Annales de la Fondation Louis de Broglie, Volume 29 no 3, 2004

Fermion production by an external field

JAUME HARO[1]

Departament de Matemàtica Aplicada I
E.T.S.E.I.B., Universitat Politècnica de Catalunya
Diagonal 647, 08028 Barcelona, Spain
Email: Jaime.Haro@upc.es

ABSTRACT. We studied the Dirac field coupled with an external uniform vector potential. We computed the pair production in a finite time t using the semiclassical approximation, and finally we compared our results with those of Schwinger.

RÉSUMÉ. Nous avons étudié le champ de Dirac couplé avec un vecteur potentiel exterieur qui est uniforme. Nous avons utilisé l'approximation semiclassique pour calculer la production de pairs pendent un temps fini, et finallement nous avons comparé nos résultats avec les résultats obtenus par Schwinger.

Key words. Pair production, Schwinger's formula, Semiclassical approach.

AMS subject classifications. 34E20, 81Q05, 81Q20.

1 Introduction

The subject of this paper is the study of pairs production at each finite time t due to the prensece of an external uniform vector potential $\vec{f}(t)$ that verifies $\frac{d}{dt}\vec{f}(t) \in \mathcal{C}_0^\infty(0,\infty)$ and $\vec{f}(t) \equiv \vec{0}$ when $t \in [0,\infty)$.

The study of pair production at finite time, in the case of the Klein-Gordon field, was studied in [14] in the context of expanding universes. Here we are interested in the pair production at finite time, in the case of Dirac field.

[1]Partially supported by DGESIC (spain), project PB98-0932-C02-01.

In the first section we consider the Dirac field defined at $[-L, L]^3$ with periodic boundary conditions. Using the Pauli's exclusion principle and the Dirac's sea hypothesis, we construct the vacuum state, the state of a particle, the state of an antiparticle, etc... Once we have constructed the Fock space, we will study the quantum dynamics of the vacuum state using semiclassical solutions to the Dirac equation, and we will see that the probability that pairs are not produced at t time, $P_\hbar(t)$, verifies

$$P_\hbar(t) \sim \exp\left(-\frac{3\alpha}{32}\frac{\mathcal{E}(t)}{mc^2}\right), \tag{1}$$

where $\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant, $\mathcal{E}(t) = \frac{(2L)^3}{8\pi}|\vec{E}(t)|^2$ is the energy of the external field at time t, and the simbol $'' \sim''$ means approximately in the sense that, $a \sim b$ if $\lim_{\hbar\to 0}(a-b) = 0$, for fixed α.

It is important to remark that for larger times, i.e., when the electric field is zero, formula (1) becomes

$$P_\hbar(t) = \exp\left(O(\hbar^\infty)\right).$$

In general, we do not have an explicit expression of the formula (1), for large times. For this reason, in section 3 we study the particular case $\vec{f}(t) = (0, 0, \chi(t))$, where

$$\chi(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ cEt & \text{if } 0 < t < T \\ cET & \text{if} \quad t > T. \end{cases}$$

For this potential, when $t > T$, using the WKB approximation in the complex plane, we obtain the following explicit expression of the formula (1)

$$\exp\left(-\frac{2TL^3E^2\alpha}{\pi^3\hbar}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left(-\frac{n\pi m^2c^4}{\hbar c|eE|}\right)\right), \tag{2}$$

i.e., we obtain Schwinger's formula ([4], [8], [9], [13], [15], [16]).

Note that, we obtain Schwinger's formula in the adiabatic approach, because the method of imaginary times ([10], [15]), i.e. the WKB approximation in the complex plane, used in the computation of the Schwinger's formula is only justified in the adiabatic approach (see [2], [12], [17]).

2 Pairs production

2.1 Dirac's equation

The Dirac equation is

$$i\hbar\dot{\psi} = c < \vec{A}, (-i\hbar\vec{\nabla} + \frac{e}{c}\vec{f}(t)) > \psi + mc^2 B\psi \equiv H_D\psi,$$

with $\frac{d}{dt}\vec{f} \in \mathcal{C}_0^\infty(0,\infty)$.

The matrices are

$$\vec{A} = \begin{pmatrix} \vec{\sigma} & \vec{0} \\ \vec{0} & -\vec{\sigma} \end{pmatrix} \quad B = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

and

$$\vec{\sigma} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right),$$

are the Pauli matrices.

