Beables for Quantum Electrodynamics

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ABSTRACT. We show that it is possible to obtain a realistic and deterministic model, based on a previous work of John Bell, which reproduces the experimental predictions of the orthodox interpretation of quantum electrodynamics.

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1   Introduction

First, we would like to recall why the measurement postulate has been introduced in the orthodox interpretation of the quantum theory. Let us assume that a system of particles is completely described by a wave-function, that always evolves according to a linear equation (such as the Schrödinger equation). We plan to measure an observable $\mathcal{A}$ of the system. To measure the observable $\mathcal{A}$, one has to build an apparatus that correlate the eigenstates of $\mathcal{A}$ to position eigenstates of the apparatus, so that any measurement is finally a measurement of positions. For example, when the spin component of a spin-$\frac{1}{2}$ particle is measured, as in the Stern-Gerlach experiment, we finally measure a point on a screen, so a position (either in the upper half-plate, either in the lower half-plate). This remark is important since it gives a privileged role to position, and so superpositions get an absolute meaning. Now if we prepare the system in a superposition of eigenstates of $\mathcal{A}$, and if we send it towards the measuring apparatus, we can obtain superpositions of macroscopic objects (measuring devices, cats, humans, and so on), on account of the linear character of the equation of motion. But we

do not observe those macroscopic superpositions, but eigenstates of the observable being measured, with probabilities given by Born's law. In order to account for the experimental results, the measurement postulate is introduced:

> When an observer measure the observable $\mathcal{A}$ of the system, he founds one of the eigenvalues of $\mathcal{A}$, with probability given by Born's law. Because of the repeatability of experiments, the wave-function of the system collapses to the corresponding eigenstate when the measurement occurs.

The measurement postulate introduces a boundary between two worlds: a quantum world, made of systems, described by wave-functions, and a classical world, made of observers, described by positions. What is wrong with that? According to the orthodox interpretation, there is no element of reality in the quantum world: the particles of the quantum world do not have any probability of being anywhere (they do not exist). Elements of the classical world do have such probabilities: they exist (their positions are elements of reality). And the orthodox interpretation offers no explanation for that. People whose main concern is quantum gravity must face this kind of problem, for it is meaningless to speak of a wave-function of the universe in the orthodox interpretation.

To get rid of the frontier between the classical and the quantum worlds, one could say that the world is described by a wave-function and by the positions of the particles that it contains. There is a theory, built along these lines; it is the non-relativistic de Broglie-Bohm pilot-wave theory. Pilot-wave theory is a realistic theory, in so far as the positions of the particles exist and are simply revealed by position measurements (the positions are beables, a term coined by Bell). If there are n particles, the universe is thus completely described by the couple $(\vec{X}(t), \Psi(t, \vec{X}))$, where $\vec{X}$ is a point in a configuration space of dimension $3n$. $\Psi(t, \vec{X})$ evolves according to the Schrödinger equation, whereas the equation of motion for $\vec{X}(t)$ is such that if we consider a set of universes with the same wave-function and initial configurations chosen according to the probability density $|\Psi(t_0, \vec{X})|^2$, then the final configurations will be distributed according to $|\Psi(t, \vec{X})|^2$ (for any later time t). So the non-relativistic pilot-wave theory gives the same experimental predictions as the orthodox interpretation of the non-relativistic quantum theory, without relying on concepts such as systems and observers.

Pilot-wave theory is weakly non-local; that means that the equation of motion for the positions of the particles is non-local, but that does not lead to any supra-luminal signaling. Now, the widespread claim is that those interpretations are ruled out by Bell's inequality ([2], chap 7) and the experiments that have been carried out later. If fact local ones are ruled out, but since non-locality is commonly claimed to be unacceptable (even weak non-locality, which does not lead to paradoxical situations), it is said that any hidden-variables theory is incompatible with the quantum theory. That is the wrong way to present the theoretical situation. To present it correctly, we have to return to the EPR paradox [5], which in essence says that some quantum correlations cannot be explained in a local way, unless we say that the quantum theory is incomplete. Since local hidden-variables theories are ruled out by Bell's inequality and experiments, the only way to explain quantum correlations is to revert to non-locality (which is in fact hidden in the collapse postulate). Then, to suppress the ill-defined measurement postulate, it is preferable to interpret the quantum theory by a non-local hidden-variables theory, such as the non-relativistic pilot-wave theory. It is worth underlining that Bell, who is often credited of the refutation of hidden-variables theories, has been one of the main advocates of the pilot-wave theory ([2], chap. 11, 14, 15 and 17).

The question that comes to mind is to know if the quantum field theory can also be interpreted as a non-local hidden-variables theory. At the time Bell wrote his paper [1] (or [2], chap. 19), Bohm had already shown that it was possible to build a realistic interpretation of any bosonic quantum field theory [3]. To achieve that goal, Bohm took the field as the beable, however he was not able to do the same for fermions. The aim of Bell was then to show that it was also possible to build a realistic interpretation of any fermionic quantum field theory, along the pilot-wave ideas [1]. Bell managed doing so but he took a different beable: the fermion number density. It is slightly different from the non-relativistic pilot-wave theory, whose beables are the positions of the particles. The model is also formulated on a spatial lattice (space is discrete but time remains continuous). His model is stochastic, but he suspected that the theory would become deterministic in the continuum limit. We have shown [4] that it is indeed the case, so that there is a deterministic Bell model.

In this article, we consider in more details the deterministic Bell model for quantum electrodynamics. First, we will give a brief presen-

tation of the non-relativistic pilot-wave theory, since the deterministic Bell model is in fact very similar to it. Then, we will give a presentation of the Dirac quantum field theory, axed around the notion of negative-energy electrons. In that approach, there are only particles of charge $-e$ but of positive and negative energy: their total number (fermion-number) is conserved. We will see that this approach is a convenient one to study localized properties (such as fermion-number density). The reinterpretation of the theory is then made in terms of positrons, as holes in the Dirac sea. In section 5, we build the deterministic Bell model for the free Dirac quantum field theory. In the next section, the results are generalized to quantum electrodynamics.

2 The non-relativistic pilot-wave theory

Let us consider a system containing n non-relativistic electrons. In the orthodox interpretation, these electrons are completely described by a wave-function $\Psi_{s_1\ldots s_n}(t, \vec{x}_1, \ldots, \vec{x}_n)$, which is the solution of the Schrödinger equation ($s_1, \ldots, s_n$ are spin-component indices). The probability density to observe the system in a configuration $(\vec{x}_1, \ldots, \vec{x}_n)$ at time t is given by

$$\rho(t, \vec{x}_1, \cdots, \vec{x}_n) = \sum_{s_1=1}^{s_1=2} \cdots \sum_{s_n=1}^{s_n=2} \Psi^*_{s_1\ldots s_n}(t, \vec{x}_1, \ldots, \vec{x}_n)\Psi_{s_1\ldots s_n}(t, \vec{x}_1, \ldots, \vec{x}_n) \ .$$

From the relation

$$\int d^3\vec{x}_1 \ldots d^3\vec{x}_n \rho(t, \vec{x}_1, \ldots, \vec{x}_n) = 1$$

and with the help of the Schrödinger equation, it can be shown, at least in the usual cases, that there exist currents $\vec{j}_1, \ldots, \vec{j}_n$ such that

$$\frac{\partial \rho(t, \vec{X})}{\partial t} + \sum_{k=1}^{k=n} \vec{\nabla}_{\vec{x}_k} \cdot \vec{j}_k(t, \vec{X}) = 0 \ . \tag{1}$$

We use $\vec{X}$ as a short notation for $(\vec{x}_1, \ldots, \vec{x}_n)$.

The main idea of the pilot-wave theory is to say that the positions of the electrons are in fact elements of reality (or beables); that means that the electrons have definite positions $\vec{x}_1(t), \ldots, \vec{x}_n(t)$ that are simply

revealed by position measurements. So the complete description of a system at time t would involve the wave-function $\Psi_{s_1 \ldots s_n}(t, \vec{x}_1, \ldots, \vec{x}_n)$, but also the n position variables $\vec{x}_1(t), \ldots, \vec{x}_n(t)$. To complete the theory, one must give the equations of motion for these objects. For the wave-function, the Schrödinger equation is retained. As far as the position variables are concerned, the only constraint on the equation of motion is that the predictions of the orthodox interpretation of the quantum theory are reproduced. Is there a general form for that equation of motion? There are only two basic objects defined at time t: the wave-function of the system and the position of the system $(\vec{X}(t) = (\vec{x}_1(t), \ldots, \vec{x}_n(t)))$. So the velocity of the system would be fixed by those objects. In other words, the equation of motion is a velocity-law:

$$\vec{X}(t + dt) = \vec{X}(t) + \vec{V}^{\Psi}(t, \vec{X})|_{\vec{X} = \vec{X}(t)} \ ,$$

where $\vec{V}^{\Psi}(t, \vec{X}) = (\vec{v}_1^{\Psi}(t, \vec{X}), \ldots, \vec{v}_n^{\Psi}(t, \vec{X}))$.

