails in particular that one does not dispose for space-time positions of the basic constitutive relations of quantum and relativity theory. That is, commutation relations between conjugate observables do not have a covariant form, or else, relativistic transformations of space-time positions cannot be obtained from an algebra of quantum generators. Such a situation leads to strong conflicts between the relativistic and quantum frameworks when

attempting to build a consistent theory including gravity [12, 13, 14].

But this situation need not be definitive. In fact, as we shall discuss here, the whole relativistic approach of space-time may be implemented within the framework of quantum field theory [15, 16, 17]. We shall describe how synchronization and localization procedures allow one to define quantum observables representing positions in space and time and satisfying canonical commutation relations with momentum and energy. Considering quantum electromagnetic fields, explicit expressions for the observables carried by one or two photons will be obtained [18]. A major difference with the classical case is the appearance of spin among localization observables. We will show that this reflects a new alternative which is imposed by quantum requirements: either space-time position observables are chosen to be hermitian, and then they have a non vanishing commutator related to spin, or they are chosen to be commuting observables, in which case they must be complex, with an imaginary part related to spin. As we shall briefly discuss, this non commutative or complex property of localization observables entails a revision of the covariance rules which underlie the formalism of general relativity. These rules usually relate the transformations of classical positions and momenta under changes of reference frames, and generally relate them to the transformations of a metric field. We shall see that they nonetheless admit generalizations under the form of quantum commutation relations [19], which thus provide extensions of the basic rules of differential geometry, under the form of algebraic relations.

## 2 Synchronization and localization

The whole construction of space-time begins with the definition of a primary notion, that of local time. The local time for a given observer is a physical observable, which is delivered by a clock located at the same place in space [3]. Local time is best provided by the most precise and accurate clocks available, that is in today's applications, by atomic clocks [5]. This allows one to define the properties which are required for classifying all events occurring at the same place, that is time simultaneity and time ordering: the time position of an event can be identified with the clock indication which coincides with it.

One then needs to extend this local notion of time to the whole space or, equivalently, to be able to identify the different local times associated with remote clocks. For that purpose, two observers need to share some information in order to compare the indications of their respective

local clocks. As emphasized by Einstein, this can be accomplished in a consistent way by the exchange of propagating signals, like electromagnetic fields. This method is also the one used by today's most efficient systems for disseminating a time reference or for synchronizing clocks all around the Earth [5]. A first observer may use electromagnetic fields as a support for encoding a time reference, representing in the most faithful way the time delivered by his clock, and then for propagating this time reference to a second observer (see Figure 1). Comparing the received time reference with the indications of his own clock, the second observer may then proceed to the identification of the two time variables, that is to the synchronization of the clocks.

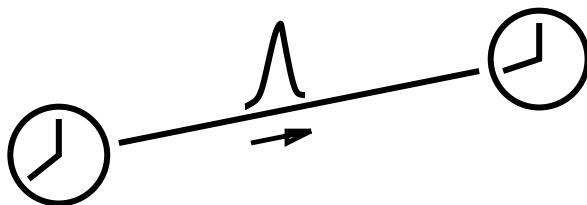


Fig 1 Information on time is encoded in electromagnetic signals shared by remote observers.

In a classical framework, one can assume that one localizes electromagnetic energy at will. The exchanged energy may be concentrated as precisely as one wishes around given values of light cone variables:

$$t_e - \frac{x_e}{c} = u_e \quad u_r = t_r - \frac{x_r}{c} \quad (1)$$

$$u_e = u_r \quad (2)$$

$t_e$  and  $t_r$  are the emission and reception times, as delivered to the emitter and receiver by their own clocks;  $x_e$  and  $x_r$  are the space coordinates of the emitter and receiver, as measured along the line of sight;  $c$  is the velocity of light. In that case, the light cone variables play the role of transfer observables: their constant value along the propagation path may be used to identify the classical time observables delivered by two remote clocks.

In a quantum framework, such an assumption no longer holds, if only because of limitations imposed on energy localization by Heisenberg inequalities. In that case, transfer observables need to be defined on the exchanged quantum fields. Such quantum observables have to provide in the most faithful way as possible time references for both observers: they should have the transformation properties of their classical analogs under frame transformations, and they should be preserved during propagation. Such quantum transfer observables may be defined, using general properties of the electromagnetic quantum field [15, 16]. An example will be given in the following, in the case where a single photon is exchanged.

Construction of space coordination follows the construction of time. Indeed, an event may be completely localized both in space and time using several transfers of time (at least the same number as the space-time dimension). For the simplicity of the discussion, let us first consider a two dimensional space-time, which can be viewed as a two dimensional projection of the real situation. Concrete realizations of space-time localization, like the Global Positioning System [5, 6], require the use of light cones, and in higher number to raise degeneracies between solutions. An event in space-time is then defined as the intersection of two electromagnetic signals, i.e. two time transfers, and may thus be characterized by the two references carried by these signals.