Then the eigenfunctions of H_D are

$$\psi^{\pm}_{1,\vec{k}}(\vec{x},t) = v^{\pm}_{1,\vec{k}}(t)\frac{e^{\frac{i\pi}{L}<\vec{k},\vec{x}>}}{(2L)^{\frac{3}{2}}}; \quad \psi^{\pm}_{-1,\vec{k}}(\vec{x},t) = v^{\pm}_{-1,\vec{k}}(t)\frac{e^{\frac{i\pi}{L}<\vec{k},\vec{x}>}}{(2L)^{\frac{3}{2}}},$$

where

$$v^{\pm}_{1,\vec{k}}(t) = \frac{1}{\sigma^{\pm}_{\vec{k}}(t)} \begin{pmatrix} a^{\pm}_{\vec{k}}(t) \\ b_{\vec{k}}(t) \\ mc^2 \\ 0 \end{pmatrix}; \quad v^{\pm}_{-1,\vec{k}}(t) = \frac{1}{\sigma^{\pm}_{\vec{k}}(t)} \begin{pmatrix} 0 \\ -mc^2 \\ b^*_{\vec{k}}(t) \\ -a^{\pm}_{\vec{k}}(t) \end{pmatrix},$$

$$a^{\pm}_{\vec{k}}(t) = \epsilon^{\pm}_{\vec{k}}(t) + \frac{c\pi k_3\hbar}{L} + ef_3(t); \quad \sigma^{\pm}_{\vec{k}}(t) = \sqrt{2\epsilon^{\pm}_{\vec{k}}(t)a^{\pm}_{\vec{k}}(t)}$$

$$b_{\vec{k}}(t) = \frac{c\pi k_1\hbar}{L} + ef_1(t) + i\left(\frac{c\pi k_2\hbar}{L} + ef_2(t)\right);$$
$$\epsilon^{\pm}_{\vec{k}}(t) = \pm\sqrt{c^2\left|\frac{\pi\vec{k}\hbar}{L} + \frac{e}{c}\vec{f}(t)\right|^2 + m^2c^4},$$

and $< \vec{u}, \vec{v} >$ is the inner product of the vectors $\vec{u}$ and $\vec{v}$.

2.2 The quantum states

In order to define the quantum states we have to apply the Dirac sea hypothesis and the Pauli exclusion principle. Then we define [6], for fixed $\vec{N} \in \mathbb{N}^3$, the following $\vec{N}-$states.

The vacuum $\vec{N}-$state at time t is:

$$|0\rangle_{\vec{N}}(t) = \frac{1}{\sqrt{2(2N+1)^3!}} v^-_{-1,-\vec{N}}(t) \wedge v^-_{1,-\vec{N}}(t) \wedge \cdots \wedge v^-_{-1,\vec{N}}(t) \wedge v^-_{1,\vec{N}}(t).$$

The $\vec{N}-$state of a particle with momentum $\vec{k}\hbar$ and helicity $\frac{\alpha\hbar}{2}$ at time t is:

$$|1^+_{\alpha,\vec{k}}\rangle_{\vec{N}}(t) = \frac{1}{\sqrt{(2(2N+1)^3+1)!}} v^+_{\alpha,\vec{k}}(t) \wedge v^-_{-1,-\vec{N}}(t) \wedge \cdots \wedge v^-_{1,\vec{N}}(t).$$

The $\vec{N}-$state of an anti-particle with momentum $\vec{k}\hbar$ and helicity $\frac{\alpha\hbar}{2}$ at time t is:

$$|1^-_{\alpha,\vec{k}}\rangle_{\vec{N}}(t) = \frac{1}{\sqrt{(2(2N+1)^3-1)!}} v^-_{-1,-\vec{N}}(t) \wedge \cdots \wedge \hat{v}^-_{-\alpha,-\vec{k}}(t), \wedge \cdots \wedge v^-_{1,\vec{N}}(t).$$

where $\hat{v}^-_{-\alpha,-\vec{k}}(t)$ means that the state $v^-_{-\alpha,-\vec{k}}(t)$ do not appear in the expression.

The $\vec{N}-$state of a pair with momentum $2\vec{k}\hbar$ and helicity $\alpha\hbar$ at time t is:

$$|1^+_{\alpha,\vec{k}}1^-_{\alpha,\vec{k}}\rangle_{\vec{N}}(t) = \frac{1}{\sqrt{2(2N+1)^3!}} v^+_{\alpha,\vec{k}}(t) \wedge v^-_{-1,-\vec{N}}(t) \wedge \cdots \hat{v}^-_{-\alpha,-\vec{k}}(t) \wedge \cdots \wedge v^-_{1,\vec{N}}(t),$$

etc...

