From Electromagnetic Duality to Extended Electrodynamics

Stoil Donev

Institute for Nuclear Research and Nuclear Energy,
Bulg.Acad.Sci., 1784 Sofia, blvd.Tzarigradsko chausee 72, Bulgaria
e-mail: sdonev@inrne.bas.bg

ABSTRACT. This paper presents the transition from Classical Electrodynamics (CED) to Extended Electrodynamics (EED) from the electromagnetic duality point of view, and emphasizes the role of the canonical complex structure in $\mathcal{R}^2$ in, both, nonrelativistic and relativistic formulations of CED and EED. We begin with summarizing the motivations for passing to EED, as well as we motivate and outline the way to be followed in pursuing the right extension of Maxwell equations. Further we give the nonrelativistic and relativistic approaches to the extension and give explicitly the new equations as well as some properties of the nonlinear vacuum solutions.

1 Introduction

Classical Electrodynamics is, obviously, not a linear theory in presence of charges and currents. Indeed, the dynamics of the charge-carriers, considered as a continuous system, is described by the following system of nonlinear partial differential equations with respect to their velocity vector field $\mathbf{v}$,

$$\mu\nabla_{\mathbf{v}}\mathbf{v} = \rho\mathbf{E} + \frac{1}{c}(\mathbf{j}\times\mathbf{B}),$$

or u in the relativistic formulation

$$\mu c^2 u^\nu \nabla_\nu u_\sigma = -F_{\sigma\nu} j^\nu,$$

where $\mathbf{j} = \rho\mathbf{v}$; $j = \rho u$, and μ is the invariant mass density. Since this velocity vector field participates in the current expressions staying on the right-hand sides of Maxwell equations:

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \mathrm{rot}\mathbf{B} - \frac{4\pi}{c}\mathbf{j}, \quad \mathrm{div}\mathbf{B} = 0, \tag{1}$$

$$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\mathrm{rot}\mathbf{E}, \qquad \mathrm{div}\mathbf{E} = 4\pi\rho, \tag{2}$$

or in relativistic notations

$$\mathbf{d}F = 0, \quad \mathbf{d} * F = 4\pi * j, \tag{3}$$

then, obviously, the whole system becomes nonlinear. And we must consider the whole system, otherwise the energy-momentum consrvation law will be violated. Further, even in the pure field case, where no charges and currents are present and the field equations are linear, Maxwell theory has its *nonlinear part*, namely, the Poynting energy-momentum balance equation

$$\frac{1}{c}\frac{\partial}{\partial t}\frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} = -\mathrm{div}\frac{\mathbf{E} \times \mathbf{B}}{4\pi},$$

and this *nonlinear* equation is of *basic importance* for the theory, even just because of its permanent and everyday use. In fact, we should hardly trust Maxwell equations if this everyday used and verified (for finite volume computations) Poynting equation was not consistent with them. But, we should not forget that *the Poynting balance equation may be consistent also with other field equations, not just with Maxwell's ones, in particular, with appropriate nonlinear ones.* It is worth noting at this moment that the duality invariance as considered in [1], is closely connected namely with the energy-momentum quantities and relations, while Maxwell field equations may be cast into $\mathcal{R}^2$ covariant form.

Let's recall now that the first steps of theoretical physics, made by Newton with his three laws of mechanics, establish, in fact, the fundamental conservation laws in the form of differential balance equations for the momentum and energy.

The *universal* and *conservative* character of these two quantities, energy and momentum, explains the remarkable power of classical mechanics. As for their importance in quantum theory we could hardly imagine it without *the Hamiltonian.* The conservation laws are the heart of all physics, because physics is doing with real objects, and we could hardly think of real physical objects at all if these objects have no any constant in time properties. Further, we could hardly understand the interaction in nature if there are no *universal* (i.e. carried by any physical object) conservative quantities like *energy* and *momentum.* Indeed, from modern physics point of view interaction in mechanics, as well as in the other branches of physics, *necessarily* requires energy-momentum exchange. The differential equation form of this energy-momentum exchange gathers together the two mutually consistent tendencies of existence: *conservation* and *alteration*. And the *alteration*, or *time-evolution*, defines, in fact, those boundaries behind which the physical system under consideration can not exist anymore. This duality between conservation and time-evolution is theoretically implemented through the *dynamical equations* of the physical system under consideration, and only those dynamical equations have to be considered as *reasonable* whose solutions have *reasonable* conservation and stability properties, e.g. the evolution of a free and time-stable object must not lead to a self-ruin, and the corresponding conservative quantities carried by the solutions should accept *finite* values, *not* infinite ones.

In view of the above we can look at the second principle of Newton in the following way:

The basic equations, governing some class of mechanical objects and their interactions, should start with establishing how the local energy-momentum exchange among these objects is performed, and all further peculiarities of their behavior to appear in the theory as correspondingly consistent relations with this basic initial fundament.