Considering that the two clocks regulating the emission of the electromagnetic signals have been synchronized, one can see that the two references carried by the signals are equivalent to two different values of the common time defined by the clocks. The latter values may be considered as coordinates and used to characterize events in space-time (see Figure 2).

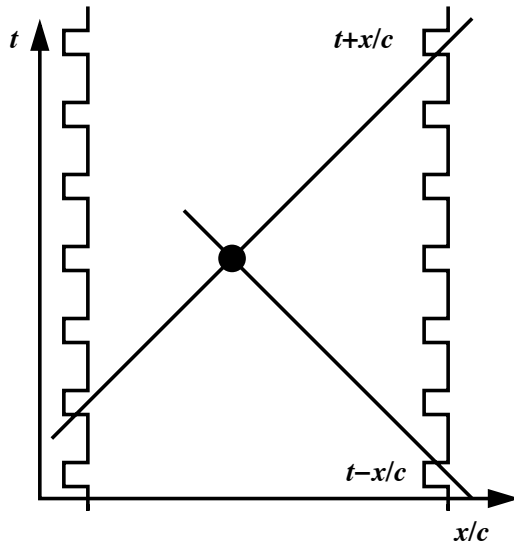


Fig 2 A space-time event is defined by electromagnetic signals propagating in different directions.

Classically, the positions of an event, both in space  $x$  and in time  $t$ , will be deduced from the light cone variables  $u_-$  and  $u_+$  of the two time transfers:

$$t - \frac{x}{c} = u_- \qquad t + \frac{x}{c} = u_+ \qquad (3)$$

Classical localization then leads to a coordination of space-time events by means of space-time coordinates taking real number values. In the quantum case, algebraic relations may still be used to deduce the space-time positions of an event from the observables of the corresponding time transfers [15]. However, as entailed by the nature of the observables entering synchronization and localization procedures, quantum coordination of space-time events can no more be performed in terms of real numbers but results in positions that are quantum operators.

From the preceding construction, it must be clear that synchronization and localization rely on the symmetries which characterize the propagation of signals, in that case, Maxwell laws for electromagnetism. This is exemplified by the universal constant  $c$ , i.e. light velocity, which enters the correspondence (3) between transfer and localization observables.



Classically, and in our four-dimensional space-time, this means that a space-time event is in fact obtained as the vertex of a light cone (Figure 3). These light cones reflect the fundamental symmetry group which underlies the propagation of electromagnetic signals.

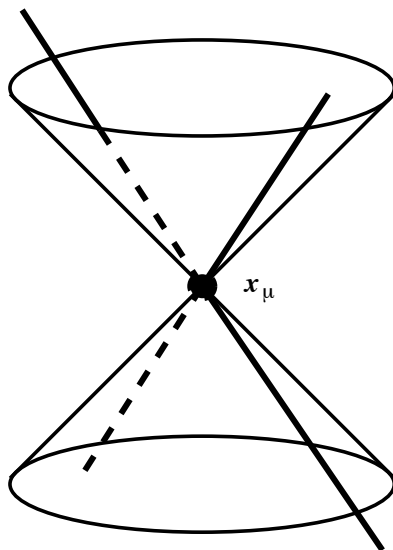


Fig 3 Classically, localization of a space-time event is associated with a light cone

In particular, causal relations can only be determined with respect to the relative positions of light cones. In a quantum context, it is no more possible to identify space-time events with vertices of light cones. However, the symmetry group of Maxwell equations still rules the propagation of electromagnetic fields. As discussed in the following, quantum synchronization and localization may be performed by merely exploiting the symmetry properties underlying the propagation of quantum fields.

### 3 Conformal algebra

Since the early beginning of relativity theory, emphasis was put on the role of symmetry groups in determining the general properties of space-

time transformations [20]. It was also realized very soon that Maxwell equations are invariant under the whole group of symmetries of the light cone [21, 22]. One can write the latter

$$\begin{aligned} (x - x')^2 &= \eta_{\mu\nu}(x - x')^\mu(x - x')^\nu = 0 \\ \mu, \nu &= 0, 1, 2, 3 \quad \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \end{aligned} \quad (4)$$

where  $\eta_{\mu\nu}$  is Minkowski metric, and where  $x$  and  $x'$  denote the coordinates of two arbitrary points on a same light cone. The linear transformations of coordinates  $x, x'$  which preserve (4) form a well known group of projective geometry, the  $SO(4, 2)$  Lie group of conformal transformations [23]. Conformal transformations are generated by 15 generators which may be interpreted as infinitesimal transformations of the reference frame. In particular, the conformal algebra contains the Poincaré algebra, which is generated by space-time translations  $P_\mu$  and by four dimensional rotations corresponding to Lorentz transformations  $J_{\mu\nu}$ :

$$\begin{aligned} (P_\mu, P_\nu) &= 0 \\ (J_{\mu\nu}, P_\rho) &= \eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu \\ (J_{\mu\nu}, J_{\rho\sigma}) &= \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\sigma}J_{\mu\rho} \end{aligned} \quad (5)$$

