

# Electromagnetic vacuum fluctuations, Casimir and Van der Waals forces

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**ABSTRACT.** Electromagnetic vacuum fluctuations have observable consequences, like the Casimir force between mirrors in vacuum. This force is now measured with good accuracy and agreement with theory when the effect of imperfect reflection of mirrors is properly taken into account. We discuss the simple case of bulk metallic mirrors described by a plasma model and show that simple scaling laws are obtained at the limits of long and short distances. The crossover between the short and long-distance laws is quite similar to the crossover between Van der Waals and Casimir-Polder forces for two atoms in vacuum. The result obtained at short distances can be understood as the London interaction between plasmon excitations at the surface of each bulk mirror.

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## 1 Introduction

An important prediction of quantum theory is the existence of irreducible fluctuations of electromagnetic fields even in vacuum, that is in the thermodynamical equilibrium state with a zero temperature. These fluctuations have a number of observable consequences in microscopic physics for example in atomic physics the Van der Waals force between atoms in vacuum.

Vacuum fluctuations also have observable mechanical effects in macroscopic physics and the archetype of these effects is the Casimir force between two mirrors at rest in vacuum. This force was predicted by H. Casimir in 1948 [1] and soon observed in different experiments which

confirmed its existence [2, 3, 4, 5]. Recent experiments have reached a good precision, in the % range, which makes possible an accurate comparison between theoretical predictions and experimental observations [6, 7].

Such a comparison is important for at least two reasons. On one hand, the Casimir force is a mechanical consequence of vacuum fluctuations which raise a serious challenge at the frontier of quantum theory with the physics of gravitation; it is therefore important to test this prediction with the greatest care and accuracy. On the other hand, several experiments are searching for hypothetical new forces predicted by the models of unification of fundamental forces and the main target of these experiments is now the distance range between the nanometer and millimeter; in this distance range, the Casimir force is the dominant interaction between neutral objects so that these experiments are essentially limited by the accuracy in the knowledge of the Casimir force. References on these topics as well as further discussions of the motivations for studying the Casimir force may be found in [8, 9].

Casimir considered an ideal configuration with two perfectly reflecting mirrors in vacuum. But the experiments are performed with real reflectors, for example metallic mirrors which show perfect reflection only at frequencies below the plasma frequency characterizing the metal. Accounting for imperfect reflection and its frequency dependence is thus essential for obtaining a reliable theoretical expectation of the Casimir force in a real situation. This is also true for other corrections to the ideal Casimir formula associated with the experimental configuration: experiments are performed at room temperature, with the effect of thermal fluctuations superimposed to that of vacuum fluctuations; in most experiments, the force is measured between a plane and a sphere, and not between two parallel planes; also, the surface state of the plates, in particular their roughness, should affect the force. Here, we will focus our attention on the evaluation of the Casimir force between two plane mirrors with arbitrary frequency dependent reflection amplitudes and ignore the other corrections associated with the effect of a non zero temperature or a non plane geometry.

The consideration of real mirrors is important not only for the analysis of experiments but also from a conceptual point of view. Real mirrors are certainly transparent at the limit of high frequencies and this allows one to dispose of the divergences associated with the infiniteness of vacuum energy. This point was already alluded to in Casimir's papers and

an important step in this direction was the Lifshitz theory of the Casimir force between two dielectric bulks [10, 11]. Here we will use the general expression of the Casimir force obtained for two plane mirrors characterized by arbitrary frequency dependent reflection amplitudes [12]. This expression is directly associated with an interpretation of the force as resulting from the radiation pressure exerted by vacuum fluctuations upon the two mirrors which form a Fabry-Perot cavity. The balance between repulsive and attractive contributions associated with resonant and antiresonant frequencies gives the net Casimir force. This method always leads to a finite result as a consequence of the causality properties and high-frequency transparency obeyed by any real mirror. In other words, the properties of real mirrors are sufficient to obtain a regular expression of the Casimir force, in spite of the infiniteness of vacuum energy.

In the next section, we will first recall this general expression of the Casimir force between real mirrors and briefly discuss its application to theory-experiment comparison. We will then focus our attention on the simplest model of metallic mirrors, namely bulk mirrors with the optical response of metals described by the plasma model. This model is not sufficient for an accurate evaluation of the Casimir force but its simplicity allows one to discuss qualitatively a lot of interesting physical features. In particular, we will discuss the analogies and differences between the results obtained with this model for the Casimir force between macroscopic mirrors and for the Casimir-Polder force between two atoms in vacuum. In both cases, simple power laws are obtained at the limits of long and short distances. The indices of these power laws are different in the two cases but the crossovers between short and long-distance laws present profound analogies. Let us briefly mention the convention we usually use to name the different interaction ranges (there is no overall accepted denomination yet). While we name the long range interaction between atoms usually Casimir-Polder force, as Casimir and Polder were the first to derive the correct expression in the retarded interaction limit, we call the short distance limit of the interaction between atomic bodies the van der Waals or London interaction, after the two physicists the first of whom predicted phenomenologically while the second derived quantum mechanically the correct interaction law.