Now we can define the quantum dynamics of the vacuum $\vec{N}-$state. Let $T^t_{\hbar,\vec{k}} v^{\pm}_{\beta,\vec{k}}(0)$ be the solution of the problem

$$(P) \qquad \begin{cases} i\hbar\dot{v} = H_{D,\vec{k}}(t)v \\ v(0) = v^{\pm}_{\beta,\vec{k}}(0), \end{cases}$$

where, $H_{D,\vec{k}}(t) = c < \vec{A}, (\frac{\pi\hbar}{L}\vec{k} + \frac{e}{c}\vec{f}(t)) > +mc^2 B$.

Then, we define

$$T^t_{\hbar}|0\rangle_{\vec{N}}(0) = \frac{1}{\sqrt{2(2N+1)^3!}} T^t_{\hbar,-\vec{N}} v^{-}_{-1,-\vec{N}}(t) \wedge T^t_{\hbar,-\vec{N}} v^{-}_{1,-\vec{N}}(t) \wedge \cdots$$
$$\cdots \wedge T^t_{\hbar,\vec{N}} v^{-}_{-1,\vec{N}}(t) \wedge T^t_{\hbar,\vec{N}} v^{-}_{1,\vec{N}}(t).$$

Now, let $P_{\vec{N},\hbar}(t)$ be the square modulus of the inner product of the vectors $|0\rangle_{\vec{N}}(t)$ and $T^t_{q,\vec{N}}|0\rangle_{\vec{N}}(0)$. Then $P_\hbar(t) = \lim_{N\to\infty} P^0_{\vec{N},\hbar}(t)$ is the probability that does not exist any pair at t time.

We write the solution of the problem (P) in the following form

$$T^t_{\hbar,\vec{k}} v^{-}_{\beta,\vec{k}}(0) = \left(\sum_{\alpha=-1}^{1} \left(a^{\alpha}_{\beta,\vec{k}} v^{-}_{\alpha,\vec{k}}(t) + b^{\alpha}_{\beta,\vec{k}} v^{+}_{\alpha,\vec{k}}(t) \right) \right) e^{-\frac{i}{\hbar}\int_0^t \epsilon^{-}_{\vec{k}}(\tau)d\tau},$$

then, we have the lemma

Lemma 2.1. *For every time t, we have*

$$P_\hbar(t) = \prod_{\vec{k}\in\mathbb{Z}^3} |det A_{\vec{k}}(t)|^2,$$

where

$$A_{\vec{k}}(t) = \begin{pmatrix} a^{1}_{1,\vec{k}}(t) & a^{-1}_{1,\vec{k}}(t) \\ a^{1}_{-1,\vec{k}}(t) & a^{-1}_{-1,\vec{k}}(t) \end{pmatrix}.$$

2.3 Semiclassical solutions of the Dirac equation

Let $T^t_\hbar v^-_{\beta,\vec{k}}(0)$ be the solution of the problem (P).

We write the solution in the following form

$$T^t_{\hbar,\vec{k}} v^-_{\beta,\vec{k}}(0) = \left(\sum_{\alpha=-1}^{1} \left(a^{\alpha}_{\beta,\vec{k}} v^-_{\alpha,\vec{k}}(t) + b^{\alpha}_{\beta,\vec{k}} v^+_{\alpha,\vec{k}}(t)\right)\right) e^{-\frac{i}{\hbar}\int_0^t \epsilon^-_{\vec{k}}(\tau)d\tau}$$
$$= \left(\sum_{\alpha=-1}^{1}\sum_{n=0}^{\infty} \left(\hbar^n a^{\alpha,n}_{\beta,\vec{k}} v^-_{\alpha,\vec{k}}(t) + \hbar^{n+1} b^{\alpha,n}_{\beta,\vec{k}} v^+_{\alpha,\vec{k}}(t)\right)\right) e^{-\frac{i}{\hbar}\int_0^t \epsilon^-_{\vec{k}}(\tau)d\tau}.$$