Looked at this way this Newton's principle can be easily and appropriately extended to description of continuous (field) objects. We know that the local energy-momentum conservation laws of every (linear or nonlinear) field theory are *nonlinear* partial differential equations, and, following the Newton's approach of pointing out dynamical equations, we must pay the corresponding respect these nonlinear equations deserve. In other words, we should establish first *how much* and *in what way* the continuous physical system under consideration is potentially able to exchange locally energy-momentum with the rest of the world, and afterwards to go on with taking into account in a consistent way its other features. This is the general approach we are going to follow in looking for an adequate nonlinearization of the pure field Maxwell equations.

On the other hand, it is much easier to work with linear dynamical field equations, especially if we have some corresponding experimental evidence for the assumption of such linear equations as a theoretical basis. But we must *not forget* the *limited* character of *any* specific experimental evidence when it is considered as a basis for fundamental assumptions. It seems more reliable to establish first the local relations, describing the balance of at least some of the universal conserved quantities, and then to go further with more precisions and specifications.

Let's recall now some features of the classical electromagnetic pure field theory. As it is well known, it is traditionally taught starting with Faraday's electromagnetic induction law and with Maxwell's magneto-electric induction law :

$$\frac{d}{dt}\int_S *\mathbf{B} = -c\int_l \mathbf{E}, \quad \frac{d}{dt}\int_S *\mathbf{E} = c\int_l \mathbf{B}.$$

Remark: Here and further we denote by $*$ the Hodge operator, defined by the corresponding metric g and volume form ω_g, through the relation

$$\alpha \wedge \beta = g(*\alpha, \beta)\omega_g,$$

where α and β are p and $n-p$ forms respectively. Also, we identify through the metric the covariant and contravariant vector and tensor fields.

The above integral relations, together with div$\mathbf{E} = 0$ and div$\mathbf{B} = 0$ lead to linear field equations for the components of $\mathbf{E}$ and $\mathbf{B}$, moreover, as a necessary condition it is obtained that any component U of these vector fields is obliged to satisfy the d'Alembert wave equation $\Box U = 0$. Now, the solutions of the field equations are meant to describe real time-stable continuous objects, usually called fields. Unfortunately, the wave equation $\Box U = 0$ predicts *strong time-instability* for any smooth enough *finite* initial condition (the Poisson's theorem), and besides, the *infinite* initial conditions (e.g. those leading to harmonic plane waves) require *infinite* energy of the corresponding solutions. It seems hardly reasonable to think of real objects carrying infinite energy, so we have to admit that the free field solutions of the Faraday-Maxwell equations do not present adequate enough models of any real continuous object because of the *finite* and *time-stable* nature of the latter.

In order to see the merits of the Poynting balance equation in this relation we consider first the well known plane wave solution of the pure field Maxwell equations in the appropriate coordinate system:

$$\mathbf{E} = \big[u(\xi + \varepsilon z), p(\xi + \varepsilon z), 0\big],$$
$$\mathbf{B} = \big[\varepsilon p(\xi + \varepsilon z), -\varepsilon u(\xi + \varepsilon z), 0\big], \varepsilon = \pm 1, \xi = ct,$$

where u and p are arbitrary differentiable functions. Even if u and p are soliton-like with respect to the coordinate z, they do not depend on the other two spatial coordinates (x, y). Hence, the solution occupies the whole $\mathcal{R}^3$, or its infinite subregion, and clearly it carries infinite integral energy

$$W = \frac{1}{4\pi}\int_{\mathcal{R}^3}\frac{\mathbf{E}^2+\mathbf{B}^2}{2}dxdydz = \frac{1}{4\pi}\int_{\mathcal{R}^3}(u^2+p^2)dxdydz = \infty.$$

In particular, the popular harmonic plane wave

$$u = U_o\cos(\omega t \pm k_z.z),\ \ p = P_o\sin(\omega t \pm k_z.z),$$
$$c^2k_z^2 = \omega^2,\ U_o = const,\ \ P_o = const,$$

clearly occupies the whole 3-space and carries infinite energy

$$W = \frac{1}{4\pi}\int_{\mathcal{R}^3}(U_o + P_o)dxdydz = \infty.$$

The plane wave solutions reflect well enough some features of the notion for energy-momentum propagation in a fixed spatial direction (the axis z in this system of coordinates), but they all are infinite, no dependence on the transverse coordinates ((x, y) in this system of coordinates) is allowed by the equations, just dependence on the running wave argument $(\xi+\varepsilon z)$ of u and p is allowed.