Furthermore, we will show that the Casimir force at short distances can be understood as the London interaction between the elementary excitations of both scatterers, that is the surface plasmons of the two bulk mirrors. These surface plasmons are oscillating electromagnetic

fields, strongly localized at the surface of a metal (evanescent waves) and associated with the collective motion of electrons.

Before entering this discussion, we may emphasize that the existence of vacuum fluctuations alters the physical conception of empty space : in contrast to classical physics, quantum theory forces us to consider vacuum as permanently filled by field fluctuations having observable effects. Van der Waals and Casimir forces are nothing but the effects of vacuum radiation pressure on microscopic or macroscopic objects at rest. Vacuum radiation pressure also induces dynamical effects for objects moving in vacuum and these effects are directly connected to the problem of relativity of motion. Discussions of these dynamical effects and references may be found in [13].

## 2 Casimir force between real mirrors

We now come to a more precise discussion of the Casimir force. Casimir calculated this force in a geometrical configuration where two plane mirrors are placed a distance  $L$  apart from each other, parallel to each other, the area  $A$  of the mirrors being much larger than the squared distance ( $A \gg L^2$ ). Considering the ideal case of perfectly reflecting mirrors Casimir obtained the following expression for the force

$$F_{\text{Cas}} = \frac{\hbar c \pi^2 A}{240 L^4} \quad (1)$$

We have chosen the sign convention found in most papers on Casimir force with a positive value of  $F_{\text{Cas}}$  corresponding to an attraction, that is also a negative pressure.

This ideal Casimir formula only depends on geometrical quantities  $A$  and  $L$  and on two fundamental constants, the speed of light  $c$  and Planck constant  $\hbar$ . This remarkable universal feature corresponds to the fact that the optical response of perfect mirrors is saturated : mirrors cannot reflect more than 100 % of the incoming light, whatever their atomic constitution may be. This makes an important difference between ideal Casimir forces and the Van der Waals forces, discussed below, which depend on atomic polarizabilities. Now experiments are performed with metallic mirrors which do not reflect all field frequencies perfectly. This has certainly to be taken into account in the comparison between theoretical estimations and experimental measurements. This also entails that the Casimir force between real mirrors depends on the

atomic structure constants which determine the optical properties of the latter.

Imperfectly reflecting mirrors will be described by scattering amplitudes which depend on the frequency, wavevector and polarization while obeying general properties of stability, high-frequency transparency and causality. The two mirrors form a Fabry-Perot cavity with the consequences well-known in classical or quantum optics : the energy density of the intracavity field is increased for the resonant frequency components whereas it is decreased for the non resonant ones. The Casimir force is but the result of the balance between the radiation pressure of the resonant and non resonant modes which push the mirrors respectively towards the outer and inner sides of the cavity [12]. This balance includes not only the contributions of ordinary waves propagating freely outside the cavity but also that of evanescent waves. These two sectors of ordinary and evanescent waves are directly connected by analyticity properties of the scattering amplitudes (see a more precise discussion below).

The Casimir force may then be written as an integral over frequencies  $\omega$ , transverse wavevectors  $\mathbf{k}$  and a sum over polarizations  $p$  of the vacuum field modes. Due to the analyticity properties, the integral may be written also over imaginary frequencies  $\omega = i\xi$  (with  $\xi$  real)

$$F = \hbar A \sum_p \int \frac{d^2\mathbf{k}}{4\pi^2} \int_0^\infty \frac{d\xi}{2\pi} 2\kappa \frac{\rho_{\mathbf{k}}^p [i\xi]}{1 - \rho_{\mathbf{k}}^p [i\xi]}$$

$$\rho_{\mathbf{k}}^p [i\xi] = r_{\mathbf{k},1}^p [i\xi] r_{\mathbf{k},2}^p [i\xi] e^{-2\kappa L} \quad , \quad \kappa = \sqrt{\mathbf{k}^2 + \frac{\xi^2}{c^2}} \quad (2)$$

In this expression,  $\rho$  represents the multiplication factor for the field after a round trip in the cavity : it is the product of the reflection amplitudes  $r_1$  and  $r_2$  of the two mirrors and of an exponential phase factor. The fraction  $\frac{\rho}{1-\rho}$  is the sum of similar factors over the number of round trips inside the Fabry-Perot cavity

$$\frac{\rho}{1-\rho} = \rho + \rho^2 + \rho^3 + \dots \quad (3)$$

In other words,  $\rho$  is the ‘open loop function’ associated with the cavity while  $\frac{\rho}{1-\rho}$  is the ‘closed loop function’ taking into account the feedback; all these quantities are evaluated at imaginary frequencies through an analytical prolongation from their values at real frequencies.