Let us see what is the constraint imposed on the velocity-law by requiring that the pilot-wave theory is in agreement with the orthodox quantum theory. Let us define $r(t, \vec{X})$ as the probability density for the system to be in configuration $\vec{X}$ at time t. The probability density is defined in the usual way: we consider a set of systems, labeled by an index α, with the same wave-function but different realizations of the beable $\vec{X}_{\alpha}(t)$. Then the probability density is defined over these many experiments α. If the positions are simply revealed by position measurements, the probability density for the system to be observed in a configuration $\vec{X}$ is equal to the probability density for the system to be in configuration $\vec{X}$. Hence, to reproduce the predictions of the orthodox quantum theory, the following relation

$$\rho(t, \vec{X}) = r(t, \vec{X})$$

must be satisfied. Let us assume that this is true for some initial time t_0; then the relation that must be satisfied is

$$\frac{\partial \rho(t, \vec{X})}{\partial t} = \frac{\partial r(t, \vec{X})}{\partial t} \ . \tag{2}$$

We first consider the right-hand part. A consequence of the conservation of the total probability

$$\int d^{3n} \vec{X} \, r(t, \vec{X})$$

is the existence of a continuity equation

$$\frac{\partial r(t, \vec{X})}{\partial t} + \vec{\nabla}_{\vec{X}} \cdot (r(t, \vec{X}) \vec{V}^{\Psi}(t, \vec{X})) = 0 \ . \tag{3}$$

With the help of eq. (1) and eq. (3), eq. (2) can be rewritten as

$$\vec{\nabla}_{\vec{X}} \cdot (r(t, \vec{X}) \vec{V}^{\Psi}(t, \vec{X})) = \sum_{k=1}^{k=n} \vec{\nabla}_{\vec{x}_k} \cdot (r(t, \vec{X}) \vec{v}_k^{\Psi}(t, \vec{X}))$$

$$= \sum_{k=1}^{k=n} \vec{\nabla}_{\vec{x}_k} \cdot \vec{j}_k(t, \vec{X}) \ .$$

Taking

$$\vec{v}_k^{\Psi}(t, \vec{X}) = \frac{\vec{j}_k(t, \vec{X})}{r(t, \vec{X})} = \frac{\vec{j}_k(t, \vec{X})}{\rho(t, \vec{X})} \ , \tag{4}$$

the predictions of the orthodox quantum theory are regained. A complete study of the pilot-wave theory can be found in [7].

3   The free Dirac quantum field theory

Our aim now is to show that the idea of positive and negative energy electrons is convenient for the study of localized properties, such as fermion-number density (or charge density).

3.1   The Dirac equation in first quantization

The classical Dirac equation is

$$i\frac{\partial \psi(t, \vec{x})}{\partial t} = -i\vec{\alpha} \cdot \vec{\nabla}\psi(t, \vec{x}) + m\beta\psi(t, \vec{x}) \ ,$$

where the hermitian matrices β and α^j ($j = 1, 2, 3$) satisfy the relations

$$\{\alpha^j, \beta\} = 0 \qquad \{\alpha^j, \alpha^k\} = 2\delta_k^j \qquad \beta^2 = 1$$

and where $\psi(t, \vec{x})$ is a four-component spinor field. A particular representation of the matrices is given by the Pauli-Dirac choice:

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \ ,$$

where the σ^j are the usual Pauli matrices.

The Dirac equation can be rewritten in a covariant form by introducing the γ matrices, defined by the relations

$$\gamma^0 = \beta = \gamma_0 \qquad\qquad \gamma^j = \alpha^j = -\gamma_j \ .$$

Doing so leads to

$$i\gamma^\mu \partial_\mu \psi(t, \vec{x}) - m\psi(t, \vec{x}) = 0 \ .$$

It can also be shown that the γ matrices satisfy the relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Then, it is easy to check that each component of the Dirac field is a solution of the Klein-Gordon equation

$$(\Box + m^2)\psi_a(t, \vec{x}) = 0 \ ,$$

with $a = 1, 2, 3, 4$. Since the free solutions of the K-G equation are $e^{-iE_{\vec{p}}t}e^{i\vec{p}\cdot\vec{x}}$ and $e^{iE_{\vec{p}}t}e^{-i\vec{p}\cdot\vec{x}}$, with $E_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$ and $\vec{p} \in \mathbb{R}^3$, the general forms of the free solutions of the Dirac equation are

$$u_s(\vec{p})e^{-iE_{\vec{p}}t}e^{i\vec{p}\cdot\vec{x}} \qquad\qquad v_s(\vec{p})e^{iE_{\vec{p}}t}e^{-i\vec{p}\cdot\vec{x}} \ ,$$

with $s = 1, 2$ and $\vec{p} \in \mathbb{R}^3$. To give an interpretation of the Dirac equation, one must find a conserved current, whose time-component is positive; the current $j^\mu = \bar{\psi}\gamma^\mu\psi$, where $\bar{\psi} = \psi^\dagger\gamma^0$, is suitable. The quantity $j^0(t, \vec{x}) = \psi^\dagger(t, \vec{x})\psi(t, \vec{x})$ is thus the probability density to find an electron at point $\vec{x}$ at time t. Normalization still remains to be discussed. If the free particle has momentum $\vec{p}_\Sigma$ in an inertial frame Σ, where the universe appears to have a volume V_Σ, then the following relation must be satisfied:

$$\int_{V_\Sigma} d^3\vec{x}\,\psi^\dagger_{\vec{p}_\Sigma}(t, \vec{x})\psi_{\vec{p}_\Sigma}(t, \vec{x}) = 1 \ ,$$

in every inertial frame. That means that the spinors must be normalized to

$$u_s^\dagger(\vec{p})u_s(\vec{p}) = \frac{E_{\vec{p}}}{Vm} \qquad\qquad v_s^\dagger(\vec{p})v_s(\vec{p}) = \frac{E_{\vec{p}}}{Vm} \ ,$$

where V is the volume of the universe in the inertial frame where the particle is at rest.

The time component of the current is positive, a task that was impossible with the Klein-Gordon equation. But the negative energy states are still there. Once interactions are taken into account, their presence would lead to the instability of the hydrogen atom, for example. To avoid this, Dirac has assumed that all the negative energy states were occupied. Hence a positive energy electron cannot transit to a negative energy state, due to the Pauli exclusion principle. That state of lowest energy is called the Dirac sea. The absence of a negative energy state of momentum $\vec{p}$ (a hole in the Dirac sea) would be seen as a particle of positive energy $E_{\vec{p}}$, momentum $-\vec{p}$ and charge e. That led eventually to the prediction of anti-particles known as positrons.

3.2   The Dirac theory in second quantization

The first step, in the construction of the corresponding quantum field theory, is to find a real classical relativistic action, that leads to the Dirac equation, when the variational principle is applied on it. The following action

$$S = \int d^3\vec{x} dt \mathcal{L}(t, \vec{x}) = \int d^3\vec{x} dt \bar{\psi}(t, \vec{x})[i\gamma^\mu \partial_\mu - m]\psi(t, \vec{x})$$

is a good candidate. It is real, up to a four-divergence. The momenta conjugate to the fields are

$$\pi_a(t, \vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_a(t, \vec{x})} = i\psi_a^*(t, \vec{x}) \qquad \pi_a^*(t, \vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_a^*(t, \vec{x})} = 0 \ .$$

The next step is quantization; classical fields become quantum fields, satisfying the equal-time canonical anti-commutation relations

$$\{\psi_a(t, \vec{x}), \psi_b^\dagger(t, \vec{y})\} = \delta_b^a \delta^3(\vec{x} - \vec{y}) \quad \{\psi_a(t, \vec{x}), \psi_b(t, \vec{y})\} = 0 \ . \tag{5}$$

Since the quantum field $\psi(t, \vec{x})$ is a solution of the Dirac equation, it is a superposition of free classical solutions with operators as coefficients:

$$\psi(t, \vec{x}) = \sqrt{\frac{1}{V}} \sum_{s,\vec{p}} \sqrt{\frac{m}{E_{\vec{p}}}} [c_s(\vec{p}) u_s(\vec{p}) e^{-iE_{\vec{p}} t} e^{i\vec{p} \cdot \vec{x}} + \zeta_s(-\vec{p}) v_s(\vec{p}) e^{iE_{\vec{p}} t} e^{-i\vec{p} \cdot \vec{x}}]$$

(summation is made over all the momentum $\vec{p} = (n_1 \frac{2\pi}{L}, n_2 \frac{2\pi}{L}, n_3 \frac{2\pi}{L})$, with $V = L^3$ and $n_1, n_2, n_3 \in \mathbb{R}^3$). Taking

$$\{c_s(\vec{p}), c_r^\dagger(\vec{q})\} = \delta_s^r \delta_{\vec{p}\vec{q}} \qquad \{\zeta_s(\vec{p}), \zeta_r^\dagger(\vec{q})\} = \delta_s^r \delta_{\vec{p}\vec{q}} \ ,$$

and all other anti-commutators vanishing, the relations (5) are regained.