Following the Newton's approach, let's check now the pure field Poynting equation, which is nonlinear and which describes *differentially* the local intrafield energy-momentum redistribution during the time evolution, whether it admits finite, i.e. spatial soliton-like solutions. Suppose that in the above system of coordinates we have $u = u(x, y, \xi + \varepsilon z)$ and $p = p(x, y, \xi + \varepsilon z)$, where the dependence on the three spatial coordinates is *arbitrary*. The corresponding $\mathbf{E}$ and $\mathbf{B}$ surely do *not* define solution to Maxwell equations, and we check if these $\mathbf{E}$ and $\mathbf{B}$ define solution to the Poynting equation. We obtain $\frac{1}{2}(\mathbf{E}^2+\mathbf{B}^2) = u^2+p^2$ and $\mathbf{E}\times\mathbf{B} = -\varepsilon(0, 0, u^2+p^2)$. Denoting $u^2+p^2 = \phi$ and the derivative of ϕ with respect to $(\xi + \varepsilon z)$ by $'$ we obtain

$$\frac{\partial}{\partial t}\Phi = c\Phi',\ \ -c\mathrm{div}(\mathbf{E}\times\mathbf{B}) = c\Phi',$$

i.e. the Poynting equation is satisfied. Now, since u and p are *arbitrary* functions of their arguments we conclude that the Poynting equation *does* admit photon-like (3+1) soliton solutions, while Maxwell equations do NOT, they predict a quick self-ruin of any finite 3-dimensional smooth enough initial field configuration. May be it is worth at this moment to say that under (3+1)-soliton we meen a *time stable continuous nondispersing finite object of 3 spatial dimensions, having internal dynamical structure, carrying finite integral energy-momentum, and the translational component of its propagation velocity is constant and is along some (straightline) direction in the 3-space.* Such solitons are called *photon-like* if they move translationally as a whole with the velocity of light, which meens that their integral energy-momentum vector has zero length in Minkowski space-time. The corresponding solutions, describing such objects, I call *(3+1) soliton-like* or just *soliton* solutions.

Hence, we are facing two alternatives: the *first* one offers *linear* equations with nonreasonable, i.e infinite, or finite but strongly time-unstable, vacuum solutions; the *second* one *could* offer reasonable finite and time-stable vacuum solutions if an *appropriate nonlinearization* of the linear Maxwell equations, consistent with the Poynting relation, is found.

Another reason to reconsider from this point of view classical Faraday-Maxwell theory comes from quantum theory, where the elementary quantum objects (free photons, free electrons, etc.) are considered as point-like, i.e. *structureless*, objects as it is in classical mechanics. But the Planck's formula $E = h\nu$ definitely *requires* any of these free objects to demonstrate *intrinsic periodic process* with frequency ν, which *no point-like free object can do*: if the object has no structure the periodicity may be caused only by an outside agent, which meens that the object is NOT free. Further, this *no structure assumption* makes the elementary quantum objects *eternal* and *undestroyable*, because there is nothing, no structure, to be destroyed; they are allowed just to change their energy-momentum under any external perturbation, therefore, no explanation of the observed transformations (e.g. anihilation) among these microobjects under collisions would be possible: *undestroyable entities can not transform into each other.* In short, the free elementary quantum objects do *not admit in principle* the point-like, i.e. the structureless, approximation, and this seems to be the *most important difference* between *classical* objects and *quantum* objects. Therefore, it seems unreasonable to try to build theory of quantum objects on the assumption that they are considered as classical (i.e. nonquantum) objects.

In our view, we have to let such inconsistencies go out of physical theories, and we must pay the corresponding respect to the structure these microobjects possess through some further reasonable and appropriate development of the theory. Therefore, any success of modern theoretical physics in doing with extended, *not* point-like, field objects, must be correspondingly respected and appreciated. In view of no enough initial experience and insight in working with such objects, it seems more reasonable to begin with a study of a specific field, than starting a straightforward attack of the most general case. The comparetively well developed classical and quantum electrodynamics makes photons the most natural and the most promising objects for this purpose.

Considerations of this kind made us favor the second of the above mentioned two alternatives. In what follows we shall briefly outline our approach to nonlinearization of Maxwell equations which was called Extended Electrodyanmics [2]. The suggestive nature of the duality properties of the electromagnetic field, which were appropriately interpreted and described in [1], will be clearly seen and thoroughly used.

2 Nonrelativistic consideration

As it was shown in [1] the dual nature of the electromagnetic field naturally leads to the nonrelativistic formulation of Maxwell equations by means of the $\mathcal{R}^2$ valued 1-form ω and of the canonical complex structure $\mathcal{I}$ of $\mathcal{R}^2$ (we use the notations in [1]):

$$\omega = \mathbf{E}\otimes\varepsilon^1 + \mathbf{B}\otimes\varepsilon^2, \tag{4}$$

$$*\mathbf{d}\omega - \frac{1}{c}\frac{\partial}{\partial t}\mathcal{I}_*(\omega) = \frac{4\pi}{c}\mathcal{I}_*(\mathcal{J}), \quad \delta\omega = -4\pi\mathcal{Q}, \tag{5}$$