At this level, the algebra (5) is defined in an abstract way, using for the definition of Lie brackets the differential operators representing infinitesimal linear transformations of the light cone parameters (4). Classically, these parameters may be identified with the coordinates obtained through space-time localization, so that the symmetries of propagation of electromagnetic fields directly correspond to the symmetries of Maxwell equations.

As is well-known, such symmetries imply the existence of physical quantities which are conserved during propagation and correspond to conservation laws:  $P_\mu$  corresponds to energy-momentum, while  $J_{\mu\nu}$  corresponds to angular momentum. At the quantum level, the generators of symmetry may be constructed from the energy-momentum tensor and identify with the quantum operators associated with the conserved quantities. This means that the Lie algebra (5) is realized within the algebra of quantum observables, with Lie brackets obtained as quantum commutators, up to a universal constant  $\hbar$  (Planck constant):

$$\begin{aligned} (A, B) &\equiv \frac{AB - BA}{i\hbar} \\ A \cdot B &\equiv \frac{AB + BA}{2} \end{aligned} \quad (6)$$

As it will appear useful in the following, we have also introduced a notation for the symmetrized quantum product.

Maxwell equations are also invariant under dilatation, with generator  $D$ , and special conformal transformations [21, 22], with generators  $C_\mu$  satisfying the following relations:

$$\begin{aligned}
 (D, P_\mu) &= P_\mu & (D, J_{\mu\nu}) &= 0 \\
 (D, C_\mu) &= -C_\mu \\
 (J_{\mu\nu}, C_\rho) &= \eta_{\nu\rho} C_\mu - \eta_{\mu\rho} C_\nu \\
 (C_\mu, C_\nu) &= 0 \\
 (P_\mu, C_\nu) &= -2\eta_{\mu\nu} D - 2J_{\mu\nu}
 \end{aligned} \tag{7}$$

Special conformal transformations allow one to generalize to uniformly accelerated frames the invariance properties which hold for Lorentz transformations, i.e. transformations to frames moving with uniform velocity [24]. Conformal symmetries not only hold at the classical level, but also for quantum electromagnetic fields [25]. Genuine quantum notions, such as the electromagnetic vacuum state and the photon number, may also be shown to be conformally invariant [26, 27, 28]. We show in the following how conformal symmetries may be exploited for defining transfer and localization observables, and determining in a universal way their transformations under changes of frame.

An important advantage of the symmetry approach is its independence on a specific underlying theory. From conformal symmetry of the theory, one deduces the existence of quantum observables satisfying the commutation relations (5) and (7). When built from the field energy-momentum tensor, these observables may take explicit expressions which depend on the elementary quantum fields, but the algebra they generate does not depend on this construction.

#### 4 Quantum transfer observables

We now define transfer observables from quantum fields. We first remark that conformal invariance is not satisfied by all massless quantum fields [29]. Representations of the conformal algebra in terms of massless quantum fields have been thoroughly studied and shown to include solutions of Klein-Gordon equations for scalar fields, solutions of Weyl equations for spin 1/2 fields, solutions of Maxwell equations for electromagnetic fields, and solutions of Bargmann-Wigner equations for massless fields

of arbitrary helicities [30]. Although the properties discussed here apply to all these fields, we shall be mainly concerned with electromagnetic fields, hence with solutions of Maxwell equations.

The simplest quantum system one may use for time transfer is made of only one photon. The electromagnetic field in a one photon state satisfies the conformal algebra, with the constraint of a vanishing mass. With this constraint, the translation generators  $P_\mu$  in (5) satisfy a supplementary algebraic condition, corresponding to the vanishing of the Poincare invariant  $P^2$ . Furthermore, to be consistent with conformal symmetry, all relations obtained from this constraint by applying conformal transformations must also be satisfied. This finally leads to a necessary and sufficient set of five algebraic relations between conformal generators:

$$\begin{aligned}
 P^2 &= 0 \\
 P^\lambda \cdot J_{\lambda\mu} + P_\mu \cdot D &= 0 \\
 2J^\lambda_\mu \cdot J_{\lambda\nu} + P_\mu \cdot C_\nu + P_\nu \cdot C_\mu &= 2\eta_{\mu\nu}(\sigma^2 - 1) \\
 C^\lambda \cdot J_{\lambda\mu} - C_\mu \cdot D &= 0 \\
 C^2 &= 0
 \end{aligned} \tag{8}$$

$\sigma$  denotes the helicity and only depends on the squared total angular momentum  $J^2 = J_{\mu\nu}J^{\mu\nu}$  and dilatation  $D^2$  ( $\eta_{\mu\nu}$  is used in the following to raise or lower indices):

$$\sigma^2 = \frac{1}{2}J^2 + D^2 + 1 \tag{9}$$

The helicity  $\sigma$  is a conformal invariant and determines all three usual Casimir invariants of the conformal algebra  $\mathcal{SO}(4, 2)$ . For one photon states, it can only take one of the two values  $\pm 1$  [18]. In general, relations (8) characterize representations of the conformal algebra on the light cone.