Expression (2) holds for dissipative mirrors as well as for non dissipative ones [14]. It is regular for any frequency dependence of the reflection amplitudes obeying natural physical conditions : causality of the amplitudes and high-frequency transparency for each mirror, stability of the closed loop function associated with the Fabry-Perot cavity. Expression (2) tends towards the ideal Casimir formula (1) as soon as the mirrors are nearly perfect for the modes contributing to the integral. Incidentally, these results show that the Casimir force between two plane mirrors can be evaluated without any renormalization technique : as guessed by Casimir in his original paper, the properties of real mirrors themselves are sufficient to enforce regularity of the Casimir force.

The condition under which the ideal Casimir result is approached will be specified below for bulk mirrors described by the plasma model. More generally, the reduction of the Casimir force (2) with respect to the ideal formula (1) due to the imperfect reflection of mirrors is described by a factor

$$\eta_F = \frac{F}{F_{\text{Cas}}} \quad (4)$$

We may proceed analogously for discussing the Casimir energy between real mirrors, evaluated by integrating the force

$$E = - \int_L^\infty F(L') \, dL' \quad (5)$$

For the energy, we use the standard convention of thermodynamics, with a negative value corresponding to a binding energy <sup>1</sup>. The force (1) between perfect mirrors is thus translated to

$$E_{\text{Cas}} = - \frac{\hbar c \pi^2 A}{720 L^3} \quad (6)$$

Meanwhile, the force (2) between real mirrors corresponds to the energy

$$E = A \sum_p \int \frac{d^2 \mathbf{k}}{4\pi^2} \int_0^\infty \frac{d\xi}{2\pi} \hbar \log(1 - \rho_{\mathbf{k}}^p[i\xi]) \quad (7)$$

This energy has its absolute value reduced by the effect of imperfect reflection and the reduction is conveniently described by a factor

$$\eta_E = \frac{E}{E_{\text{Cas}}} \quad (8)$$

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<sup>1</sup>This convention is opposite to that used in our papers quoted in the list of references but it is better adapted to the forthcoming discussions.

This factor plays an important role in the discussion of the most precise recent experiments.

In these experiments, the force has been measured between a sphere and a plane and not between two plane parallel mirrors. The Casimir force in this geometry is estimated from the proximity force approximation [15] which amounts to integrate the contributions of the various inter-plate distances as if they were independent. Although the accuracy of this approximation remains to be mastered, it is usually thought to give a reliable approximation. The force in the sphere-plane geometry is then given by the radius  $R$  of the sphere and by the Casimir energy as evaluated in the plane-plane configuration for the distance  $L$  of closest approach

$$F_{\text{sphere-plane}} = \frac{2\pi R}{A} |E_{\text{plane-plane}}| \quad (9)$$

Using the results (6-8) obtained in the plane-plane geometry, one finally obtains the Casimir force in the sphere-plane geometry

$$F_{\text{sphere-plane}} = \frac{\hbar c \pi^3 R}{360 L^3} \eta_E \quad (10)$$

This theoretical expression is used for comparison with recent precise measurements of the force [6]. It accounts for the effects of imperfect reflection and plane-sphere geometry which have a significant impact on the value of the Casimir force. This is not the case for the effects of thermal fluctuations and surface roughness which only have a marginal influence ( $< 1\%$ ) in the same experiments. As a consequence, the good agreement ( $\simeq 1\%$ ) obtained in the theory-experiment comparison can be considered as a confirmation of the existence and properties of the Casimir force as well as a test of the corrections associated with imperfect reflection and plane-sphere geometry.

We want to emphasize that a precise description of the optical response of the mirrors is necessary to reach an accurate evaluation of the effect of imperfect reflection. If an accuracy of the order of 1% is aimed at, this description should take into account the knowledge of optical data of the metals on a wide frequency range [16]. In particular, the plasma model is not sufficient for such an accurate evaluation. As announced in the Introduction however, we will use this simple model in the following to get interesting results about the comparison of Casimir and Casimir-Polder forces.