Let us now give an interpretation of the operators c and ζ. The hamiltonian is obtained through the Legendre transformation

$$
\begin{aligned}
H_{\mathcal{D}} &= \int d^3\vec{x}\,\psi^\dagger(t,\vec{x})(-i\vec{\alpha}\cdot\vec{\nabla} + m\beta)\psi(t,\vec{x}) \\
&= \sum_{s,\vec{p}} \sqrt{|\vec{p}|^2 + m^2}\,(c_s^\dagger(\vec{p})c_s(\vec{p}) - \zeta_s^\dagger(\vec{p})\zeta_s(\vec{p})) \ .
\end{aligned}
$$

The momentum is

$$
\vec{P} = \int d^3\vec{x}\,\psi^\dagger(t,\vec{x})(-i\vec{\nabla})\psi(t,\vec{x}) = \sum_{s,\vec{p}} \vec{p}\,(c_s^\dagger(\vec{p})c_s(\vec{p}) + \zeta_s^\dagger(\vec{p})\zeta_s(\vec{p})) \ .
$$

We define the fermion-number as the operator

$$
F = \int d^3\vec{x}\,\psi^\dagger(t,\vec{x})\psi(t,\vec{x}) = \sum_{s,\vec{p}} (c_s^\dagger(\vec{p})c_s(\vec{p}) + \zeta_s^\dagger(\vec{p})\zeta_s(\vec{p})) \ ,
$$

$\psi^\dagger(t,\vec{x})\psi(t,\vec{x})$ being the fermion-number density. Fermion-number means number of fermions, that is number of positive-energy electrons plus number of negative-energy electrons. The fermion-number is conserved ($[H_{\mathcal{D}}, F] = 0$); the corresponding current is $j^\mu = \bar{\psi}\gamma^\mu\psi = (F, \vec{J})$ with $\partial_\mu j^\mu = 0$. Since there are only electrons of charge $-e$, the fact that the fermion-number is conserved is equivalent to the charge conservation. The current $\vec{J}$ is

$$
\vec{J} = \int d^3\vec{x}\,\psi^\dagger(t,\vec{x})\vec{\alpha}\psi(t,\vec{x}) = \sum_{s,\vec{p}} \vec{p}\,(c_s^\dagger(\vec{p})c_s(\vec{p}) - \zeta_s^\dagger(\vec{p})\zeta_s(\vec{p})) \ .
$$

The charge density is $Q(\vec{x}) = -eF(\vec{x})$ and the charge current is $-e\vec{J}$.

We define a vacuum $|0_{\mathcal{D}}\rangle$ as a state destroyed by the operators c and ζ:

$$
c_s(\vec{p})|0_{\mathcal{D}}\rangle = 0 \qquad \zeta_s(\vec{p})|0_{\mathcal{D}}\rangle = 0 \qquad \forall\, s, \vec{p} \ .
$$

This vacuum $|0_{\mathcal{D}}\rangle$ is not the Dirac sea (the Dirac sea is obtained from $|0_{\mathcal{D}}\rangle$ by filling all the negative-energy states). Then $c_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle$ and $\zeta_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle$ are naturally interpreted as a one-electron states. Here is an

array with the properties of the one-electron states

$$H_{\mathcal{D}}c_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle = E_{\vec{p}}c_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle \qquad H_{\mathcal{D}}\zeta_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle = -E_{\vec{p}}\zeta_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle$$
$$Fc_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle = c_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle \qquad F\zeta_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle = \zeta_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle$$
$$\vec{P}c_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle = \vec{p}c_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle \qquad \vec{P}\zeta_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle = \vec{p}\zeta_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle$$
$$\vec{J}c_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle = \vec{p}c_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle \qquad \vec{J}\zeta_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle = -\vec{p}\zeta_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle$$

We see that the negative-energy electrons are not appropriated to the momentum space: it is difficult to give an interpretation of the state $\zeta_s^\dagger(\vec{p})|0_{\mathcal{D}}\rangle$ since it has a charge $-e$, momentum $\vec{p}$ and a charge current $e\vec{p}$. However, these states are well suited to the study of localized properties. To see that, let us switch to the Schrödinger picture; the Schrödinger fields are the Heisenberg fields taken at time $t = 0$

$$\psi_S(\vec{x}) = \psi(t = 0, \vec{x}) = \psi(\vec{x})$$

and we have the Schrödinger equation

$$i\frac{\partial|\Psi(t)\rangle}{\partial t} = H_{\mathcal{D}}|\Psi(t)\rangle \ .$$

The fermion-number $F = \int d^3\vec{x}\psi^\dagger(\vec{x})\psi(\vec{x})$ is the total number of positive-energy and negative-energy electrons, and the density of electrons (or fermion-number density) is thus given by

$$\psi^\dagger(\vec{x})\psi(\vec{x}) = \psi_a^\dagger(\vec{x})\psi_a(\vec{x}) \ .$$

What are the eigenstates of the the fermion-number density? Let us start from a state $|\Phi\rangle$, such that

$$\psi^\dagger(\vec{x})\psi(\vec{x})|\Phi\rangle = f(\vec{x})|\Phi\rangle \ , \tag{6}$$

and apply on operator $\psi_b^\dagger(\vec{y})$ on $|\Phi\rangle$. From eq. (6) and from the relations (5), it can be seen that

$$\psi^\dagger(\vec{x})\psi(\vec{x})\psi_b^\dagger(\vec{y})|\Phi\rangle = (\delta(\vec{x} - \vec{y}) + f(\vec{x}))\psi_b^\dagger(\vec{y})|\Phi\rangle \ ,$$

so that any of the four operators $\psi_b^\dagger(\vec{y})$ creates a quantum of the fermion-number at point $\vec{y}$. Thus the signification of $\psi_b^\dagger(\vec{y})$ is that it creates an electron at point $\vec{y}$. In the same way, it can be seen that the operators

$\psi_b(\vec{y})$ are annihilators of electrons at point $\vec{y}$. To obtain the eigenstates of the fermion-number density, one has to start from a state which is annihilated by any operator $\psi_b(\vec{y})$ and apply various creators $\psi_{a_j}^\dagger(\vec{x}_j)$ on it. That state is simply the vacuum $|0_\mathcal{D}\rangle$:

$$\psi_b(\vec{x})|0_\mathcal{D}\rangle = 0 \;\; \forall \vec{x} \in \mathbb{R}^3 \;\; \forall b \in \{1, 2, 3, 4\} \; .$$

Here is an array with the first eigenstates of the fermion-number density.

$F = 0$	$|0_\mathcal{D}\rangle$
$F = 1$	$\psi_{a_1}^\dagger(\vec{x}_1)|0_\mathcal{D}\rangle \; \forall \; a_1 \in \{1, 2, 3, 4\} \; \forall \vec{x}_1 \in \mathbb{R}^3$
$F = 2$	$\psi_{a_1}^\dagger(\vec{x}_1)\psi_{a_2}^\dagger(\vec{x}_2)|0_\mathcal{D}\rangle \; \forall \; a_1, a_2 \in \{1, 2, 3, 4\} \; \forall \vec{x}_1, \vec{x}_2 \in \mathbb{R}^3$
$\dots$	$\dots$

The structure of the Fock space is then

$$\mathcal{F} = \mathcal{H}_1 \oplus (\mathcal{H}_1 \otimes \mathcal{H}_1)_{as} \oplus (\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_1)_{as} \oplus \dots \; .$$

where $\mathcal{H}_1$ is the Hilbert space with fermion-number number equal to 1 (thus completely spanned by the orthonormal basis $\{\psi_a^\dagger(\vec{x})|0_\mathcal{D}\rangle \; \forall \; a, \vec{x}\}$), where $\oplus$ is the direct sum, $\otimes$ the direct-product, and as means that the space is restricted to its antisymmetric part.