$$\mathcal{Q} = \rho_e\otimes\varepsilon^1 + \rho_m\otimes\varepsilon^2, \quad \mathcal{J} = \mathbf{j}_e\otimes\varepsilon^1 + \mathbf{j}_m\otimes\varepsilon^2, \tag{6}$$

where δ is the coderivative, $(\varepsilon^1, \varepsilon^2)$ is the canonical basis of $\mathcal{R}^2$,

$$\mathcal{I}_*\omega = \mathbf{E}\otimes\mathcal{I}(\varepsilon^1) + \mathbf{B}\otimes\mathcal{I}(\varepsilon^2) = \mathbf{E}\otimes\varepsilon^2 - \mathbf{B}\otimes\varepsilon^1,$$

and $\rho_e, \rho_m, \mathbf{j}_e, \mathbf{j}_m$ are electric density, magnetic density, electric current and magnetic current, respectively.

The above formulae imply that the field has two differentially interrelated but algebraically distinguished components. Following our approach, we look now on this circumstance from energy-momentum exchange point of view. Clearly, the field ω is potentially able to exchange energy-momentum with any other physical system, which also is potentially able to exchange energy-momentum with the field, and this energy-momentum exchange may, in general, be carried out through each of its 2 vector components $\mathbf{E}$ and $\mathbf{B}$. Further, an intra-field energy-momentum exchange between the two vector components of the field may also take place, and this third intra-field exchange may be responsible for the intrinsic spin momentum of the field. Hence, we have to describe these *three* potentially possible and independent exchange processes, where *independent* means that, in general, any of these exchanges may go without the other two to occur. In particular, the intra-field exchange may occur in the pure field case.

Such a situation can be modelled by pointing out a 3-dimensional vector space W, where each of the dimensions will account for only one of these exchange processes. Since our field object ω takes values in $\mathcal{R}^2$ it is natural to try to connect algebraically W with $\mathcal{R}^2$. The simplest such space appears to be the symmetrized tensor product $Sym(\mathcal{R}^2\otimes\mathcal{R}^2)=\mathcal{R}^2\vee\mathcal{R}^2$, which is 3-dimensional. So, since $\vee:\mathcal{R}^2\times\mathcal{R}^2\rightarrow\mathcal{R}^2\vee\mathcal{R}^2$ is a bilinear map we can multiply two $\mathcal{R}^2$ valued differential forms through $\vee$, as it was pointed out in [1]. The three components of the obtained $\mathcal{R}^2\vee\mathcal{R}^2$ valued differential form are meant to represent the densities of the above mentioned energy-momentum exchange quantities.

We begin working out this idea through equations (5), of course. Let's multiply the left-hand side of the first (5) equation from the right by ω through $\vee$ and take the euclidean $*$ from the left. We obtain

$$
\begin{aligned}
\vee\Big(\,\mathbf{d}\omega-\frac{1}{c}\frac{\partial}{\partial t}\mathcal{I}_*\omega,\omega\Big)&=\left[\left(\mathrm{rot}\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right)\times\mathbf{E}\right]\otimes\varepsilon^1\vee\varepsilon^1\\
&+\left[\left(\mathrm{rot}\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right)\times\mathbf{B}\right]\otimes\varepsilon^2\vee\varepsilon^2\\
&+\left[\left(\mathrm{rot}\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right)\times\mathbf{E}+\left(\mathrm{rot}\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right)\times\mathbf{B}\right]\otimes\varepsilon^1\vee\varepsilon^2
\end{aligned}
$$

Now we take $\mathcal{I}_*$ from the left of the second (5) equation and multiply from the right by $-\mathcal{I}_*\omega$ through $\vee$. We obtain

$$\begin{aligned} -\vee(\mathcal{I}_*\delta\omega,\mathcal{I}_*\omega) =& \mathbf{B}\mathrm{div}\mathbf{B}\otimes\varepsilon^1\vee\varepsilon^1+\mathbf{E}\mathrm{div}\mathbf{E}\otimes\varepsilon^2\vee\varepsilon^2 \\ &-(\mathbf{B}\mathrm{div}\mathbf{E}+\mathbf{E}\mathrm{div}\mathbf{B})\otimes\varepsilon^1\vee\varepsilon^2. \end{aligned}$$

We sum up now these two relations:

$$\begin{aligned} &*\vee\left(*\mathbf{d}\omega-\frac{1}{c}\frac{\partial}{\partial t}\mathcal{I}_*\omega,\omega\right)-\vee(\mathcal{I}_*\delta\omega,\mathcal{I}_*\omega)= \\ &\left[\left(\mathrm{rot}\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right)\times\mathbf{E}+\mathbf{B}\mathrm{div}\mathbf{B}\right]\otimes\varepsilon^1\vee\varepsilon^1 \\ &+\left[\left(\mathrm{rot}\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right)\times\mathbf{B}+\mathbf{E}\mathrm{div}\mathbf{E}\right]\otimes\varepsilon^2\vee\varepsilon^2+ \\ &+\left[\left(\mathrm{rot}\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right)\times\mathbf{E}+\left(\mathrm{rot}\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right)\times\mathbf{B}-\mathbf{B}\mathrm{div}\mathbf{E}-\mathbf{E}\mathrm{div}\mathbf{B}\right]\otimes\varepsilon^1\vee\varepsilon^2. \end{aligned}$$