### 3 Reflection on bulk mirrors described by the plasma model

For bulk mirrors, the reflection amplitudes are simply given by the Fresnel laws corresponding to the vacuum/metal interface with different expressions for the two polarizations TE and TM

$$r^{\text{TE}} = \frac{\kappa - \kappa_m}{\kappa + \kappa_m} \quad , \quad r^{\text{TM}} = \frac{\kappa_m - \varepsilon_m \kappa}{\kappa_m + \varepsilon_m \kappa}$$

$$\kappa_m = \sqrt{\mathbf{k}^2 + \varepsilon_m \frac{\xi^2}{c^2}} \quad , \quad \kappa = \sqrt{\mathbf{k}^2 + \frac{\xi^2}{c^2}} \quad (11)$$

$\varepsilon_m$  is the dielectric function of the metal given by the plasma model

$$\varepsilon_m [i\xi] = 1 + \frac{\omega_P^2}{\xi^2} \quad , \quad \lambda_P = \frac{2\pi c}{\omega_P} \quad (12)$$

with  $\omega_P$  the plasma frequency and  $\lambda_P$  the plasma wavelength;  $\kappa$  is the quantity already defined in (2);  $\kappa_m$  is the expression defined analogously in the metal with the dielectric function  $\varepsilon_m$ .

These expressions are well known but their analyticity properties deserve a special attention in the context of the present discussion. Causality entails that the reflection amplitudes are analytical functions of the frequency in the ‘physical domain’ defined in the complex plane by a positive real part for  $\xi$  [17]

$$\omega \equiv i\xi \quad , \quad \Re \xi > 0 \quad (13)$$

Analyticity has to be understood with  $\mathbf{k}$  and  $p$  fixed, the branch of the square roots being taken so that

$$\Re \kappa_m > 0 \quad , \quad \Re \kappa > 0 \quad (14)$$

The sectors of ordinary and evanescent waves lie on the boundary of this domain : they indeed correspond to real frequencies  $\omega$ , that is also to purely imaginary values for  $\xi$ . They are distinguished by the values of  $\kappa$  which are purely imaginary for ordinary waves ( $\omega \geq c|\mathbf{k}|$ ), but real for evanescent waves ( $\omega < c|\mathbf{k}|$ ). In the latter case,  $\kappa$  is just the inverse of the penetration length in vacuum of waves coming from the refracting medium.

As already alluded to, analyticity also connects these two sectors to the sector of imaginary frequencies ( $\xi$  real). Precisely, expressions



of the reflection amplitudes or of the loop functions in this sector are obtained from similar expressions written for real frequencies through an analytical continuation. This property was in fact used to write the force (2) and energy (7) as integrals over imaginary frequencies [14]. Note in particular that the exponential factor  $\exp(-2\kappa L)$  appearing in the loop function  $\rho$  describes the frustration of total reflection on each vacuum/metal interface due to the evanescent propagation of the field through the length  $L$  of the cavity. This explains why the radiation pressure of vacuum modes is not identical on the internal and external sides of each mirror and, therefore, why the Casimir force has finally a non null value.

It is also worth discussing in more details the modulus of the reflection amplitude which is expected to be smaller than unity as a consequence of unitarity of the scattering on a mirror

$$|r| \leq 1 \quad (15)$$

This property is certainly true in the sector of ordinary waves where it is effectively a direct consequence of unitarity, for lossy as well as lossless mirrors [14]. It is easily seen (11) that it is also true in the sector of imaginary frequencies where  $\kappa$ ,  $\kappa_m$  and  $\varepsilon_m$  are real and positive. But the case of evanescent waves requires a closer examination and the result of this examination depends on the polarization.

For TE waves, it follows from (14) that (15) holds also in the evanescent sector. It is then a consequence of the Phragmén-Lindelöf theorem [18] that (15) is true on the whole physical domain (13). Then the open loop function  $\rho$  also has its modulus smaller than unity in this domain which ensures stability of the closed loop gain  $\frac{\rho}{1-\rho}$ . This stability means that neither the mirrors forming the cavity nor the vacuum fields enclosed in this cavity have the ability to sustain the oscillation which would be associated with a pole of  $\frac{\rho}{1-\rho}$  in the physical domain [19]. For TM waves, this stability property is also true and no self-sustained oscillation of the Fabry-Perot cavity occurs. However, (15) does not hold in the evanescent sector since the reflection amplitudes reach large values corresponding to the existence of surface plasmon resonances (see a more precise discussion below). This means that the closed loop amplitude is still stable although the open loop amplitude has a modulus larger than unity. In more mathematical terms, the oscillation condition  $\rho = 1$ , which corresponds to a pole of the closed loop function  $\frac{\rho}{1-\rho}$ , is

never met in the physical domain (13) although  $|\rho|$  may exceed unity in this domain.