Since $[H_\mathcal{D}, F] = 0$ and in accordance with the well-known super-selection rule that forbids superpositions of states with different values of the fermion-number, we know that $|\Psi(t)\rangle$ is an eigenstate of the fermion-number. Let us consider the case where there is only one quantum of the fermion-number:

$$F|\Psi(t)\rangle = |\Psi(t)\rangle \; .$$

Then $|\Psi(t)\rangle$ can be decomposed along the eigenstates of the fermion-number density $(\psi^\dagger(\vec{x})\psi(\vec{x}))$ with fermion-number equal to 1; these eigenstates are

$$\psi_a^\dagger(\vec{x})|0_\mathcal{D}\rangle \; \vec{x} \in \mathbb{R}^3, \quad a \in \{1, 2, 3, 4\} \; .$$

Thus, in our case,

$$|\Psi(t)\rangle = \sum_a \int d^3\vec{x} \, \Psi_a(t, \vec{x}) \psi_a^\dagger(\vec{x})|0_\mathcal{D}\rangle \; .$$

Inserting the previous equation in the Schrödinger equation, using the relations (5), and the definition of the hamiltonian, one finds that

$$i\frac{\partial \Psi(t,\vec{x})}{\partial t} = -i\vec{\alpha} \cdot \vec{\nabla}\Psi(t,\vec{x}) + m\beta\Psi(t,\vec{x}) \ ,$$

which is the Dirac equation. So the link is made between the first and the second quantization.

Usually, the theory formalism is rewritten by introducing the positrons, thanks to the the substitutions

$$\zeta_s(\vec{p}) \rightarrow d_s^\dagger(-\vec{p}) \qquad\qquad \zeta_s^\dagger(\vec{p}) \rightarrow d_s(-\vec{p}) \ .$$

Then another vacuum has to be defined (let us call it $|0\rangle$):

$$c_s(\vec{p})|0\rangle = 0 \qquad\qquad d_s(\vec{p})|0\rangle = 0 \qquad\qquad \forall \ s, \vec{p} \ .$$

That vacuum $|0\rangle$ is the usual vacuum. The operator $\zeta_s^\dagger(\vec{p})\zeta_s(\vec{p})$ is replaced by $d_s(-\vec{p})d_s^\dagger(-\vec{p})$, which is also equal to $-d_s^\dagger(-\vec{p})d_s(-\vec{p}) + 1$. Making that substitution in the previous observables leads to

$$H_\mathcal{D} = \sum_{s,\vec{p}} \sqrt{|\vec{p}|^2 + m^2}(c_s^\dagger(\vec{p})c_s(\vec{p}) + d_s^\dagger(\vec{p})d_s(\vec{p}))$$

$$F = \sum_{s,\vec{p}} (c_s^\dagger(\vec{p})c_s(\vec{p}) - d_s^\dagger(\vec{p})d_s(\vec{p}))$$

$$\vec{P} = \sum_{s,\vec{p}} \vec{p}(c_s^\dagger(\vec{p})c_s(\vec{p}) + d_s^\dagger(\vec{p})d_s(\vec{p}))$$

$$\vec{J} = \sum_{s,\vec{p}} \vec{p}(c_s^\dagger(\vec{p})c_s(\vec{p}) - d_s^\dagger(\vec{p})d_s(\vec{p})) \ ,$$

up to some constants which have been dropped. Another observable is usually defined, when we talk of positrons and electrons; it is the particle-number

$$N = \sum_{s,\vec{p}} (c_s^\dagger(\vec{p})c_s(\vec{p}) + d_s^\dagger(\vec{p})d_s(\vec{p})) \ .$$

The properties of the one-particle states are

$$H_\mathcal{D}c_s^\dagger(\vec{p})|0\rangle = \sqrt{|\vec{p}|^2 + m^2}c_s^\dagger(\vec{p})|0\rangle \qquad H_\mathcal{D}d_s^\dagger(\vec{p})|0\rangle = \sqrt{|\vec{p}|^2 + m^2}d_s^\dagger(\vec{p})|0\rangle$$

$$Fc_s^\dagger(\vec{p})|0\rangle = c_s^\dagger(\vec{p})|0\rangle \qquad\qquad Fd_s^\dagger(\vec{p})|0\rangle = -d_s^\dagger(\vec{p})|0\rangle$$

$$\vec{P}c_s^\dagger(\vec{p})|0\rangle = \vec{p}c_s^\dagger(\vec{p})|0\rangle \qquad\qquad \vec{P}d_s^\dagger(\vec{p})|0\rangle = \vec{p}d_s^\dagger(\vec{p})|0\rangle$$

$$\vec{J}c_s^\dagger(\vec{p})|0\rangle = \vec{p}c_s^\dagger(\vec{p})|0\rangle \qquad\qquad \vec{J}d_s^\dagger(\vec{p})|0\rangle = -\vec{p}d_s^\dagger(\vec{p})|0\rangle \ .$$

Thus, when we study properties related to the momentum space, the electrons and positrons point of view is the best one. However, the localized-particle states are not so natural. These states are defined as the eigenstates of the operator

$$n(\vec{x}) = \sum_{s=1}^{s=2} [C_s^\dagger(\vec{x})C_s(\vec{x}) + D_s^\dagger(\vec{x})D_s(\vec{x})] , \tag{7}$$

where

$$C_s(\vec{x}) = \frac{1}{\sqrt{(2\pi)^3}} \int d^3\vec{p}\, c_s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \quad D_s(\vec{x}) = \frac{1}{\sqrt{(2\pi)^3}} \int d^3\vec{p}\, d_s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} .$$

Eigenstates of the particle density are obtained from the vacuum $|0\rangle$ by applying several creators $C_{s_j}^\dagger(\vec{x}_j)$ or $D_{s_k}^\dagger(\vec{x}_k)$ on it. These states are the Newton-Wigner states.

The particle-number is different from the fermion-number

$$F = \sum_{s,\vec{p}} (c_s^\dagger(\vec{p})c_s(\vec{p}) + d_s(\vec{p})d_s^\dagger(\vec{p})) = C + \sum_{s,\vec{p}} (c_s^\dagger(\vec{p})c_s(\vec{p}) - d_s^\dagger(\vec{p})d_s(\vec{p}))$$

(where C is an infinite constant). It is worth mentioning that the particle-number does not commute with the fermion-number density $\psi^\dagger(\vec{x})\psi(\vec{x})$. It is possible to find well-behaved functions f such that

$$[\int d^3\vec{x} f(\vec{x})\psi^\dagger(\vec{x})\psi(\vec{x}), N] \neq 0 .$$

The proof is given in appendix A. In the electrons and positrons point of view, the charge density is defined as

$$Q(\vec{x}) = -e : \psi^\dagger(\vec{x})\psi(\vec{x}) : ,$$

where the dots mean normal-ordering. A corollary is that eigenstates of the charge density are not eigenstates of the particle-number.

4   Localized measurements

In the non-relativistic pilot-wave theory, the hidden variables are the positions of the particles. In the quantum field theory context, what hidden variables must we choose? The first natural answer seems to be

the particle density. We will now argue why the particle-density (eq. (7)) is not a good choice for a beable.

The first point is that the observable $n(\vec{x})$ appears very artificial, since it is not made of the fields $\psi(\vec{x})$ and $\psi^\dagger(\vec{x})$. Remember that any measurement is made with a measuring apparatus that correlate eigenstates of the observable being measured to eigenstates of the apparatus. So, when we look at localized properties, we are interested in hamiltonian densities. Of course, $n(\vec{x})$ is hermitian, but that does not imply that it has a physical meaning. In fact, when we measure localized properties, if we want to attain high precision, we have to use high energy (small wavelengths), and that leads to pairs creation. So it seems that the idea of an eigenstate containing a particle localized in a small region, looses his meaning when the region is sufficiently small. There is a second argument that shows that $n(\vec{x})$ does not have physical meaning: if we start from a state $C_s^\dagger(\vec{x})|0\rangle$ at time t and if we let it evolve according to the Schrödinger equation for a small time Δt, then it can be shown that it has a small probability to be outside the light-cone of the event $(\vec{x}, t)$ (see [6] and also [8]). The third point is more related to the general ideas of the pilot-wave theory. In fact, the crucial ingredient of the pilot-wave theory is the existence of a continuity equation linked to the chosen beable. Since the particle number is not conserved, we do not expect such an equation for the particle density.