The light cone condition between components of the energy-momentum still allows one to define conjugate observables of the spatial components  $P_i$ . They play the role of position observables for the photon [18]:

$$U_i = \frac{1}{P_0} \cdot J_{0i} \quad i = 1, 2, 3 \tag{10}$$

Positions (10) for the photon are defined as quantum observables and generalize, in terms of quantum operators, the classical expressions which

where used by Einstein in his discussion of the inertia of energy [31]. When expressed in terms of the field energy-momentum tensor, they take the form of barycentric coordinates for the field energy. As expected, only three space-time positions may be defined for the photon. Indeed, the latter can only be localized in space and time up to an arbitrariness proportional to its energy-momentum  $P_\mu$ . The three degrees of freedom may be associated with the space-time observables  $U_i$ , which represent three transfer observables.

Alternatively, for fields satisfying the conformal massless condition (8), momenta  $P_\mu$  and transfer observables  $U_i$ , together with the helicity  $\sigma$ , may be considered as building a complete set of space-time degrees of freedom. The conformal generators (5) and (7) (with condition (8)) may be recovered as algebraic expressions of these observables:

$$\begin{aligned}
 J_{ij} &= P_i \cdot U_j - P_j \cdot U_i - \sigma \epsilon_{ijk} \frac{P^k}{P_0} \\
 D &= P \cdot U + \frac{1}{2} \\
 C_0 &= -P_0 \cdot U^2 + \frac{\sigma^2}{P_0} \\
 C_i &= 2D \cdot U_i - P_i \cdot U^2 + 2\sigma \epsilon_{ijk} \frac{P^k}{P_0} U^j - \sigma^2 \frac{P_i}{P_0^2}
 \end{aligned} \tag{11}$$

When ignoring helicity dependent terms, the conformal generators (11) take a classical form, provided the classical coordinates on the light cone and their translations are replaced by quantum positions and momenta. This classical correspondence justifies the interpretation of the transfer observables  $U_i$  as quantum generalizations of the classical light cone coordinates (3). However, in the full quantum case, further terms appear in (11) which depend on the helicity. Quantum expressions take into account the intrinsic angular momentum, or spin, carried by the photon, which are completely described by its helicity. These terms imply in particular that the photon cannot be identified with an idealized classical ray.

As a direct consequence of their definition (10), in the enveloping algebra of conformal generators, the transfer observables have non null commutation relations, which are related to helicity:

$$(U_i, U_j) = \sigma \epsilon_{ijk} \frac{P^k}{P_0^3} \tag{12}$$

This property shows that the transfer observables one must use in a quantum framework cannot be treated as classical coordinates. Indeed, as usual in a quantum framework, relations (12) lead to Heisenberg inequalities which, in the case of non vanishing helicity, prohibit a perfectly accurate and simultaneous knowledge of all components of the transfer observables. Relations (11) and (12) further entail that chirality must play a primary role in the definition of space-time observables.

## 5 Quantum localization observables

As previously discussed, full localization of an event in space and time requires the use of several time transfers: the space-time positions of an event are defined by means of fields which propagate in different directions. In that case, the total energy-momentum of the fields which are involved in localization corresponds to a non vanishing mass. From the total energy-momentum tensor of the fields which characterize the event, and more precisely from the associated symmetry generators, it is then possible to define the localization observables  $X_\mu$  and  $S_\mu$  (using the symmetrized product defined in (6)):

$$X_\mu = \frac{P^\lambda \cdot J_{\lambda\mu} + P_\mu \cdot D}{P^2}$$

$$S_\mu = -\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}P^\nu J^{\lambda\rho} \qquad S_{\mu\nu} = \epsilon_{\mu\nu\lambda\rho}\frac{P^\lambda}{P^2}S^\rho \quad (13)$$