This discussion can be made more precise for bulk mirrors described by the plasma model. To this aim, we first consider reflection of TM waves on one such mirror. The corresponding reflection amplitude  $r^{\text{TM}}$  given by (11) is seen to diverge when  $\kappa_m + \varepsilon_m \kappa = 0$ , which defines the surface plasmon resonance condition [20]. With the plasma model, this condition can be solved as an expression for the frequency in terms of the transverse wavevector <sup>2</sup>

$$\omega_{\text{plasmon}}^2 = \frac{\omega_{\text{P}}^2 + 2c^2 \mathbf{k}^2 - \sqrt{\omega_{\text{P}}^4 + 4c^4 \mathbf{k}^4}}{2} \quad (16)$$

The surface plasmon frequency  $\omega_{\text{plasmon}}$  is real and lies on the boundary  $\Re\xi = 0$  of the physical domain (13). When dissipation is taken into account, for example by considering the Drude model instead of the plasma model, the pole of the closed loop amplitude is pushed from this boundary  $\Re\xi = 0$  into the unphysical domain  $\Re\xi < 0$ . It follows that the divergence encountered with the plasma model is transformed into a resonance. In the vicinity of this resonance, the modulus  $|r^{\text{TM}}|$  of the reflection amplitude exceeds unity but the closed loop function remains stable since its pole lies outside the physical domain.

When two mirrors are considered, their surface plasmons are coupled by evanescent propagation through the cavity so that their frequencies are displaced. Their displacement can be seen as responsible for the interaction between the two mirrors [21, 22]. This approach will be discussed in more detail below since it leads to an interesting interpretation of the Casimir force at short distances.

#### 4 Power laws in the limits of long and short distances

Using the simple mirror model of the preceding section, we now discuss the relation between the effect of imperfect reflection and the explored distance range. The evaluation of the reduction factor (4) for bulk mirrors described by the plasma model has been presented in [16, 23]. In this simple case, the reduced force  $\eta_{\text{F}}$  is a function of a single parameter, namely the reduced distance  $\frac{L}{\lambda_{\text{P}}}$ . We draw this function on Figure 1.

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<sup>2</sup>Note that the condition  $\kappa_m - \varepsilon_m \kappa = 0$ , differing by a mere sign, corresponds to a vanishing reflection amplitude and defines the Brewster frequency with the following expression  $\omega_{\text{Brewster}}^2 = \frac{\omega_{\text{P}}^2 + 2c^2 \mathbf{k}^2 + \sqrt{\omega_{\text{P}}^4 + 4c^4 \mathbf{k}^4}}{2}$ .

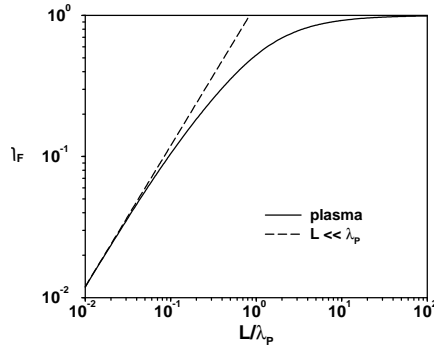


Figure 1: The solid curve represents the variation of the force reduction factor  $\eta_F$  as a function of the reduced distance  $\frac{L}{\lambda_P}$  :  $\eta_F$  goes to unity at large distances whereas it varies linearly with  $\frac{L}{\lambda_P}$  at short distances; the latter behaviour is represented by the dashed line.

We first note that the factor  $\eta_F$  gets close to unity at large distances

$$L \gg \lambda_P \quad , \quad \eta_F \simeq 1 \quad , \quad F \simeq \frac{\hbar c A \pi^2}{240 L^4} \quad (17)$$

This corresponds to a well known interpretation : large distances  $L \gg \lambda_P$  correspond to low frequencies  $\omega \ll \omega_P$  for which metallic mirrors described by the plasma model are nearly perfect reflectors; this is why the ideal Casimir formula (1) is a very good approximation of the real force (2) in this limit.

Otherwise, the factor  $\eta_F$  is smaller than unity and describes the reduction of the force due to the imperfect reflection of the metallic mirror at high frequencies. In the limiting case of distances small with respect to the plasma wavelength, the reduction becomes quite significant since the factor  $\eta_F$  varies linearly with the small factor  $\frac{L}{\lambda_P}$

$$L \ll \lambda_P \quad , \quad \eta_F \simeq \alpha \frac{L}{\lambda_P} \quad , \quad F \simeq \alpha \frac{\hbar c A \pi^2}{240 \lambda_P L^3} \quad (18)$$

The dimensionless constant  $\alpha$  was evaluated numerically in [16, 23] and it will be given a more detailed interpretation in the following.