We obtain, after having equalized the powers of $\hbar$, the system

$$\dot{a}^{s,0}_{\beta,\vec{k}} + \sum_{\alpha=-1}^{1} a^{\alpha,0}_{\beta,\vec{k}} < \dot{v}^-_{\alpha,\vec{k}}(t), v^-_{s,\vec{k}}(t) >= 0.$$

$$\dot{a}^{s,n}_{\beta,\vec{k}} + \sum_{\alpha=-1}^{1} \left(a^{\alpha,n}_{\beta,\vec{k}} < \dot{v}^-_{\alpha,\vec{k}}(t), v^-_{s,\vec{k}}(t) > + b^{\alpha,n-1}_{\beta,\vec{k}} < \dot{v}^+_{\alpha,\vec{k}}(t), v^-_{s,\vec{k}}(t) >\right) = 0.$$

$$i\sum_{\alpha=-1}^{1} a^{\alpha,0}_{\beta,\vec{k}} < \dot{v}^-_{\alpha,\vec{k}}(t), v^+_{s,\vec{k}}(t) > -2\epsilon^+_{\vec{k}}(t) b^{s,0}_{\beta,\vec{k}} = 0.$$

$$i\dot{b}^{s,n-1}_{\beta,\vec{k}} + i\sum_{\alpha=-1}^{1} \left(a^{\alpha,n}_{\beta,\vec{k}} < \dot{v}^-_{\alpha,\vec{k}}(t), v^+_{s,\vec{k}}(t) > + b^{\alpha,n-1}_{\beta,\vec{k}} < \dot{v}^+_{\alpha,\vec{k}}(t), v^+_{s,\vec{k}}(t) > \right)$$
$$- 2\epsilon^+_{\vec{k}}(t) b^{s,n}_{\beta,\vec{k}} = 0.$$

Note that, we obtain the solution of the system by recurrence. In fact, we solve the first equation with initial data $a^{\alpha,0}_{\beta,\vec{k}}(0) = \delta_{\alpha,\beta}$, and then, we solve the others equations.

We use the relation $(\epsilon^+_{\vec{k}})^2 \leq C(\epsilon^+_{\vec{k}}(t))^2$, where $\epsilon^+_{\vec{k}} = \sqrt{c^2\left|\frac{\pi\hbar\vec{k}}{L}\right|^2 + m^2c^4}$, $C = 2(1 + \frac{e^2\|\vec{f}\|^2_\infty}{mc^2})$, and the following lemma

Lemma 2.2. *For $\alpha, \beta = -1, 1$, there is a function $g \in \mathcal{C}^\infty_0(0,\infty)$, independent on $\vec{k}$ and $\hbar$, such that*

$$|<\dot{v}^{\mp}_{\alpha,\vec{k}}(t),v^{\pm}_{\beta,\vec{k}}(t)>|\leq\frac{g(t)}{\epsilon^{+}_{\vec{k}}};\quad |<\dot{v}^{\mp}_{\alpha,\vec{k}}(t),v^{\mp}_{\beta,\vec{k}}(t)>|\leq g(t)$$

$$|\frac{d}{dt}<\dot{v}^{\mp}_{\alpha,\vec{k}}(t),v^{\pm}_{\beta,\vec{k}}(t)>|\leq\frac{g(t)}{\epsilon^{+}_{\vec{k}}};\quad |\frac{d}{dt}<\dot{v}^{\mp}_{\alpha,\vec{k}}(t),v^{\mp}_{\beta,\vec{k}}(t)>|\leq g(t).$$

Therefore, we obtain the

Proposition 2.3. *For $m=1,2$ there are a constant $\bar{C}$ and a function $G\in\mathcal{C}^{\infty}_0(0,\infty)$, independents on $\vec{k}$ and $\hbar$, such that*

$$|a^{\alpha,0}_{\beta,\vec{k}}|\leq 1;\quad |\dot{a}^{\alpha,0}_{\beta,\vec{k}}|\leq G(t);\quad |b^{\alpha,0}_{\beta,\vec{k}}|\leq\frac{\bar{C}}{(\epsilon^{+}_{\vec{k}})^2};\quad |\dot{b}^{\alpha,0}_{\beta,\vec{k}}|\leq\frac{G(t)}{(\epsilon^{+}_{\vec{k}})^2}$$

$$|a^{\alpha,m}_{\beta,\vec{k}}|\leq\frac{\bar{C}}{(\epsilon^{+}_{\vec{k}})^{2+m}};\quad |\dot{a}^{\alpha,m}_{\beta,\vec{k}}|\leq\frac{G(t)}{(\epsilon^{+}_{\vec{k}})^{2+m}};$$
$$|b^{\alpha,m}_{\beta,\vec{k}}|\leq\frac{\bar{C}}{(\epsilon^{+}_{\vec{k}})^{2+m}};\quad |\dot{b}^{\alpha,m}_{\beta,\vec{k}}|\leq\frac{G(t)}{(\epsilon^{+}_{\vec{k}})^{2+m}}.$$