In contrast to their classical counterparts (3), the localizations observables defined by (13) contain spin observables, either represented by a vector  $S_\mu$  (Pauli-Lubanski vector [32]) or by an equivalent tensor  $S_{\mu\nu}$ . Equivalently, these observables may be obtained by requiring that the generators of the Weyl algebra, i.e. Poincaré generators and dilatation, take their usual expressions:

$$J_{\mu\nu} = P_\mu \cdot X_\nu - P_\nu \cdot X_\mu + S_{\mu\nu}$$

$$D = P^\mu \cdot X_\mu \quad (14)$$

Commutation relations of the quantum operators  $X_\mu$  defined by (13) are completely determined by the conformal algebra (5) and (7). As in the case of transfer observables, the quantum operators  $X_\mu$  define conjugate observables of the energy-momentum  $P_\mu$ , and represent quantum position observables. In this case however, four space-time positions are defined, which include a time operator  $X_0$ , conjugate to the energy  $P_0$ :

$$(P_\mu, X_\nu) = -\eta_{\mu\nu} \quad (15)$$

This result contradicts the common opinion that such a time operator cannot exist [10, 11]. In fact, some assumptions which sustain this opinion can be seen to fail in the present case. The impossibility crucially relies on a self-adjointness assumption concerning the time operator. But, as a result of its definition (13), the time operator, although hermitian, fails to be self-adjoint: its definition domain explicitly excludes an important part of the Hilbert space, namely states having a vanishing mass (like zero or one photon states). Non self-adjoint operators are well known to be quite acceptable representations of physical observables [33]. They even appear to be unavoidable when trying to represent localization in a non-commutative space-time [34]. Thus defined, the position observables allow one to write their conjugate relations with momenta in a Lorentz covariant way (15), or equivalently, to realize relativistic transformations within the algebra of quantum observables.

Indeed, positions transform classically under Lorentz transformations and dilatation:

$$\begin{aligned}(J_{\mu\nu}, X_\rho) &= \eta_{\nu\rho}X_\mu - \eta_{\mu\rho}X_\nu \\ (D, X_\mu) &= -X_\mu\end{aligned}\tag{16}$$

This does not mean that observable space-time positions may be treated as classical coordinates. In fact, as was the case for transfer observables, localization observables appear to have a non vanishing commutator:

$$(X_\mu, X_\nu) = \frac{S_{\mu\nu}}{P^2}\tag{17}$$

where  $S_{\mu\nu}$  is the spin tensor defined in (13). The identification of spin with the position commutator asserts the necessity to include spin among space-time localization observables. As for transfer observables, chirality enters in a basic way the definition of space-time positions.

Recalling the fundamental relation between causality and the building elements of space-time, it may be useful to provide some geometrical interpretation of the emergence of spin among localization observables. As previously remarked, the quantum fields used in synchronization or localization are built with photons, and cannot be identified with classical light rays. Their non vanishing helicity already leads to irreducible uncertainties, related to Heisenberg inequalities, which precludes their representation under the form of infinitely thin lines. As exhibited by a semi-classical interpretation of the constitutive relations (8) and (13),

the quantum fields used in localization cannot intersect exactly. The size of the overlapping region is directly related to the total helicity and to the spin born by the quantum fields [18]. A more realistic representation may be given to the localization geometrical setting under the form of a hyperboloid, with a waist size related to the spin and helicity of the fields, and to the inverse squared mass (see Figure 4). This simple representation illustrates the difference between the quantum localization procedure and its classical analog represented in Figure 3.

One remarks that the existence of position observables crucially relies on the non vanishing mass of the total field used for localization (13). Furthermore, a mass observable may be defined, following Einstein [31], from the energy-momentum  $P_\mu$  of the fields

$$\begin{aligned} M^2 &= P^\mu P_\mu \\ (P_\mu, M^2) &= (J_{\mu\nu}, M^2) = 0 \end{aligned} \quad (18)$$

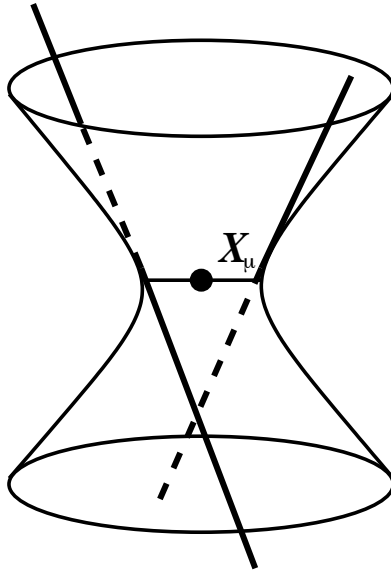


Fig 4 In a quantum world, localization of a space-time event is better described by a hyperboloid