At this point, it is worth comparing the variation with distance of the Casimir force with that of the Van der Waals force between two atoms

in vacuum. Casimir and Polder [24] indeed showed that the latter force obeys power laws in the two limits of short and long distances, with the exponent being changed by one unit when going from one limit to the other and the crossover taking place when the interatomic distance  $L$  crosses the typical atomic wavelength  $\lambda_A$ . The same behaviours are also observed for the Casimir force between two metallic mirrors with the plasma wavelength  $\lambda_P$  playing the role of  $\lambda_A$ .

Since this comparison allows one to get interesting insight on the Casimir force, we briefly recall the results known for the Casimir-Polder force in the next section and then discuss the analogies and differences between both cases.

## 5 Reminders : the Casimir-Polder force between two atoms

In the present section we remind a few interesting results about the Van der Waals force between two atoms. We first rewrite the general expression obtained by Casimir and Polder [24] as an interaction energy [25]

$$E_{\text{CP}} = -\frac{\hbar c}{\pi L^2} \int_0^\infty d\kappa \alpha_A^2 [i c \kappa] \left( \kappa^4 + \frac{2\kappa^3}{L} + \frac{5\kappa^2}{L^2} + \frac{6\kappa}{L^3} + \frac{3}{L^4} \right) e^{-2\kappa L}$$

$$\alpha_A [i c \kappa] = \sum_{n>0} \frac{E_n A_n}{E_n^2 + \hbar^2 c^2 \kappa^2} \quad (19)$$

The interaction energy is an integral over imaginary frequencies  $\omega = i c \kappa$ ,  $L$  is the distance between the two atoms and  $\alpha_A$  represents the frequency dependent polarizability of the atoms<sup>3</sup>. The polarizability is written here in the dissipation free approximation and it depends on the energy  $E_n$  of the  $n$ -th atomic state measured with respect to the ground state and on a coefficient  $A_n$  proportional to the square modulus of the dipole matrix element associated with the transition. In the sequel of the paper, we discuss the profound analogies as well as a few significant differences appearing in the comparison of expression (19) and (7).

We want first to stress a fundamental analogy related to the very significance of long-range interactions such as Casimir-Polder or Casimir effects. Feinberg and Sucher [26] have written a still more general expression of the Casimir-Polder energy with magnetic polarizabilities taken into account besides the electric polarizabilities which appear in (19).

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<sup>3</sup>Expression (19) is written for two identical atoms; otherwise,  $\alpha_A^2$  should be replaced by the product of the polarizabilities of the two atoms.

Since magnetic terms are smaller than electric ones, we ignore them here but however refer to their paper for a very general discussion of long-range interactions which decrease as power laws of the distance. This makes an important difference with molecular interactions which decrease as exponential laws and, for this reason, dominate at typical atomic distances. An interesting fact is that long range interactions, in contrast to molecular interactions, may be deduced from the scattering amplitudes evaluated for the two scatterers separately. In more formal words, the two-centers scattering amplitudes may be written in terms of the one-center scattering amplitudes evaluated on shell for the photons and of free field propagation factors. For example, the Casimir-Polder energy (19) between atoms is written in terms of the polarizabilities  $\alpha_A$  while the Casimir energy (7) between mirrors is written in terms of the reflection amplitudes  $r$ . In both cases, free field propagation between the two centers is described by the exponential factor  $\exp(2\kappa L)$  where  $2\kappa$  is the momentum transferred on each scattering. In both cases too, the interaction energy can be written equivalently as an integral over real or imaginary frequencies, the equivalence being a consequence of causality properties.

These fundamental analogies should be appreciated in contrast with some significant differences. In particular, the atoms are point-like scatterers well coupled to the spherical waves centered on them whereas mirrors are plane scatterers which fit the definition of plane waves. Hence, the expression (7) of the Casimir force shows explicitly the summation over transverse wavevectors  $\mathbf{k}$  and polarizations  $p$  while the field propagation directions and polarizations have been traced over in expression (19) of the Casimir-Polder energy. It also follows from the point-like character of atoms that their mutual coupling through the field is less efficient than for mirrors. In other words, the two atoms form a poor-finesse cavity so that the higher order interferences terms, which play an important role in the Fabry-Perot cavity, can be disregarded in the two-atoms problem. This difference is made explicit by expanding the logarithm in (7) as

$$\log(1 - \rho) = - \left( \rho + \frac{\rho^2}{2} + \frac{\rho^3}{3} + \dots \right) \quad (20)$$

The lowest-order term in this expansion varies as  $e^{-2\kappa L}$  with distance and thus corresponds to the term present in (19). In contrast, the higher-order terms appearing in (20) and, therefore, in (7) are not accounted

for in the perturbative expression (19).