Now, by virtue of this proposition, we show that the semiclassical solution of the problem P, has the following form

$$\tilde{v}^{-}_{\beta,\vec{k}}(t)=\left(\sum_{\alpha=-1}^{1}\sum_{n=0}^{2}(\hbar^n a^{\alpha,n}_{\beta,\vec{k}}v^{-}_{\alpha,\vec{k}}(t)+\hbar^{n+1}b^{\alpha,n}_{\beta,\vec{k}}v^{+}_{\alpha,\vec{k}}(t))\right)e^{-\frac{i}{\hbar}\int_0^t\epsilon^{-}_{\vec{k}}(\tau)d\tau}.$$

In fact, we have

$$(i\hbar\dot{\psi}-H_{D,\vec{k}}(t))\tilde{v}^{-}_{\beta,\vec{k}}(t)=i\hbar^4\left(\sum_{s,\alpha=-1}^{1}b^{\alpha,2}_{\beta,\vec{k}}<\dot{v}^{+}_{\alpha,\vec{k}},v^{-}_{s,\vec{k}}>v^{-}_{s,\vec{k}}(t)\right.$$
$$\left.+\sum_{s=-1}^{1}(\dot{b}^{s,2}_{\beta,\vec{k}}+\sum_{\alpha=-1}^{1}b^{\alpha,2}_{\beta,\vec{k}}<\dot{v}^{+}_{\alpha,\vec{k}},v^{+}_{s,\vec{k}}>)v^{+}_{s,\vec{k}}(t)\right)e^{-\frac{i}{\hbar}\int_0^t\epsilon^{-}_{\vec{k}}(\tau)d\tau}.$$

Consequently, we deduce from the lemma 2.2 and the proposition 2.3, that

$$||(i\hbar\dot{\psi} - H_{D,\vec{k}}(t))\tilde{v}^{-}_{\beta,\vec{k}}(t)||_2^2 \leq \hbar^8 \frac{\tilde{G}^2(t)}{(\epsilon^{+}_{\vec{k}})^8}, \tag{3}$$

where, $\tilde{G}(t) \in \mathcal{C}^{\infty}_0(0,\infty)$ is a function independent on $\vec{k}$ and $\hbar$.

From (3) we deduce that ([7], [11])

$$||T^t_{\hbar,\vec{k}} v^{-}_{\beta,\vec{k}}(0) - \tilde{v}^{-}_{\beta,\vec{k}}(t)||_2 \leq \frac{\hbar^3}{(\epsilon^{+}_{\vec{k}})^4} \int_0^{\infty} |\tilde{G}(\tau)| d\tau \equiv \frac{K\hbar^3}{(\epsilon^{+}_{\vec{k}})^4},$$

and we obtain

$$a^{\alpha}_{\beta,\vec{k}} = \sum_{n=0}^{2} \hbar^n a^{\alpha,n}_{\beta,\vec{k}} + \hbar^3 A^{\alpha}_{\beta,\vec{k}}(t), \quad b^{\alpha}_{\beta,\vec{k}} = \sum_{n=0}^{1} \hbar^{n+1} a^{\alpha,n}_{\beta,\vec{k}} + \hbar^3 B^{\alpha}_{\beta,\vec{k}}(t),$$

with $|A^{\alpha}_{\beta,\vec{k}}(t)| \leq \frac{K}{(\epsilon^{+}_{\vec{k}})^4}$ i $|B^{\alpha}_{\beta,\vec{k}}(t)| \leq \frac{\bar{C}+K}{(\epsilon^{+}_{\vec{k}})^4}$.

2.4 Properties of the semiclassical solutions

Now, we will enunciate the basic properties of the semiclassical solutions in order to obtain formula (1) afterwards. Let $\vec{a}^{n}_{\beta,\vec{k}} = (a^{-1,n}_{\beta,\vec{k}}, a^{1,n}_{\beta,\vec{k}})$, then:

Lemma 2.4. *For $\alpha, \beta = -1, 1$, we have*

$$< \vec{a}^{0}_{\beta,\vec{k}}(t), \vec{a}^{0}_{\alpha,\vec{k}}(t) >= \delta_{\beta,\alpha} \quad \forall t \in \mathbb{R}.$$

$$< \vec{a}^{1}_{\beta,\vec{k}}(t), \vec{a}^{0}_{\alpha,\vec{k}}(t) > + < \vec{a}^{0}_{\beta,\vec{k}}(t), \vec{a}^{1}_{\alpha,\vec{k}}(t) >= 0 \quad \forall t \in \mathbb{R}.$$