The mass observable is invariant under all Poincaré transformations, but it nonetheless cannot be identified with a pure number: according to the conformal algebra (7), it transforms under dilatation and special conformal transformations:

$$\begin{aligned} (D, M^2) &= 2M^2 \\ (C_\mu, M^2) &= 4M^2 \cdot X_\mu \end{aligned} \tag{19}$$

with  $X_\mu$  defined as in (13). Equivalently, quantum positions can be defined from the shift of the mass observable under transformations to accelerated frames (19). Rewriting the transformation (19) for the mass and for a small but finite acceleration  $a^\mu$ , the quantum red shift law then takes the same form as the classical Einstein law [19]:

$$\begin{aligned} \Delta &= \frac{a^\mu}{2} C_\mu \\ (\Delta, M) &= M \cdot \Phi \\ \Phi &= a^\mu X_\mu \end{aligned} \tag{20}$$

The accelerated mass is proportional to the rest mass and to a gravitational potential  $a \cdot X$  depending linearly on the position measured along the acceleration. It may also be read as a conformal metric factor arising in transformations to accelerated frames and depending on observables  $X_\mu$  in the same way as the classical metric factor depends on classical coordinates [35].

## 6 Complex positions

Localization of events in space-time makes spin observables emerge besides position observables. At this point, it is worth remarking that one may define an equivalent set of localization observables which satisfy remarkable commutation properties. Introducing a complex structure under the form of an involution ( $(\pm i)^2 = -1$ ) which commutes with the whole conformal algebra, new localization observables may be defined in the following way:

$$\begin{aligned} x_\mu^\pm &= X_\mu \mp i \frac{S_\mu - \sigma P_\mu}{P^2} \\ s_{\mu\nu}^\pm &= S_{\mu\nu} \pm i \frac{P_\mu S_\nu - P_\nu S_\mu}{P^2} \end{aligned} \tag{21}$$

$\sigma$  is an arbitrary scalar, i.e. conformal invariant, which leaves commutation relations unchanged. An example will be given in the following,

but it is already clear from (21) that  $\sigma$  must have the same chirality properties as spin, i.e. it must be a pseudo-scalar. For each choice of the complex structure ( $\pm i$ ), there corresponds a set of observables  $x_\mu^\pm$  and  $s_{\mu\nu}^\pm$  which are complex and satisfy commutation relations with a canonical form:

$$\begin{aligned} (P_\mu, x_\nu^\pm) &= -\eta_{\mu\nu} \\ (x_\mu^\pm, x_\nu^\pm) &= 0 \\ (P_\mu, s_{\nu\rho}^\pm) &= (x_\mu^\pm, s_{\nu\rho}^\pm) = 0 \\ (s_{\mu\nu}^\pm, s_{\rho\sigma}^\pm) &= \eta_{\nu\rho}s_{\mu\sigma}^\pm + \eta_{\mu\sigma}s_{\nu\rho}^\pm - \eta_{\mu\rho}s_{\nu\sigma}^\pm - \eta_{\nu\sigma}s_{\mu\rho}^\pm \end{aligned} \quad (22)$$

Within each set of complex observables, different position components commute between themselves and commute with all spin components. The latter furthermore obey the commutation relations of a Lorentz representation. Each set of complex observables thus realizes, in a quantum framework, the Poisson brackets algebra of usual classical space-time degrees of freedom. Of course, these new localization observables are not hermitian, and furthermore they do not commute with their adjoints:

$$(x_\mu^+, x_\nu^-) = \frac{P_\mu}{P^2}(x_\nu^+ - x_\nu^-) + \frac{P_\nu}{P^2}(x_\mu^+ - x_\mu^-) - 2i\sigma \frac{\eta_{\mu\nu}}{P^2} \quad (23)$$

It follows that properties which characterize classical coordinates cannot be met simultaneously: one can define either hermitian localization observables, with a real spectrum but with non canonical commuting properties, or canonical localization observables, with commuting position components, but which cannot be hermitian. In the quantum case, this dilemma takes the form of two equivalent sets of observables constituting the basic elements of space-time. These two sets are in one-to-one correspondence through a complex conjugation.

As entailed by the definition of complex observables (21), commutation relations between adjoint operators (23) explicitly show that the complex structure is linked to chirality. This is confirmed by the relation between the complex structure and spin orientation: the two complex spins (21) may be seen to be either self-dual or anti-self-dual under four dimensional duality:

$$s_{\mu\nu}^\pm = \pm \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} s^{\pm\rho\sigma} \quad (24)$$

where  $\epsilon_{\mu\nu\rho\sigma}$  is the completely antisymmetrical Lorentz invariant tensor. Let us note that this connection reveals a universal property, as it also

holds for the spin quantum observables which may be defined for a Dirac electron [36]. Indeed, quantum position and spin observables (21) may also be used to describe a Dirac electron in space-time, and obtain a quantum description both of the associated Clifford algebra and of the electron motion [37]. In that case, the involution entering the definition of the complex structure ( $\pm i$ ) identifies with the usual  $\gamma_5$  matrix which characterizes the spin orientation of the electron.