In the non-relativistic pilot-wave theory, the hidden variable is the particle density, but since there are only electrons, the particle density is proportional to the charge density. Then can we take the charge density as the beable? That has been suggested by Bell, in his interpretation of the lattice fermionic quantum field theories [1]. To be more precise, Bell suggested that the fermion-number density $\psi^\dagger(\vec{x})\psi(\vec{x})$ could be given the beable status. All the critics that we made of the particle density do not apply to the fermion-number density. The fermion-number density does not commute with the particle number. If we start from a state $\psi_a^\dagger(\vec{x})|0_\mathcal{D}\rangle$ at time t, the charge will stay in the light cone. And finally, the fermion-number is conserved, so we expect a continuity equation.

5   The Bell model for the free Dirac quantum field theory

Since the fermion-number is conserved ($[H_\mathcal{D}, F] = 0$), and in accordance with the well-known super-selection rule, the pilot-state $|\Psi(t)\rangle$ is an eigenstate of the fermion-number; let us define the corresponding eigenvalue by ω: $F|\Psi(t)\rangle = \omega|\Psi(t)\rangle$. So the pilot-state can be decom-

posed along the eigenstates of the fermion-number density with fermion-number equal to ω:

$$|\Psi(t)\rangle = \frac{1}{\omega!}\int d^3\vec{x}_1\cdot\cdot d^3\vec{x}_\omega \Psi_{a_1\cdot\cdot a_\omega}(t,\vec{x}_1,\cdot\cdot,\vec{x}_\omega)\psi_{a_1}^\dagger(\vec{x}_1)\cdot\cdot\psi_{a_\omega}^\dagger(\vec{x}_\omega)|0_{\mathcal{D}}\rangle$$

(from now on, we use Einstein's convention: any repeated index is summed over). It is thus possible, from the Schrödinger equation, to obtain an equation satisfied by $\Psi_{a_1\cdot\cdot a_\omega}(t,\vec{x}_1,\cdot\cdot,\vec{x}_\omega)$. It can be shown that this equation is

$$i\frac{\partial\Psi_{a_1\cdot\cdot a_\omega}(t,\vec{X})}{\partial t} = \sum_{j=1}^{j=\omega}[-i\vec{\alpha}_{a_j a}\cdot\vec{\nabla}_{\vec{x}_j} + m\beta_{a_j a}]\Psi_{a_1\cdot\cdot(a_j\to a)\cdot\cdot a_\omega}(t,\vec{X}) \ , \tag{8}$$

where $\vec{X}$ is a short notation for $(\vec{x}_1,\dots,\vec{x}_\omega)$. We consider the case $\omega = 2$. It can be easily generalized. The pilot-state is

$$|\Psi(t)\rangle = \frac{1}{2!}\int d^3\vec{x}_1 d^3\vec{x}_2 \Psi_{a_1 a_2}(t,\vec{x}_1,\vec{x}_2)\psi_{a_1}^\dagger(\vec{x}_1)\psi_{a_2}^\dagger(\vec{x}_2)|0_{\mathcal{D}}\rangle \ .$$

Let us apply the hamiltonian on $|\Psi(t)\rangle$:

$$H_{\mathcal{D}}|\Psi(t)\rangle = \frac{1}{2!}\int d^3\vec{x}_1 d^3\vec{x}_2 d^3\vec{x} \Psi_{a_1 a_2}(t,\vec{x}_1,\vec{x}_2)$$
$$[i\vec{\nabla}\psi_a^\dagger(\vec{x})\cdot\vec{\alpha}_{ab} + m\psi_a^\dagger(\vec{x})\beta_{ab}]\psi_b(\vec{x})\psi_{a_1}^\dagger(\vec{x}_1)\psi_{a_2}^\dagger(\vec{x}_2)|0_{\mathcal{D}}\rangle$$
$$= \frac{1}{2!}\int d^3\vec{x}_1 d^3\vec{x}_2 d^3\vec{x} \Psi_{a_1 a_2}(t,\vec{x}_1,\vec{x}_2)[i\vec{\nabla}\psi_a^\dagger(\vec{x})\cdot\vec{\alpha}_{ab} + m\psi_a^\dagger(\vec{x})\beta_{ab}]$$
$$(\delta_{a_1 b}\delta(\vec{x} - \vec{x}_1)\psi_{a_2}^\dagger(\vec{x}_2) - \delta_{a_2 b}\delta(\vec{x} - \vec{x}_2)\psi_{a_1}^\dagger(\vec{x}_1))|0_{\mathcal{D}}\rangle$$
$$= \frac{1}{2!}\int d^3\vec{x}_1 d^3\vec{x}_2 \Psi_{a_1 a_2}(t,\vec{x}_1,\vec{x}_2)$$
$$[i\vec{\alpha}_{aa_1}\cdot\vec{\nabla}\psi_a^\dagger(\vec{x_1})\psi_{a_2}^\dagger(\vec{x}_2)|0_{\mathcal{D}}\rangle - i\vec{\alpha}_{aa_2}\cdot\vec{\nabla}\psi_a^\dagger(\vec{x_2})\psi_{a_1}^\dagger(\vec{x}_1)|0_{\mathcal{D}}\rangle$$
$$+ m\beta_{aa_1}\psi_a^\dagger(\vec{x_1})\psi_{a_2}^\dagger(\vec{x}_2)|0_{\mathcal{D}}\rangle - m\beta_{aa_2}\psi_a^\dagger(\vec{x_2})\psi_{a_1}^\dagger(\vec{x}_1)|0_{\mathcal{D}}\rangle]$$
$$= \frac{1}{2!}\int d^3\vec{x}_1 d^3\vec{x}_2 \Psi_{a_1 a_2}(t,\vec{x}_1,\vec{x}_2)$$
$$[i\vec{\alpha}_{aa_1}\cdot\vec{\nabla}\psi_a^\dagger(\vec{x_1})\psi_{a_2}^\dagger(\vec{x}_2)|0_{\mathcal{D}}\rangle + i\vec{\alpha}_{aa_2}\cdot\psi_{a_1}^\dagger(\vec{x}_1)\vec{\nabla}\psi_a^\dagger(\vec{x_2})|0_{\mathcal{D}}\rangle$$
$$+ m\beta_{aa_1}\psi_a^\dagger(\vec{x_1})\psi_{a_2}^\dagger(\vec{x}_2)|0_{\mathcal{D}}\rangle + m\beta_{aa_2}\psi_{a_1}^\dagger(\vec{x}_1)\psi_a^\dagger(\vec{x_2})|0_{\mathcal{D}}\rangle] \ . \tag{9}$$

The second equality comes from the relations (5) and from the fact that $|0_\mathcal{D}\rangle$ is annihilated by any operator $\psi_a(\vec{x})$. The third equality results from an integration over $\vec{x}$. The fourth equality is a consequence of the anti-commutation of the fields $\psi^\dagger$. Integrating by parts eq. (9) and renaming the dummy indices gives

$$H_\mathcal{D}|\Psi(t)\rangle = \frac{1}{2!}\int d^3\vec{x}_1 d^3\vec{x}_2[-i\vec{\alpha}_{a_1 a}\cdot\vec{\nabla}_{\vec{x}_1}\Psi_{aa_2}(t,\vec{x}_1,\vec{x}_2)$$
$$-i\vec{\alpha}_{a_2 a}\cdot\vec{\nabla}_{\vec{x}_2}\Psi_{a_1 a}(t,\vec{x}_1,\vec{x}_2)+m\beta_{a_1 a}\Psi_{aa_2}(t,\vec{x}_1,\vec{x}_2)$$
$$+m\beta_{a_2 a}\Psi_{a_1 a}(t,\vec{x}_1,\vec{x}_2)]\psi_a^\dagger(\vec{x_1})\psi_{a_2}^\dagger(\vec{x}_2)|0_\mathcal{D}\rangle .$$

And since

$$i\frac{\partial\Psi(t)\rangle}{\partial t} = \frac{i}{2!}\int d^3\vec{x}_1 d^3\vec{x}_2\frac{\partial\Psi_{a_1 a_2}(t,\vec{x}_1,\vec{x}_2)}{\partial t}\psi_{a_1}^\dagger(\vec{x}_1)\psi_{a_2}^\dagger(\vec{x}_2)|0_\mathcal{D}\rangle ,$$

the Schrödinger equation is equivalent to the above-mentioned result (eq. (8)) for the case $\omega = 2$.