Now we do the same operations with the right-hand sides of equations (5) and obtain

$$\begin{aligned} &\frac{4\pi}{c}*\vee(\mathcal{I}_*\mathcal{J},\omega)+4\pi\vee(\mathcal{I}_*\mathcal{Q},\mathcal{I}_*\omega)= \\ &4\pi\left[\left(\frac{1}{c}(-\mathbf{j}_m\times\mathbf{E})+\rho_m\mathbf{B}\right)\otimes\varepsilon^1\vee\varepsilon^1+\left(\frac{1}{c}(\mathbf{j}_e\times\mathbf{B})+\rho_e\mathbf{E}\right)\otimes\varepsilon^2\vee\varepsilon^2+\right. \\ &\left.+\left(\frac{1}{c}(-\mathbf{j}_m\times\mathbf{B}+\mathbf{j}_e\times\mathbf{E})-\rho_m\mathbf{E}-\rho_e\mathbf{B}\right)\otimes\varepsilon^1\vee\varepsilon^2\right] \end{aligned}$$

Hence, the nonlinear equation

$$*\vee\left(*\mathbf{d}\omega-\frac{1}{c}\frac{\partial}{\partial t}\mathcal{I}_*\omega,\omega\right)-\vee(\mathcal{I}_*\delta\omega,\mathcal{I}_*\omega)=\frac{4\pi}{c}*\vee(\mathcal{I}_*\mathcal{J},\omega)+4\pi\vee(\mathcal{I}_*\mathcal{Q},\mathcal{I}_*\omega) \tag{7}$$

gives the following three equations

$$\left(\mathrm{rot}\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right)\times\mathbf{E}+\mathbf{B}\mathrm{div}\mathbf{B}=\frac{4\pi}{c}(-\mathbf{j}_m\times\mathbf{E})+4\pi\rho_m\mathbf{B}, \tag{8}$$

$$\left(\mathrm{rot}\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right)\times\mathbf{B}+\mathbf{E}\mathrm{div}\mathbf{E}=\frac{4\pi}{c}(\mathbf{j}_e\times\mathbf{B})+4\pi\rho_e\mathbf{E}, \tag{9}$$

$$\left(\mathrm{rot}\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right)\times\mathbf{E}+\left(\mathrm{rot}\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right)\times\mathbf{B}-\mathbf{B}\mathrm{div}\mathbf{E}-\mathbf{E}\mathrm{div}\mathbf{B}=$$
$$=\frac{4\pi}{c}\left(-\mathbf{j}_m\times\mathbf{B}+\mathbf{j}_e\times\mathbf{E}\right)-4\pi\rho_m\mathbf{E}-4\pi\rho_e\mathbf{B}. \tag{10}$$

It is clearly seen that the right-hand side of equation (10) becomes zero every time when the electric and magnetic currents and charges are zero, so it depends algebraically on the energy-momentum exchanges described by the first two equations, and this is not in a full accordance with our assumption that all of the three exchanges are independent on each other. Besides, generally speaking, we could imagine that there exist some new, unknown kind of media built not of electric and magnetic charges as in the classical case, but still able to exchange energy momentum with the field, e.g. a gravitational field. This suggests the following formal generalization: 4 algebraically independent vector fields $\mathbf{a}^i, i=1,2,3,4$ and 4 functions $a^i, i=1,2,3,4$ to be introduced, so that in the most general case we shall have

$$\left(\mathrm{rot}\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right)\times\mathbf{B}+\mathbf{E}\mathrm{div}\mathbf{E}=\mathbf{a}^1\times\mathbf{B}+a^1\mathbf{E} \tag{11}$$

$$\left(\mathrm{rot}\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right)\times\mathbf{E}+\mathbf{B}\mathrm{div}\mathbf{B}=\mathbf{a}^4\times\mathbf{E}-a^4\mathbf{B} \tag{12}$$

$$\left(\mathrm{rot}\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right)\times\mathbf{E}+\left(\mathrm{rot}\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right)\times\mathbf{B}-\mathbf{B}\mathrm{div}\mathbf{E}-\mathbf{E}\mathrm{div}\mathbf{B}=$$
$$=\mathbf{a}^2\times\mathbf{B}+\mathbf{a}^3\times\mathbf{E}+a^2\mathbf{E}-a^3\mathbf{B}. \tag{13}$$

Obviously, equations (8)-(10) correspond to

$$\mathbf{a}^1=\mathbf{a}^3=\frac{4\pi}{c}\mathbf{j}_e;\ \mathbf{a}^2=\mathbf{a}^4=\frac{4\pi}{c}\mathbf{j}_m;\ a^1=-a^3=4\pi\rho_e;\ a^2=a^4=-4\pi\rho_m. \tag{14}$$

We have to say, that, in general, the vector fields $\mathbf{a}^i$ and the corresponding functions a^i are not subject to the condition to satisfy corresponding continuity equations, although this is not forbidden, this depends on the features of the medium which participates in the energy-momentum exchange with the field through its own $\mathbf{a}^i, a^i$.