Relations (21) and (24) establish a universal connection between the choice of a complex structure and the choice of a space-time orientation. This property confirms the already noted fundamental role played by helicity and spin in defining the basic elements of space-time: they appear as internal degrees of freedom implementing an intrinsic chiral nature of space-time, related to the existence of an underlying complex structure.

## 7 Two photons system

As an illustration of the previous results, we briefly discuss the simplest quantum field system allowing localization, that is the two photons states. The conformal algebra for the two photons system follows from the energy-momentum stress tensor of the system. Each generator of the conformal algebra for the total system is simply obtained as the sum of the corresponding generators associated with each photon. A detailed study of the conformal algebra shows that the three Casimir invariants of  $\mathcal{SO}(4, 2)$  are not independent, but only depend on two conformal invariants [18]: the total spin modulus  $s$  and the total helicity  $\sigma$  of the two photons system:

$$-\frac{S^\mu S_\mu}{P^2} = \frac{1}{2} S_{\mu\nu} S^{\mu\nu} = s(s+1)$$

$$\sigma_1 + \sigma_2 = \sigma \tag{25}$$

$\sigma_1$  and  $\sigma_2$  are the two conformal invariants corresponding to the helicities of the two photons. Conformal invariance of the total spin modulus  $s$  appears as a remarkable property holding for two photons states only.

Assuming that the two photons are not collinear i.e. that their total mass does not vanish, localization observables (13) and their complex analogs (21) may be defined. Then, as in the case of a single photon (11), one may use the energy-momentum and a set of localization observables to rewrite all generators of the conformal algebra. It is remarkable that the expressions for all conformal generators take their classical form,

whatever the choice of set of complex observables:

$$\begin{aligned}
 J_{\mu\nu} &= P_\mu \cdot x_\nu^\pm - P_\nu \cdot x_\mu^\pm + s_{\mu\nu}^\pm \\
 D &= P^\mu \cdot x_\mu^\pm \mp i\sigma \\
 C_\mu &= 2D \cdot x_\mu^\pm - P_\mu \cdot x^{\pm 2} + 2x^{\pm\lambda} s_{\lambda\mu}^\pm
 \end{aligned}
 \tag{26}$$

For each choice of a complex structure, all observables entering expressions (26) satisfy canonical commutation relations (22), so that complex observables defined by (21) appear as the quantum localization observables which are closest to their classical analogs. One may remark that, in the two photons case, the arbitrary conformal invariant  $\sigma$  appearing in the definition of complex observables (21) identifies with the total helicity.

A major consequence of both the canonical commutation relations between complex localization observables (22) and of the classical expressions taken by all conformal generators (26) is that complex observables transform classically under all conformal transformations. In particular, transformations of complex positions under special conformal transformations take simple classical forms. For transformations to uniformly accelerated frames, with finite acceleration  $a^\mu$ , simple expressions are readily obtained:

$$x_\mu^{\pm'} = \frac{x_\mu^\pm + \frac{1}{2}a_\mu x^{\pm 2}}{1 + ax^\pm + \frac{a^2}{4}x^{\pm 2}}
 \tag{27}$$

This result holds for each complex set, so that transformations of hermitian position  $X_\mu$  and spin  $S_\mu$  are easily deduced and seen to differ from the analogous classical transformations.

## 8 Metric effects

As entailed by their general definition through synchronization and localization procedures, the behavior of space-time observables under frame transformations is universally determined by symmetry groups. Quantum positions have been seen to transform in the same way as their classical analogs under Poincaré and dilatation generators, that is under Weyl transformations (see (15) and (16)). But, as previously noticed, these transformations do not take a classical form any more, when special conformal generators are considered. In that case, conformal symmetry still leads to universal expressions, but which can be seen to mix positions and spin observables. For a transformation  $\Delta$  to a frame moving

with a small but finite uniform acceleration  $a^\mu$  (defined according to (20)), they read:

$$\begin{aligned}
 (\Delta, P_\mu) &= a^\nu (X_\nu \cdot P_\mu - X_\mu \cdot P_\nu + S_{\mu\nu}) + a_\mu X \cdot P \\
 (\Delta, X_\mu) &= -a^\nu (X_\nu \cdot X_\mu - \frac{S_\nu \cdot S_\mu}{(P^2)^2}) + \frac{a_\mu}{2} (X^2 + \frac{s(s+1)}{P^2}) \\
 (\Delta, S_\mu) &= a^\nu (X_\nu \cdot S_\mu - X_\mu \cdot S_\nu) + a_\mu X \cdot S
 \end{aligned}
 \tag{28}$$

If one ignores the spin dependent term in the transformation of momentum, spin and momentum transformations exhibit the same connection as in the classical case: the transformation of spin is a linear operator taking the same expression as the linear operator associated with the transformation of momentum (see (28)). Similarly, if spin dependent contributions are ignored, positions transform classically, provided quantum position operators with a symmetrized product are used. This means in particular that the spin independent part of positions shift has for differential the previous linear operator entering the transformations of spin and momentum (see (28)). Thus, ordinary covariance properties would still hold at the quantum level, were it not for spin dependent corrections. However, the spin dependent term in the transformation of momentum is a direct consequence of the conformal algebra (see (7)), while those in the transformation of positions follow from the complex or non commutative nature of position observables. It should already be clear from these commutation properties that the covariance rules which lie at the basis of ordinary differential geometry cannot be applied to quantum observables without change.