Corollary 2.5. *We consider*

$$A^0_{\vec{k}}(t) = \begin{pmatrix} a^{1,0}_{1,\vec{k}}(t) & a^{-1,0}_{1,\vec{k}}(t) \\ a^{1,0}_{-1,\vec{k}}(t) & a^{-1,0}_{-1,\vec{k}}(t) \end{pmatrix},$$

then $|det A^0_{\vec{k}}(t)| = 1$.

Lemma 2.6. *For every time t, we have*

$$|\vec{a}^1_{\beta,\vec{k}}(t)|^2 + < \vec{a}^2_{\beta,\vec{k}}(t), \vec{a}^0_{\beta,\vec{k}}(t) > + < \vec{a}^0_{\beta,\vec{k}}(t), \vec{a}^2_{\beta,\vec{k}}(t) >=$$
$$-\frac{1}{8(\epsilon^+_{\vec{k}}(t))^2} \sum_{\alpha=-1}^{1} (| < \dot{v}^-_{\alpha,\vec{k}}(t), v^+_{\alpha,\vec{k}}(t) > |^2 + | < \dot{v}^-_{\alpha,\vec{k}}(t), v^+_{-\alpha,\vec{k}}(t) > |^2)$$
$$= -\frac{c^2 e^2}{16(\epsilon^+_{\vec{k}}(t))^6} \left(\left| \left(\frac{c\pi\hbar\vec{k}}{L} + e\vec{f}(t) \right) \wedge \vec{E}(t) \right|^2 + m^2 c^4 |\vec{E}(t)|^2 \right),$$

where, $\vec{E}(t) = \frac{1}{c}\dot{\vec{f}}(t)$ is the electric field and $\vec{u} \wedge \vec{p}$ is the vectorial product of the vectors $\vec{u}$ and $\vec{p}$.

Lemma 2.7. *Using the lemmas 2.4 and 2.6, for every time t, we have*

$|detA_{\vec{k}}(t)| \leq 1$

$|detA_{\vec{k}}(t)|^2 = 1 - \frac{\hbar^2 c^2 e^2}{8(\epsilon^+_{\vec{k}}(t))^6} \left(\left| \left(\frac{c\pi\hbar\vec{k}}{L} + e\vec{f}(t) \right) \wedge \vec{E}(t) \right|^2 + m^2 c^4 |\vec{E}(t)|^2 \right) + \hbar^3 \tilde{E}_{\vec{k}}(t),$

with $|\tilde{E}_{\vec{k}}(t)| \leq \frac{\tilde{L}}{(\epsilon^+_{\vec{k}})^4}$, where $\tilde{L}$ is a constant independent on $\vec{k}$ and $\hbar$.

2.5 Pairs production at finite time

Here we deduce the formula

$$P_\hbar(t) \sim \exp\left(-\frac{3\alpha}{32} \frac{1}{mc^2} \mathcal{E}(t) \right). \tag{4}$$

By virtue of the lemma 2.7, we have

$$|detA_{\vec{k}}(t)|^2 = 1 - \frac{\hbar^2 c^2 e^2}{8(\epsilon^+_{\vec{k}}(t))^6} \left(\left| \left(\frac{c\pi\hbar\vec{k}}{L} + e\vec{f}(t) \right) \wedge \vec{E}(t) \right|^2 + m^2 c^4 |\vec{E}(t)|^2 \right)$$
$$+ \hbar^3 \tilde{E}_{\vec{k}}(t),$$