We mention that choosing appropriately the quantities $(\mathbf{a}^i, a^i)$ in our equations we can obtain, for example, the extension of Maxwell pure field equations considered by B.Lehnert [3]. In fact, if we put $\mathbf{a}^2 = \mathbf{a}^4 = 0$, $a^2 = a^4 = 0$, $\mathrm{div}\mathbf{B} = 0$, $\mathbf{a}^1 = \mathbf{a}^3 = const.\mathbf{j}$, $a^1 = a^3 = \mathrm{div}\mathbf{E} = \sigma$, $\mathbf{j} = \sigma\mathbf{C}$, and $\mathbf{C}^2 = c^2$, where c is the velocity of light in vacuum, then the solutions of Lehnert's extension of Maxwell equations constitute a class of solutions of our equations. The only difference is in the interpretation: B.Lehnert consideres these equations as free-field, while in our approach they will appear as describing some special kind of medium.

Let's write down explicitly the corresponding vacuum equations:

$$\left(\mathrm{rot}\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B} + \mathbf{E}\mathrm{div}\mathbf{E} = 0 \tag{15}$$

$$\left(\mathrm{rot}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\right) \times \mathbf{E} + \mathbf{B}\mathrm{div}\mathbf{B} = 0 \tag{16}$$

$$\left(\mathrm{rot}\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{E} + \left(\mathrm{rot}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\right) \times \mathbf{B} - \mathbf{B}\mathrm{div}\mathbf{E} - \mathbf{E}\mathrm{div}\mathbf{B} = 0. \tag{17}$$

We note, that Faraday-Maxwell theory separates its own sector of solutions in the frame our more general nonlinear approach based on equations (11)-(13), nothing from that theory is lost and may be used in every situation where it is considered to provide a good enough approximation. But, for example, in case of describing soliton-like behavior of electromagnetic radiation in free space we have to turn to the nonlinear sector of solutions of (15)-(17). As for the consistency of the pure field equations (15)-(17) with the Poynting pure field equation, it follows from (15)-(17) that

$$\mathbf{E}.\left(\mathrm{rot}\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) = 0,\ \ \mathbf{B}.\left(\mathrm{rot}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\right) = 0, \tag{18}$$

and from these two relations the Poynting pure field equation follows.

Finally, in order to get some initial impression about the nonlinear vacuum solutions of (15)-(17) we note some of their properties. First, talking about nonlinear vacuum solutions of (15)-(17) we mean those vacuum solutions which satisfy the following nonequalities:

$$\mathrm{rot}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \neq 0, \quad \mathrm{rot}\mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} \neq 0, \quad \mathbf{div E} \neq 0, \quad \mathbf{div B} \neq 0. \tag{19}$$

Now it is easy to verify that, besides (18), the nonlinear vacuum solutions satisfy the following relations:

$$\mathbf{E}.\mathbf{B} = 0, \ \mathbf{E}^2 = \mathbf{B}^2, \ \mathbf{B}.\left(\mathrm{rot}\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) = \mathbf{E}.\left(\mathrm{rot}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\right).$$

These relations guarantee that $|\mathbf{E}|$ and $|\mathbf{B}|$ of the nonlinear solutions of (15)-(17) are invariant with respect to the duality transformations.

Consider now the vector fields

$$\vec{\mathcal{E}} = \mathrm{rot}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} + \frac{\mathbf{E}\times\mathbf{B}}{|\mathbf{E}\times\mathbf{B}|}\mathrm{div}\mathbf{B}, \tag{20}$$

$$\vec{\mathcal{B}} = \mathrm{rot}\mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} - \frac{\mathbf{E}\times\mathbf{B}}{|\mathbf{E}\times\mathbf{B}|}\mathrm{div}\mathbf{E}. \tag{21}$$

Under a duality transformation they transform like $\mathbf{E}$ and $\mathbf{B}$ respectively. Equations (15)-(17) are equivalent respectively to

$$\vec{\mathcal{E}}\times\mathbf{E} = 0, \ \vec{\mathcal{B}}\times\mathbf{B} = 0, \ \vec{\mathcal{E}}\times\mathbf{B} + \vec{\mathcal{B}}\times\mathbf{E} = 0. \tag{22}$$

It follows:

$$\vec{\mathcal{E}} = f\mathbf{E}, \ \vec{\mathcal{B}} = f\mathbf{B}, \ |\vec{\mathcal{E}}| = |\vec{\mathcal{B}}|,$$

where f is a function. These properties allow the important concept of *scale factor* $L(\mathbf{E},\mathbf{B})$ for any nonlinear vacuum solution to be defined by

$$L(\mathbf{E},\mathbf{B}) = \frac{1}{|f|} = \frac{|\mathbf{E}|}{|\vec{\mathcal{E}}|} = \frac{|\mathbf{B}|}{|\vec{\mathcal{B}}|}. \tag{23}$$

Clearly, $L(\mathbf{E},\mathbf{B})$ is also invariant with respect to duality transformations. Hence, every nonlinear solution defines its own scale.