We now go one step further in the discussion of (19) by considering the large and short distance limits. In the large distance limit where  $L$  is greater than the wavelengths of the various atomic transitions, retardation effect plays a dominant role. This means that the exponential factor  $e^{-2\kappa L}$  restricts significant contributions to the integral (19) to low values of  $\kappa$  for which the polarizability remains nearly equal to its static value  $\alpha_A [0]$ . Hence, the interaction energy is obtained by evaluating a universal integral

$$E_{\text{CP}} = -\frac{23}{4} \frac{\hbar c}{\pi L^7} \alpha_A^2 [0] \quad , \quad L \gg \lambda_A$$

$$\frac{23}{4} \equiv \int_0^\infty du (u^4 + 2u^3 + 5u^2 + 6u + 3) e^{-2u} \quad (21)$$

This result bears some similarity with the Casimir energy evaluated at large distances which depends only on the static value  $r [0]$  of reflection amplitudes. In the Casimir case, this value goes to unity and then completely disappears from the ideal Casimir formula (6). In the Casimir-Polder case in contrast,  $\alpha_A [0]$  determines the global magnitude of the interaction energy. The difference between the two power laws can be attributed to dimensional arguments with  $\alpha_A$  having the dimension of a volume.

We then consider the short distance limit  $L \ll \lambda_A$  where the retardation effect is negligible. This means that the exponential factor may be discarded and also entails that the last term in the parenthesis appearing in (19) dominates the other ones. The interaction energy then scales as  $\frac{1}{L^6}$ , a result well known for the Van der Waals interaction calculated by London with retardation effects ignored [27]

$$E_{\text{CP}} = -\frac{3\hbar c}{\pi L^6} \int_0^\infty d\kappa \alpha_A^2 [i c \kappa] \quad , \quad L \ll \lambda_A \quad (22)$$

Using the expression of the frequency-dependent polarizability (see 19) and the integral

$$\int_0^\infty \frac{a}{a^2 + x^2} \frac{b}{b^2 + x^2} dx = \frac{\pi}{2} \frac{1}{a + b} \quad , \quad a, b > 0 \quad (23)$$

one rewrites the London expression as

$$E_{\text{CP}} = -\frac{3}{2L^6} \sum_{n, n' > 0} \frac{A_n A_{n'}}{E_n + E_{n'}} \quad , \quad L \ll \lambda_A \quad (24)$$

At this point we may stress that the change of exponent in the power laws is effectively similar in the Casimir and Casimir-Polder cases : the Casimir energy scales as  $\frac{1}{L^3}$  at large distances and  $\frac{1}{L^2\lambda_P}$  at short distances while the Casimir-Polder energy scales as  $\frac{1}{L^7}$  at large distances and  $\frac{1}{L^6\lambda_A}$  at short distances. Keeping in mind that the global change of exponents is explained by dimensional arguments, the change of exponent at the crossover is effectively the same. In order to stress this point, we could introduce a  $\eta$  factor in the Casimir-Polder case as the ratio of the general expression (19) to the long-distance expression (21)

$$\eta_E^{\text{CP}} = \frac{4}{23} \int_0^\infty dK \frac{\alpha_A^2 [i\kappa K]}{\alpha_A^2 [0]} (K^4 + 2K^3 + 5K^2 + 6K + 3) e^{-2K}$$

$$K \equiv \kappa L \tag{25}$$

This factor goes to unity at large distances and varies linearly with  $L \ll \lambda_A$  at short distances; it thus varies roughly as the  $\eta$  factor defined above for the Casimir problem.

In the next section, we show that this analogy may be pushed one step further, allowing one to give an interesting interpretation of the Casimir force at short distances as resulting from the London (non retarded) interaction between the elementary excitations in the two scatterers, that is the surface plasmons which live at the interface between each bulk mirror and vacuum [28].

## 6 London interaction between surface plasmons

In this last section, we study the Casimir energy between metallic plates described by the plasma model at the limit of short distances  $L \ll \lambda_P$ .