Consequently, the probability that the vacuum remains unchaged verifies

$$P_\hbar(t) \sim$$
$$\prod_{\vec{k}\in\mathbb{Z}^n}\left(1-\frac{\hbar^2c^2e^2}{8(\epsilon^+_{\vec{k}}(t))^6}\left(\left|\left(\frac{c\pi\hbar\vec{k}}{L}+e\vec{f}(t)\right)\wedge\vec{E}(t)\right|^2+m^2c^4|\vec{E}(t)|^2\right)\right)$$
$$\sim \exp\left(-\sum_{\vec{k}\in\mathbb{Z}^n}\frac{\hbar^2c^2e^2}{8(\epsilon^+_{\vec{k}}(t))^6}\left(\left|\left(\frac{c\pi\hbar\vec{k}}{L}+e\vec{f}(t)\right)\wedge\vec{E}(t)\right|^2+m^2c^4|\vec{E}(t)|^2\right)\right)$$
$$\sim \exp\left(-\frac{c^2e^2L^3}{8\hbar\pi^3}\int_{\mathbb{R}^3}\frac{|c\vec{p}\wedge\vec{E}(t)|^2+m^2c^4|\vec{E}(t)|^2}{(c^2|\vec{p}|^2+m^2c^4)^3}d\vec{p}\right)$$
$$= \exp\left(-\frac{3\alpha(2L)^3|\vec{E}(t)|^2}{256\pi mc^2}\right)=\exp\left(-\frac{3\alpha}{32}\frac{\mathcal{E}(t)}{mc^2}\right).$$

3 Schwinger's result

Here we consider the external uniform vector potential $\vec{f}(t)=(0,0,\chi(t))$, where

$$\chi(t)=\begin{cases} 0 & \text{if} \quad t<0 \\ cEt & \text{if } 0<t<T \\ cET & \text{if} \quad t>T, \end{cases}$$

and the spatial domain $[-L,L]^3$.

Then, $\forall t>T$, the probability that the vacuum state remains unchanged at time t is given by the Schwinger's formula

$$\exp\left(-\frac{2TL^3E^2\alpha}{\pi^3\hbar}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left(-\frac{n\pi m^2c^4}{\hbar c|eE|}\right)\right).$$

We deduce this result using the relativistic tunneling effect ([1], [10], [15]), i.e., using the WKB method in the complex plane [2].

If $0<\tau<T$, the classical Hamiltonian is

$$H(\tau)=\pm\sqrt{c^2p_\perp^2+c^2(p_3+eE\tau)^2+m^2c^4},$$

where $p_\perp=(p_1,p_2)$.

The dynamic equations are

$$\dot{x}_i = \frac{c^2 p_i}{H(\tau)}; \quad i = 1, 2$$

$$\dot{x}_3 = \frac{c^2(p_3 + eE\tau)}{H(\tau)}$$

$$\dot{\vec{p}} = \vec{0}.$$

For a particle with negative kinetic energy and momentum $\vec{p}$, we have

$$x_3(\tau) = x_3(0) + \frac{1}{eE}\left(\sqrt{c^2|\vec{p}|^2 + m^2c^4} - \sqrt{c^2p_\perp^2 + c^2(p_3 + eE\tau)^2 + m^2c^4}\right).$$

We note that, if $0 < \frac{-p_3}{eE} < T$, then $x_3\left(\frac{-p_3}{eE}\right)$ is a classical turning point. Therefore, at $\frac{-p_3}{eE}$ there is a probability that the particle has positive kinetic energy, and then, if $\tau > \frac{-p_3}{eE}$, its dynamics would be

$$\begin{aligned} x_3(\tau) =& x_3(0) + \frac{1}{eE}\left(\sqrt{c^2|\vec{p}|^2 + m^2c^4} - 2\sqrt{c^2p_\perp^2 + m^2c^4}\right. \\ &\left. + \sqrt{c^2p_\perp^2 + c^2(p_3 + eE\tau)^2 + m^2c^4}\right). \end{aligned}$$

The average number of produced pairs at time $t > T$ with momentum $(p_\perp, p_3)$, namely $\omega(p_\perp, p_3)$, is given in the adiabatic approach by the penetration factor

$$\omega(p_\perp, p_3) \sim \exp\left(-\frac{2}{\hbar}\int_{\tau_-}^{\tau_+} \sqrt{c^2p_\perp^2 + m^2c^4 + c^2(p_3 + eE\tau)^2 d\tau}\right),$$

where $\tau_\pm = \frac{-p_3 \pm i\sqrt{p_\perp^2 + m^2c^2}}{eE}$, and $\tau_- < \tau < \tau_+$.

This penetration factor is obtained using the method of imaginary times [13], (note that, in a mathematical language this method is called "Over-Barrier Reflection" [2], [12]).