The probability density to observe the universe in a configuration $(\vec{x}_1,\ldots,\vec{x}_\omega)$ is

$$\rho(t,\vec{x}_1,\ldots,\vec{x}_\omega) = \Psi_{a_1\cdots a_\omega}^*(t,\vec{x}_1,\cdots,\vec{x}_\omega)\Psi_{a_1\cdots a_\omega}(t,\vec{x}_1,\cdots,\vec{x}_\omega) . \tag{10}$$

With the help of equation (8), it can be shown that we have the following continuity equation

$$\frac{\partial\rho(t,\vec{X})}{\partial t} + \sum_{k=1}^{k=\omega}\vec{\nabla}_{\vec{x}_k}\cdot\vec{j}_k(t,\vec{X}) = 0 , \tag{11}$$

where

$$\vec{j}_k(t,\vec{x}_1,\ldots,\vec{x}_\omega) = \Psi_{a_1\cdots a_\omega}^*(t,\vec{x}_1,\cdots,\vec{x}_\omega)\vec{\alpha}_{a_j a}\Psi_{a_1\cdots(a_j\to a)\cdots a_\omega}(t,\vec{x}_1,\cdots\vec{x}_\omega) .$$

Let us go on with the proof: with the help of eq. (8), the time-derivative

of $\rho(t, \vec{X})$ (eq. (10)) is equal to

$$\dot\rho(t, \vec{X}) = -\Psi^*_{a_1\cdot\cdot a_\omega}(t, \vec{X}) \sum_{j=1}^{j=\omega} [\vec\alpha_{a_j a} \cdot \vec\nabla_{\vec{x}_j} + im\beta_{a_j a}] \Psi_{a_1\cdot\cdot(a_j\to a)\cdot\cdot a_\omega}(t, \vec{X})$$

$$-\sum_{j=1}^{j=\omega} \vec\nabla_{\vec{x}_j} \Psi^*_{a_1\cdot\cdot(a_j\to a)\cdot\cdot a_\omega}(t, \vec{X}) \cdot (\vec\alpha_{a_j a})^* \Psi_{a_1\cdot\cdot a_\omega}(t, \vec{X})$$

$$+im\sum_{j=1}^{j=\omega} \Psi^*_{a_1\cdot\cdot(a_j\to a)\cdot\cdot a_\omega}(t, \vec{X})(\beta_{a_j a})^* \Psi_{a_1\cdot\cdot a_\omega}(t, \vec{X})$$

Noting that $(\vec\alpha_{a_j a})^* = (\vec\alpha^\dagger)_{a a_j} = \vec\alpha_{a a_j}$, that $(\beta_{a_j a})^* = (\beta^\dagger)_{a a_j} = \beta_{a a_j}$, and renaming the dummy indices, we see that the terms containing a factor m cancel each other out. So there remains

$$\dot\rho(t, \vec{X}) = -\Psi^*_{a_1\cdot\cdot a_\omega}(t, \vec{X}) \sum_{j=1}^{j=\omega} [\vec\alpha_{a_j a} \cdot \vec\nabla_{\vec{x}_j}] \Psi_{a_1\cdot\cdot(a_j\to a)\cdot\cdot a_\omega}(t, \vec{X})$$

$$-\sum_{j=1}^{j=\omega} \vec\nabla_{\vec{x}_j} \Psi^*_{a_1\cdot\cdot(a_j\to a)\cdot\cdot a_\omega}(t, \vec{X}) \cdot \vec\alpha_{a a_j} \Psi_{a_1\cdot\cdot a_\omega}(t, \vec{X}) \ ,$$

and the conclusion (eq. (11)) follows directly.

All this expressions are totally similar to those found in the non-relativistic pilot-wave theory. The positions of the fermions (electrons of positive and negative energy) play the role of the positions of the particles in the non-relativistic theory. Their number is conserved in both cases; that leads to continuity equations, eq. (11) being the analog of eq. (1). In the Bell model of the free Dirac quantum field theory, the universe is described by a pilot-state, solution of the Schrödinger equation, and by a point $(\vec{x}_1(t), \ldots, \vec{x}_\omega(t))$. Proceeding just as in the non-relativistic pilot-wave theory (see eq. (4)), it is easy to see that if the kth fermion is moving according to the velocity-law

$$\vec{v}_k(t) = \frac{\Psi^*_{a_1\cdot\cdot a_\omega}(t, \vec{x}_1, \cdot\cdot, \vec{x}_\omega)\vec\alpha_{a_k a}\Psi_{a_1\cdot\cdot(a_k\to a)\cdot\cdot a_\omega}(t, \vec{x}_1, \cdot\cdot \vec{x}_\omega)}{\Psi^*_{b_1\cdot\cdot b_\omega}(t, \vec{x}_1, \cdot\cdot, \vec{x}_\omega)\Psi_{b_1\cdot\cdot b_\omega}(t, \vec{x}_1, \cdot\cdot, \vec{x}_\omega)} \ ,$$

then all the predictions of the orthodox quantum field theory are regained. Of course, for any physical state (any state obtained from $|0\rangle$ by creating a finite number of electrons and positrons), ω is infinite. But

the number of fermions ω is countable if the volume of the universe is finite. Another remark is that is also totally equivalent to say that the charge-density is the element of reality.

6   A deterministic Bell model for quantum electrodynamics

The complete hamiltonian for quantum electrodynamics is

$$H = H_{\mathcal{D}} + H_{\Gamma} - e \int d^3\vec{x}\, \bar{\psi}(\vec{x})\gamma^\mu \psi(\vec{x}) A_\mu(\vec{x}) \ ,$$

where $H_{\mathcal{D}}$ is the free Dirac hamiltonian and H_{Γ} is the free photon hamiltonian. The pilot-state is defined as the solution of the Schrödinger equation

$$i\frac{\partial |\Psi(t)\rangle}{\partial t} = H|\Psi(t)\rangle \ .$$

Since $[H, F] = 0$, and according to the charge super selection rule, the pilot-state is an eigenstate of the fermion-number; let us define the corresponding eigenvalue by ω

$$F|\Psi(t)\rangle = \omega|\Psi(t)\rangle \ .$$

A complete basis for that particular class of states is then given by

$$|\gamma\rangle \otimes \psi_{a_1}^\dagger(\vec{x}_1)\ldots \psi_{a_\omega}^\dagger(\vec{x}_\omega)|0_{\mathcal{D}}\rangle$$
$$\vec{x}_1, \ldots, \vec{x}_\omega \in \mathbb{R}^3 \quad a_1, \ldots, a_\omega \in \{1, 2, 3, 4\} \quad \gamma \in \{\gamma\} \ ,$$

where the set of all $|\gamma\rangle$ form a complete basis of the photons Fock space and where $\psi_{a_1}^\dagger(\vec{x}_1)\ldots \psi_{a_\omega}^\dagger(\vec{x}_\omega)|0_{\mathcal{D}}\rangle$ is an eigenstate of the fermion density with fermion-number equal to ω. Hence we have that

$$|\Psi(t)\rangle = \sum_{a_1=1}^{a_1=4} \cdots \sum_{a_\omega=1}^{a_\omega=4} \sum_\gamma \int d^3\vec{x}_1 \cdots d^3\vec{x}_\omega \Psi_{a_1 \ldots a_\omega}^\gamma(t, \vec{x}_1, \ldots, \vec{x}_\omega)$$
$$|\gamma\rangle \otimes \psi_{a_1}^\dagger(\vec{x}_1)\ldots \psi_{a_\omega}^\dagger(\vec{x}_\omega)|0_{\mathcal{D}}\rangle \ .$$