To this aim, we first show that its expression (7) can be greatly simplified at this limit. Indeed, values of  $\kappa$  contributing significantly to the integral correspond to  $\kappa L$  of the order of unity : large values have their contribution suppressed by the exponential factor  $\exp(-2\kappa L)$  while small values correspond to a small measure in the integration over transverse wavevector. Since  $L \ll \lambda_P$ , this condition also means that  $\kappa\lambda_P \gg 1$ . Meanwhile, a non vanishing value of  $r$  implies  $\xi \lesssim \omega_P$ , that is also  $\frac{\xi}{c}\lambda_P \lesssim 1$ . In these conditions,  $\kappa$  and  $\kappa_m$  are both approximately equal to  $|\mathbf{k}|$ , the TE reflection amplitude is negligible whereas the TM reflection amplitude takes a simple Lorentzian form

$$r^{\text{TM}} \simeq \frac{\omega_S^2}{\omega^2 - \omega_S^2} \quad , \quad \omega_S^2 \equiv \frac{\omega_P^2}{2} \tag{26}$$

The frequency  $\omega_S$  is the constant value of the surface plasmon frequency  $\omega_{\text{plasmon}}$  obtained by putting the condition  $|\mathbf{k}| \lambda_P \gg 1$  in (16).

For a cavity made with two identical mirrors, the open loop function then takes the simple form

$$\rho^{\text{TM}} = \left( \frac{\omega_S^2}{\omega^2 - \omega_S^2} \right)^2 e^{-2|\mathbf{k}|L} \quad (27)$$

The poles of the closed loop function, given by the condition  $\rho^{\text{TM}} = 1$ , therefore correspond to the frequencies

$$\omega_{S\pm} = \omega_S \sqrt{1 \pm e^{-|\mathbf{k}|L}} \quad (28)$$

This expression shows how the surface plasmons corresponding to the two mirrors are displaced from their original frequencies  $\omega_S$  due to their coupling through the cavity. It also provides an interesting interpretation of the Casimir interaction energy which can be written from (27) as

$$E=A \int \frac{d^2\mathbf{k}}{4\pi^2} \left( \frac{\hbar\omega_{S+}}{2} + \frac{\hbar\omega_{S-}}{2} - 2\frac{\hbar\omega_S}{2} \right) \quad (29)$$

The Casimir energy is nothing but the result of the shift of zero-point energies of the surface plasmons due to their coupling through the cavity field [28]. This result has been obtained with retardation effects neglected : this follows from the assumptions used in the derivation and is also apparent in the fact that the exponential factors depend on the transverse wavevectors but not on the frequencies of the field.

We now conclude by giving an analytical form of this London-like expression of the Casimir energy at short distances. We come back to equation (7) that we expand with the help of (20). We ignore the vanishing TE contribution and obtain

$$\begin{aligned} E &= -\hbar A \sum_{n=1}^{\infty} \int \frac{d^2\mathbf{k}}{4\pi^2} \frac{e^{-2n|\mathbf{k}|L}}{n} I_n \\ I_n &= \int_0^{\infty} \frac{d\xi}{2\pi} (r^{\text{TM}}[i\xi])^{2n} = \frac{\omega_S}{4} \frac{(4n-3)!!}{(4n-2)!!} \\ \frac{(4n-3)!!}{(4n-2)!!} &\equiv \frac{1.3.5\dots(4n-3)}{2.4.6\dots(4n-2)} \end{aligned} \quad (30)$$



Collecting these results finally leads to the Casimir energy at short distances

$$E \simeq -\frac{\hbar c A}{16\sqrt{2}L^2\lambda_P} \sum_{n=1}^{\infty} \frac{1}{n^3} \frac{(4n-3)!!}{(4n-2)!!} \quad (31)$$

This result may be equivalently expressed in terms of the Casimir force or of the force reduction factor (18) with now a formal expression for the numerical coefficient  $\alpha$

$$\begin{aligned} \alpha &= \frac{30}{\sqrt{2}\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^3} \frac{(4n-3)!!}{(4n-2)!!} \\ &= \frac{15}{\sqrt{2}\pi^2} \left( 1 + \frac{5}{64} + \frac{7}{384} + \dots \right) \simeq 1.193 \end{aligned} \quad (32)$$

This reproduces the result of [16, 23]. Note that the integral  $I_1$  corresponding to the lowest order term  $n = 1$  has the same form as the integral (23) used in the calculation of the London expression (22), with the surface plasmon frequency playing the role of an atomic resonance frequency. The other contributions  $n > 1$  represent the effect of higher-order interferences in the Fabry-Perot cavity. It is worth emphasizing that they have the same scaling law versus distance as the lowest order term  $n = 1$  and contribute significantly ( $\sim 10\%$ ) to the final result.

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