It is easy to verify that

$$\omega(p_\perp, p_3) \sim \exp\left(-\frac{\pi(c^2p_\perp^2 + m^2c^4)}{\hbar c|eE|}\right).$$

Therefore, the probability that pairs with spin $\frac{1}{2}$ and momentum $(p_\perp, p_3)$ ara not produced, using the Exclusion Principle, is

$$1 - \exp\left(-\frac{\pi(c^2p_\perp^2 + m^2c^4)}{\hbar c|eE|}\right).$$

Then, the probability that the vacuum remains unchanged is

$$\left(\prod 1 - \exp\left(-\frac{\pi(c^2p_\perp^2 + m^2c^4)}{\hbar c|eE|}\right)\right)^2 =$$
$$\exp\left(2\sum \log\left(1 - \exp\left(-\frac{\pi(c^2p_\perp^2 + m^2c^4)}{\hbar c|eE|}\right)\right)\right),$$

because there are two differents states for particles with spin $\frac{1}{2}$.

The sum is over all $p_\perp = \frac{\pi\hbar}{L}(k_1, k_2)$ with $(k_1, k_2) \in \mathbb{Z}^2$ and over p_3. But the time required by the particle to arrive at the turning point is $-\frac{p_3}{eE}$ if $0 < \frac{-p_3}{eE} < T$. Therefore, the particles that arrive at the turning point verify that p_3 is between 0 and eET. Therefore, since $\hbar$ is small, the sum is approximately an integral, and using the logarithm Taylor's series, we have

$$\exp\left(-\frac{2TL^3|eE|}{(\pi\hbar)^3}\sum_{n=1}^{\infty}\frac{1}{n}\int_{\mathbb{R}^2}\exp\left(-\frac{n\pi(c^2p_\perp^2 + m^2c^4)}{\hbar c|eE|}\right)dp_\perp\right) =$$
$$\exp\left(-\frac{2TL^3E^2\alpha}{\pi^3\hbar}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left(-\frac{n\pi m^2c^4}{\hbar c|eE|}\right)\right).$$

Consequently, the Schwinger's formula gives the probability that the vacuum state remains unchanged for large times, i.e., for times such that the electric field is zero ($\chi(t) = cET$). If we compute the probability that the vacuum state remains unchanged for $0 < t < T$ we obtain the formula (1).

References

[1] J.M.EISENBERG, G.KÄLBERMANN, Pair production in transport equations; Physical Review D35 no. 5, pag. 368-375 (1988).

[2] M.V.FEDORYUK, Asymptotic Analisis; Springer-Verlag (1993).

[3] S.A.FULLING, Aspects of Quantum Field Theory in Curved Space-Time; London Mathematical Society Stident Text 17 (1985).

[4] W.GREINER, B.MÜLLER, J.RAFELSKI, Quantum Electrodynamics of Strong Fields; Springer-Verlang (1985).

[5] J.HARO, El límit clàssic de la mecànica quàntica; Tesi Doctoral, U.A.B. (1997).

[6] J.HARO, Étude classique de l'équation de Dirac; Ann. Fond. Louis de Broglie 23, no. 3-4, pag. 166-172, (1998).

[7] J.HARTHONG, Études sur la mécanique quantique; Asterisque, 111, (1984).

[8] B.R. HOLSTEIN, Strong field pair production; Am. J. Phys. 67, no. 6, pag. 499-507, (1999).

[9] C.ITZYKSON, J.B.ZUBER, Quantum field theory; McGraw-Hill International Editions, (1980).

[10] S.M.MARINOV, V.S.POPOV, Electron-Positron Pair Creation from Vacuum Induced by Variable Electric Field; Fortschritte der Physik 25, pag. 373-400 (1977).

[11] V.P.MASLOV and M.V.FEDORIUK, Semi-classical approximation in quantum mechanics; D.Riedel Publishing Compay, Dordrecht, Holland (1981).

[12] R.E.MEYER, Exponential Asymptotics; SIAM Review 22, no. 2, pag. 213-224 (1980).

[13] A.I..NIKISHOV, Barrier scattering in field theory removal of Klein paradox; Nuclear Physics B21, pag. 346-358 (1970).

[14] L.PARKER, Quantized Fields and Particle Creation in Expanding Universes.I; Physical Review 183 no. 5, pag. 1057-1068 (1969).

[15] V.S.POPOV, Pair production in a variable external field (Quasiclassical approximation); Sov. Phys. JETP 34, pag. 709-718 (1972).

[16] J.S.SCHWINGER, On Gauge Invariance and Vacuum Polarization; Physical Review 82, no. 5, pag. 664-679, (1951).

[17] W.WASOW, Adiabatic invariance of a simple oscillator; SIAM J. Math. Anal. 4, no. 1, pag. 78-88 (1973).

(Manuscrit reçu le 7 novembre 2002)