Inserting that expression in the Schrödinger equation, it is possible to show that the equation satisfied by $\Psi^\gamma_{a_1 \cdots a_\omega}(t, \vec{x}_1, \dots, \vec{x}_\omega)$ is

$$i\frac{\partial \Psi^\gamma_{a_1 \cdots a_\omega}(t, \vec{X})}{\partial t} = \sum_{j=1}^{j=\omega}[-i\vec{\alpha}_{a_j a} \cdot \vec{\nabla}_{\vec{x}_j} + m\beta_{a_j a}]\Psi^\gamma_{a_1 \cdots (a_j \to a) \cdots a_\omega}(t, \vec{X})$$

$$- \sum_{j=1}^{j=\omega} \sum_{\gamma'} \langle\gamma|e(\beta\gamma^\mu)_{a_j a} A_\mu(\vec{x}_j)|\gamma'\rangle \Psi^{\gamma'}_{a_1 \cdots (a_j \to a) \cdots a_\omega}(t, \vec{X})$$

$$+ \sum_{\gamma'} \langle\gamma|H_\Gamma|\gamma'\rangle \Psi^{\gamma'}_{a_1 \cdots a_\omega}(t, \vec{X}) . \tag{12}$$

The probability density to observe the system in a configuration $\vec{X}$ at time t is

$$\rho(t, \vec{X}) = \Psi^{\gamma*}_{a_1 \cdots a_\omega}(t, \vec{X})\Psi^\gamma_{a_1 \cdots a_\omega}(t, \vec{X})$$

(sum over any repeated index). Let us calculate $\dot\rho(t, \vec{X})$ with the help of eq. (12). We have that $\dot\rho(t, \vec{X}) =$

$$- \Psi^{\gamma*}_{a_1 \cdots a_\omega}(t, \vec{X}) \sum_{j=1}^{j=\omega}[\vec{\alpha}_{a_j a} \cdot \vec{\nabla}_{\vec{x}_j} + im\beta_{a_j a}]\Psi^\gamma_{a_1 \cdots (a_j \to a) \cdots a_\omega}(t, \vec{X})$$

$$- \sum_{j=1}^{j=\omega} \vec{\nabla}_{\vec{x}_j} \Psi^{\gamma*}_{a_1 \cdots (a_j \to a) \cdots a_\omega}(t, \vec{X}) \cdot (\vec{\alpha}_{a_j a})^* \Psi^\gamma_{a_1 \cdots a_\omega}(t, \vec{X})$$

$$+ im \sum_{j=1}^{j=\omega} \Psi^{\gamma*}_{a_1 \cdots (a_j \to a) \cdots a_\omega}(t, \vec{X})(\beta_{a_j a})^* \Psi^\gamma_{a_1 \cdots a_\omega}(t, \vec{X})$$

$$+ i \sum_{\gamma\gamma'} \Psi^{\gamma*}_{a_1 \cdots a_\omega}(t, \vec{X}) \sum_{j=1}^{j=\omega} \langle\gamma|e(\beta\gamma^\mu)_{a_j a} A_\mu(\vec{x}_j)|\gamma'\rangle \Psi^{\gamma'}_{a_1 \cdots (a_j \to a) \cdots a_\omega}(t, \vec{X})$$

$$- i \sum_{\gamma\gamma'} \Psi^{\gamma*}_{a_1 \cdots a_\omega}(t, \vec{X})\langle\gamma|H_\Gamma|\gamma'\rangle \Psi^{\gamma'}_{a_1 \cdots (a_j \to a) \cdots a_\omega}(t, \vec{X})$$

$$- i \sum_{\gamma\gamma'} \sum_{j=1}^{j=\omega} \Psi^{\gamma'*}_{a_1 \cdots (a_j \to a) \cdots a_\omega}(t, \vec{X})\langle\gamma'|e(\beta\gamma^\mu)^*_{a a_j} A^\dagger_\mu(\vec{x}_j)|\gamma\rangle \Psi^\gamma_{a_1 \cdots a_\omega}(t, \vec{X})$$

$$+ i \sum_{\gamma\gamma'} \Psi^{\gamma'*}_{a_1 \cdots (a_j \to a) \cdots a_\omega}(t, \vec{X})\langle\gamma'|H^\dagger_\gamma|\gamma\rangle \Psi^\gamma_{a_1 \cdots a_\omega}(t, \vec{X}) .$$

Noting that the matrix β is hermitian and that the dummy indices can be renamed, it is easy to check that the terms containing a factor m cancel each other out. In the same way, on account of the hermiticity of H_Γ, $A_\mu^\dagger(\vec{x})$ and $(\beta\gamma^\mu)$, and due to the possible renaming of the dummy indices, the terms containing photon fields cancel each other out, so that

$$\dot{\rho}(t,\vec{X}) = -\Psi_{a_1\cdots a_\omega}^{\gamma *}(t,\vec{X}) \sum_{j=1}^{j=\omega} [\vec{\alpha}_{a_j a} \cdot \vec{\nabla}_{\vec{x}_j}] \Psi_{a_1\cdots(a_j\to a)\cdots a_\omega}^{\gamma}(t,\vec{X})$$

$$-\sum_{j=1}^{j=\omega} \vec{\nabla}_{\vec{x}_j} \Psi_{a_1\cdots(a_j\to a)\cdots a_\omega}^{\gamma *}(t,\vec{X}) \cdot \vec{\alpha}_{a a_j} \Psi_{a_1\cdots a_\omega}^{\gamma}(t,\vec{X}) .$$

That last equation can be rewritten as

$$\frac{\partial \rho(t,\vec{X})}{\partial t} + \sum_{k=1}^{k=\omega} \vec{\nabla}_{\vec{x}_k} \cdot \vec{j}_k(t,\vec{X}) = 0 ,$$

where

$$\vec{j}_k(t,\vec{x}_1,\ldots,\vec{x}_\omega) = \Psi_{a_1\cdots a_\omega}^{\gamma *}(t,\vec{x}_1,\cdots,\vec{x}_\omega)\vec{\alpha}_{a_j a}\Psi_{a_1\cdots(a_j\to a)\cdots a_\omega}^{\gamma}(t,\vec{x}_1,\cdots\vec{x}_\omega) .$$

Then, if the kth fermion moves with velocity

$$\vec{v}_k(t) = \left.\frac{\Psi_{a_1\cdots a_\omega}^{\gamma *}(t,\vec{X})\vec{\alpha}_{a_k a}\Psi_{a_1\cdots(a_k\to a)\cdots a_\omega}^{\gamma}(t,\vec{X})}{\Psi_{b_1\cdots b_\omega}^{\gamma *}(t,\vec{X})\Psi_{b_1\cdots b_\omega}^{\gamma}(t,\vec{X})}\right|_{\vec{X}=\vec{X}(t)} ,$$

all the predictions of the orthodox interpretation of quantum electrodynamics are regained, if any measurement amounts to a measurement of the charge density.

7   Conclusion and open questions

We hope to have shown that it is possible to have a picture of what is going on in quantum electrodynamics, by taking the fermion-number density, or charge density, as an element of reality. Of course, this is still a work in progress. The point of view we have adopted is not convenient for practical calculations, but that was not the goal we were pursuing.

A line of research that should be followed is the study of the non-relativistic limit of the Bell model for quantum electrodynamics, in order to show if we obtain the non-relativistic pilot-wave theory in that

limit. Another line of research is the application of these ideas to bosonic quantum field theories. On one hand, in the Klein-Gordon quantum field theory, there is no state annihilated by a charge annihilator, so it is an argument against the Bell model for the Klein-Gordon quantum field theory, at least in the present form of the Bell model. On the other hand, charge is conserved, so we expect a continuity equation for the charge. Perhaps that the solution will come by looking at things locally.

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A   Calculation of the commutator $[N, \psi^\dagger(\vec{x})\psi(\vec{x})]$

We want to show that

$$[N, \psi^\dagger(\vec{x})\psi(\vec{x})] \neq 0 , \tag{13}$$

where

$$N = \sum_r \int d^3\vec{k}[c_r^\dagger(\vec{k})c_r(\vec{k}) + d_r^\dagger(\vec{k})d_r(\vec{k})] , \tag{14}$$

even we think about the fields as distributions. We use the following relation (F stands for fermion):$[F_1 F_2, F_3 F_4]$

$$=F_1[F_2, F_3 F_4] + [F_1, F_3 F_4]F_2$$
$$=F_1\{F_2, F_3\}F_4 - F_1 F_3\{F_2, F_4\} + \{F_1, F_3\}F_4 F_2 - F_3\{F_1, F_4\}F_2 .$$

Let us recall the expressions of the spinor fields:

$$\psi(\vec{x}) = \sqrt{\frac{1}{(2\pi)^3}} \sum_s \int d^3\vec{p} \sqrt{\frac{m}{E_{\vec{p}}}}[u_s(\vec{p})e^{i\vec{p}\cdot\vec{x}}c_s(\vec{p}) + v_s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}d_s^\dagger(\vec{p})]$$

$$\psi^\dagger(\vec{x}) = \sqrt{\frac{1}{(2\pi)^3}} \sum_s \int d^3\vec{p} \sqrt{\frac{m}{E_{\vec{p}}}}[u_s^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}c_s^\dagger(\vec{p}) + v_s^\dagger(\vec{p})e^{i\vec{p}\cdot\vec{x}}d_s(\vec{p})] .$$