We examine now the vector fields $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ as measures of the internal (spin) angular momentum. Clearly, we have in view only the nonlinear solutions of (15)-(17). The third equation of (23) says that $\vec{\mathcal{E}}\times\mathbf{B} = -\vec{\mathcal{B}}\times\mathbf{E}$. Both sides of this relation measure the same changes of momentum: the left-hand side says how much momentum is transferred from $\mathbf{E}$ to $\mathbf{B}$, and the right-hand side says how much momentum is transferred from $\mathbf{B}$ to $\mathbf{E}$, and these quantities are equal in magnitude. Note that these mutual transfers are made through the rotations rot$\mathbf{E}$ and rot$\mathbf{B}$ of $\mathbf{E}$ and $\mathbf{B}$, so we could interpret these continuous processes of intrafield momentum transfers as an appearance of the intrinsic rotational (spin) features of the solution. Further, besides these rotational degrees of freedom, the nature of the solution makes the field propagate along the spatial direction defined by the Poynting vector, so it carries energy-momentum along $\mathbf{E}\times\mathbf{B}$. Therefore, we may expect the time derivative of the projection of $\vec{\mathcal{E}}\times\mathbf{B}$ on the direction of propagation to be equal to $\mathrm{div}(\vec{\mathcal{E}}\times\mathbf{B})$. Hence, we may assume

$$\frac{1}{c}\frac{\partial}{\partial t}\frac{(\vec{\mathcal{E}}\times\mathbf{B}).(\mathbf{E}\times\mathbf{B})}{|\mathbf{E}\times\mathbf{B}|} = -\mathrm{div}(\vec{\mathcal{E}}\times\mathbf{B}). \tag{24}$$

In other words, we consider the quantity

$$\vec{\mathcal{H}} = (\vec{\mathcal{E}}\times\mathbf{B}).\frac{\mathbf{E}\times\mathbf{B}}{|\mathbf{E}\times\mathbf{B}|} \tag{25}$$

as a measure of the spin momentum density, so its time change has to come from somewhere, and equation (24) defines this change as the divergence of $\vec{\mathcal{E}}\times\mathbf{B} = -\vec{\mathcal{B}}\times\mathbf{E}$. Equation (24) has to be considered, of course, as an additional relation to equations (15)-(17) when these intra-rotational properties of the solutions are considered as important enough to be quantitatively accounted for.

3 Relativistic consideration

We recall from [1] the concept of $(*,\mathcal{I})$-equivariant $\mathcal{R}^2$ valued differential form Γ:

$$(*,\mathcal{I})\Gamma = *\Gamma_1\otimes\mathcal{I}(\varepsilon^1) + *\Gamma_2\otimes\mathcal{I}(\varepsilon^2) = \Gamma_1\otimes\varepsilon^1 + \Gamma_2\otimes\varepsilon^2 = \Gamma,$$

where $(\varepsilon^1,\varepsilon^2)$ is the canonical basis of $\mathcal{R}^2$. Further we consider Minkowski space-time with metric tensor $\eta_{\mu\nu}$, signature $(-,-,-,+)$ and corresponding $*$-operator.

The relativistic nonlinearization respects the following principles:

-the basic field is $\mathcal{R}^2$-valued $(*,\mathcal{I})$-equivariant differential 2-form Ω on Minkowski space-time,

-the basic differential vacuum equations for Ω must appear as one 3-dimensional relation for one object,

-the physical sense of the basic differential equations in presence of external fields must be local balance of universal conserved quantities,

-Faraday-Maxwell theory must be entirely present in the nonlinear generalization.