But this does not mean that the covariance rules which play a crucial role in the formalism of general relativity must be completely abandoned. Indeed, the algebraic formalism, which is best suited to the quantum framework, also bears in itself some constitutive rules which are amenable to extensions of the covariance rules. Conformal invariance implies in particular that the fundamental commutator between momenta and positions is a pure number (15). Then, as a direct consequence of the conformal Lie algebra, Jacobi identities between any triple of operators, and in particular between momenta  $P_\mu$ , positions  $X_\nu$  and acceleration  $\Delta$  are satisfied, leading to the following identities:

$$\begin{aligned}
 (\Delta, (P_\mu, X_\nu)) &= 0 \\
 (P_\mu, (\Delta, X_\nu)) &= (X_\nu, (\Delta, P_\mu))
 \end{aligned}
 \tag{29}$$

Relations (29) are easily seen to include the properties which were previously discussed in analogy with covariance rules. Indeed, as spin and momentum commute, the left-hand side of the second equality of (29) just coincides with the linear differential associated with the spin independent part of the position shift. Similarly, as positions act as conjugate operators of momenta, the right-hand side of (29) coincides with the linear operator describing the transformation of momentum. Relations (29), which follow from the conformal invariance of the canonical commutator between positions and momenta can thus be considered as an algebraic extension, within the quantum framework, of the classical covariance rules of differential geometry [19].

As previously discussed, the transformation of mass under a special conformal transformation takes the same form in the quantum framework as in the classical one (20), so that it can even be used as an equivalent definition of position observables. It is also remarkable that, although involving non classical terms in their explicit expressions, both sides of the second equality of (29) satisfy another basic identity associated with general relativity [19]:

$$(P_\mu, (\Delta, X_\nu)) + (P_\nu, (\Delta, X_\mu)) = 2\eta_{\mu\nu}\Phi \\ \Phi = a^\rho X_\rho \tag{30}$$

Relations (30) represent algebraic extensions of the classical relations which determine, in general relativity, the changes of the metric field in terms of changes of coordinates. These relations give in particular a quantum description of Einstein effect, by relating the variations of clock rates to the gravitational field. Relations (30) and (20) are written here in terms of quantum position operators instead of classical coordinates. They may thus be considered as defining a quantum generalization of the metric field which describes gravitation.

## 9 Conclusion

We have shown that the synchronization and localization procedures underlying the definition of space-time according to relativity may be implemented in a quantum framework. This meets the logical necessity of describing physical positions in space-time as quantum operators. Moreover, the resulting framework gains in simplicity with respect to the classical case. Indeed, the classical framework uses at least two different algebraic structures, one describing physical observables, usually given

by a commutative algebra over real numbers, and another one describing their space-time transformations, usually given by a Lie algebra of symmetries. In contrast, within the quantum framework, only one algebra is needed, as the Lie algebra of symmetries is embedded in the non commutative algebra of quantum observables.

A significant by-product of the previous scheme is the definition of a time operator. This allows one to write fully covariant canonical relations between positions and momenta, and to circumvent some recurrent obstructions to a fully algebraic implementation of relativistic transformations, as required by quantum theory [12, 13, 14]. Another important feature of the quantum implementation is the occurrence of further localization observables, namely spin observables, besides positions in space-time. Spin may be seen either to express the non vanishing commutator of hermitian quantum positions, or the complex imaginary part of commuting quantum positions. This doubling of position observables reflects the existence of a complex structure linked to space-time orientation, so that chirality enters the description of space-time at a basic level. A similar connection has also been remarked to underlie the geometry associated with Dirac fields [38], and also a consistent description of the motion of a Dirac electron [37].

Non commutativity does not imply to abandon the rules, entailed by ordinary differential geometry, which have proved useful to the formalism of general relativity. We have shown here that purely algebraic extensions of the covariance rules and of their connection to the metric field may be given in terms of quantum positions. This encourages one to look for a quantum algebraic formulation of the founding principles, such as the equivalence between gravitation and accelerated motion. One must nonetheless remark that localization in physical space-time, as discussed here, must be distinguished from the locality properties in parameter space which are used in classical theory as well as in standard quantum field theory. This entails that the relations between causality and localization in space-time should be reconsidered.

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