By using the anti-commutation relations

$$\{c_s(\vec{k}), c_r^\dagger(\vec{p})\} = \delta_{sr}\delta^3(\vec{k} - \vec{p}) \qquad \{d_s(\vec{k}), d_r^\dagger(\vec{p})\} = \delta_{sr}\delta^3(\vec{k} - \vec{p}) ,$$

and all other anti-commutators vanishing, we find that

$$\{\psi_a^\dagger(\vec{x}), c_r(\vec{k})\} = \sqrt{\frac{1}{(2\pi)^3}}\sqrt{\frac{m}{E_{\vec{k}}}}u_{ar}^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}} \quad \{\psi_a(\vec{x}), c_r(\vec{k})\} = 0$$

$$\{\psi_a(\vec{x}), c_r^\dagger(\vec{k})\} = \sqrt{\frac{1}{(2\pi)^3}}\sqrt{\frac{m}{E_{\vec{k}}}}u_{ar}(\vec{k})e^{i\vec{k}\cdot\vec{x}} \quad \{\psi_a^\dagger(\vec{x}), c_r^\dagger(\vec{k})\} = 0$$

$$\{\psi_a(\vec{x}), d_r(\vec{k})\} = \sqrt{\frac{1}{(2\pi)^3}}\sqrt{\frac{m}{E_{\vec{k}}}}v_{ar}(\vec{k})e^{-i\vec{k}\cdot\vec{x}} \quad \{\psi_a^\dagger(\vec{x}), d_r(\vec{k})\} = 0$$

$$\{\psi_a^\dagger(\vec{x}), d_r^\dagger(\vec{k})\} = \sqrt{\frac{1}{(2\pi)^3}}\sqrt{\frac{m}{E_{\vec{k}}}}v_{ar}^\dagger(\vec{k})e^{i\vec{k}\cdot\vec{x}} \quad \{\psi_a(\vec{x}), d_r^\dagger(\vec{k})\} = 0 ,$$

so that $[\psi_a^\dagger(\vec{x})\psi_a(\vec{x}), \sum_r \int d^3\vec{k}c_r^\dagger(\vec{k})c_r(\vec{k})]$

$$= \sum_r \int d^3\vec{k}\big(\psi_a^\dagger(\vec{x})\{\psi_a(\vec{x}), c_r^\dagger(\vec{k})\}c_r(\vec{k}) - c_r^\dagger(\vec{k})\{\psi_a^\dagger(\vec{x}), c_r(\vec{k})\}\psi_a(\vec{x})\big)$$

$$= \frac{m^2}{(2\pi)^3}\sum_{s,r}\int \frac{d^3\vec{p}\,d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}}[u_s^\dagger(\vec{p})u_r(\vec{k})e^{-i(\vec{p}-\vec{k})\cdot\vec{x}}c_s^\dagger(\vec{p})c_r(\vec{k})$$

$$+ v_s^\dagger(\vec{p})u_r(\vec{k})e^{i(\vec{p}+\vec{k})\cdot\vec{x}}d_s(\vec{p})c_r(\vec{k}) - u_r^\dagger(\vec{k})u_s(\vec{p})e^{i(\vec{p}-\vec{k})\cdot\vec{x}}c_r^\dagger(\vec{k})c_s(\vec{p})$$

$$- u_r^\dagger(\vec{k})v_s(\vec{p})e^{-i(\vec{p}+\vec{k})\cdot\vec{x}}c_r^\dagger(\vec{k})d_s^\dagger(\vec{p})]\ .$$

Since r, s, $\vec{p}$ and $\vec{k}$ are dummy variables, we find that

$$[\psi_a^\dagger(\vec{x})\psi_a(\vec{x}), \sum_r \int d^3\vec{k}c_r^\dagger(\vec{k})c_r(\vec{k})] = \frac{m^2}{(2\pi)^3}\sum_{s,r}$$

$$\int \frac{d^3\vec{p}\,d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}}[v_s^\dagger(\vec{p})u_r(\vec{k})e^{i(\vec{p}+\vec{k})\cdot\vec{x}}d_s(\vec{p})c_r(\vec{k})]-$$

$$\frac{m^2}{(2\pi)^3}\sum_{s,r}\int \frac{d^3\vec{p}\,d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}}[u_r^\dagger(\vec{k})v_s(\vec{p})e^{-i(\vec{p}+\vec{k})\cdot\vec{x}}c_r^\dagger(\vec{k})d_s^\dagger(\vec{p})]\ .$$

In the same way, we obtain $[\psi_a^\dagger(\vec{x})\psi_a(\vec{x}), \sum_r \int d^3\vec{k}d_r^\dagger(\vec{k})d_r(\vec{k})]$

$$= \sum_r \int d^3\vec{k}\big(-\psi_a^\dagger(\vec{x})d_r^\dagger(\vec{k})\{\psi_a(\vec{x}), d_r(\vec{k})\} + \{\psi_a^\dagger(\vec{x}), d_r^\dagger(\vec{k})\}d_r(\vec{k})\psi_a(\vec{x})\big)$$

$$= \frac{m^2}{(2\pi)^3}\sum_{s,r}\int \frac{d^3\vec{p}\,d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}}[-u_s^\dagger(\vec{p})v_r(\vec{k})e^{-i(\vec{p}+\vec{k})\cdot\vec{x}}c_s^\dagger(\vec{p})d_r^\dagger(\vec{k})$$

$$- v_s^\dagger(\vec{p})v_r(\vec{k})e^{i(\vec{p}-\vec{k})\cdot\vec{x}}d_s(\vec{p})d_r^\dagger(\vec{k}) + v_r^\dagger(\vec{k})u_s(\vec{p})e^{i(\vec{p}+\vec{k})\cdot\vec{x}}d_r(\vec{k})c_s(\vec{p})$$

$$+ v_r^\dagger(\vec{k})v_s(\vec{p})e^{-i(\vec{p}-\vec{k})\cdot\vec{x}}d_r(\vec{k})d_s^\dagger(\vec{p})]\ .$$

This can be simplified to

$$[\psi_a^\dagger(\vec{x})\psi_a(\vec{x}), \sum_r \int d^3\vec{k}d_r^\dagger(\vec{k})d_r(\vec{k})] =$$

$$-\frac{m^2}{(2\pi)^3}\sum_{s,r}\int \frac{d^3\vec{p}\,d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}}[u_s^\dagger(\vec{p})v_r(\vec{k})e^{-i(\vec{p}+\vec{k})\cdot\vec{x}}c_s^\dagger(\vec{p})d_r^\dagger(\vec{k})]+$$

$$\frac{m^2}{(2\pi)^3}\sum_{s,r}\int \frac{d^3\vec{p}\,d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}}[v_r^\dagger(\vec{k})u_s(\vec{p})e^{i(\vec{p}+\vec{k})\cdot\vec{x}}d_r(\vec{k})c_s(\vec{p})]\ .$$

Putting the two results together, we get

$$[\psi_a^\dagger(\vec{x})\psi_a(\vec{x}), N] =$$

$$-\frac{2m^2}{(2\pi)^3}\sum_{s,r}\int\frac{d^3\vec{p}d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}}[u_s^\dagger(\vec{p})v_r(\vec{k})e^{-i(\vec{p}+\vec{k})\cdot\vec{x}}c_s^\dagger(\vec{p})d_r^\dagger(\vec{k})]+$$

$$\frac{2m^2}{(2\pi)^3}\sum_{s,r}\int\frac{d^3\vec{p}d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}}[v_r^\dagger(\vec{k})u_s(\vec{p})e^{i(\vec{p}+\vec{k})\cdot\vec{x}}d_r(\vec{k})c_s(\vec{p})] \ ,$$

which is not equal to zero, even if we think about fields as distributions. If we start from the state $d_s^\dagger(p_0)c_s^\dagger(p_0)|0\rangle$, it is clear that there are well-behaved functions f such that

$$\langle 0|\int d^3\vec{x}f(\vec{x})[\psi^\dagger(\vec{x})\psi(\vec{x}), N]|d_s^\dagger(p_0)c_s^\dagger(p_0)|0\rangle \neq 0 \ .$$

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