The requirement for $(*,\mathcal{I})$-equivariance of Ω, where $\mathcal{I}$ is the canonical complex structure in $\mathcal{R}^2$, implies that in the canonical orthonormal basis $(\varepsilon^1,\varepsilon^2)$ in $\mathcal{R}^2$ we have $\Omega = F\otimes\varepsilon^1 + *F\otimes\varepsilon^2$. Now, these basis vectors determine two subspaces $\{\varepsilon^1\}$ and $\{\varepsilon^2\}$ of $\mathcal{R}^2$ and the corresponding projections $pr_1 : \mathcal{R}^2 \rightarrow \{\varepsilon^1\}$; $pr_2 : \mathcal{R}^2 \rightarrow \{\varepsilon^2\}$, $\mathcal{R}^2 = \{\varepsilon^1\}\oplus\{\varepsilon^2\}$. These two projection operators extend to projections π_1 and π_2 in the $\mathcal{R}^2$-valued differential forms on M:

$$\pi_1\Omega = \pi_1(\Omega^1\otimes k_1 + \Omega^2\otimes k_2) = \Omega^1\otimes\pi_1 k_1 + \Omega^2\otimes\pi_1 k_2 =$$

$$= \Omega^1\otimes\pi_1(a\varepsilon_1 + b\varepsilon^2) + \Omega^2\otimes\pi_1(m\varepsilon_1 + n\varepsilon^2) = (a\Omega^1 + m\Omega^2)\otimes\varepsilon^1.$$

Similarly,

$$\pi_2\Omega = (b\Omega_1 + n\Omega_2)\otimes\varepsilon^2.$$

In particular, for our Ω we have simply

$$\pi_1(F\otimes\varepsilon^1 + *F\otimes\varepsilon^2) = F\otimes\varepsilon^1,\ \pi_2(F\otimes\varepsilon^1 + *F\otimes\varepsilon^2) = *F\otimes\varepsilon^2.$$

We consider now the expression $\vee(\delta\Omega, *\Omega)$. In our basis $(\varepsilon^1,\varepsilon^2)$ we obtain

$$\begin{aligned}\vee(\delta\Omega,*\Omega) &= \vee(\delta F\otimes\varepsilon^1 + \delta * F\otimes\varepsilon^2, *F\otimes\varepsilon^1 + ** F\otimes\varepsilon^2) = \\ &= (\delta F\wedge *F)\otimes\varepsilon^1\vee\varepsilon^1 + (\delta * F\wedge ** F)\otimes\varepsilon^2\vee\varepsilon^2 \\ &\quad + (\delta **F\wedge F + \delta * F\wedge *F)\otimes\varepsilon^1\vee\varepsilon^2.\end{aligned}$$

The first two components of this expression determine how much energy-momentum the field is potentially able to exchange with the external field, and the third component determines how much energy-momentum may be redistributed between F and $*F$. Hence, if the field Ω is free, all these three components must be zero, and we obtain our free field equations (recall $**F=-F$):

$$\delta F\wedge *F=0,\ \ \delta *F\wedge F=0,\ \ \delta F\wedge F-\delta *F\wedge *F=0. \qquad (26)$$

In components, these equations in terms of the coderivative δ read

$$F_{\mu\nu}(\delta F)^{\nu}=0,\ \ (*F)_{\mu\nu}(\delta *F)^{\nu}=0,\ \ F_{\mu\nu}(\delta *F)^{\nu}+(*F)_{\mu\nu}(\delta F)^{\nu}=0. \qquad (27)$$

These equations, in view of the energy-momentum relations

$$Q_{\mu}^{\nu}=\frac{1}{4\pi}\left[\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta_{\mu}^{\nu}-F_{\mu\sigma}F^{\nu\sigma}\right]=\frac{1}{8\pi}\left[-F_{\mu\sigma}F^{\nu\sigma}-(*F)_{\mu\sigma}(*F)^{\nu\sigma}\right], \qquad (28)$$

and

$$\nabla_{\nu}Q_{\mu}^{\nu}=-\frac{1}{4\pi}\left[F_{\mu\nu}(\nabla_{\sigma}F^{\sigma\nu})+(*F)_{\mu\nu}(\nabla_{\sigma}(*F)^{\sigma\nu})\right] \qquad (29)$$

in Maxwell theory make possible using the standard energy-momentum tensor of Maxwell theory as energy-momentum tensor of this extended electromagnetic theory.

In the same way, in terms of the exterior derivative $\mathbf{d}$ we have

$$\begin{aligned}&(*F)^{\mu\nu}(\mathbf{d}*F)_{\mu\nu\sigma}=0,\ \ F^{\mu\nu}(\mathbf{d}F)_{\mu\nu\sigma}=0,\\&(*F)^{\mu\nu}(\mathbf{d}F)_{\mu\nu\sigma}+F^{\mu\nu}(\mathbf{d}*F)_{\mu\nu\sigma}=0.\end{aligned} \qquad (30)$$

The coordinate free form of (26) reads:

$$\vee(\delta\Omega,*\Omega)=0. \qquad (31)$$

The above equations are equivalent to the nonrelativistic equations (15)-(17) when $F_{\mu\nu}$ are correspondingly interpreted.

Now, assume that Ω propagates in presence of an external field which is able to exchange energy-momentum with Ω. This means that the external field must have its own "tools" to participate in the interaction. In classical electrodynamics the usual external field is "continuously distributed charged particles". Its interaction tools are the electric and magnetic charges and currents, and the particles exchange energy-momentum with the field along two independent ways: with F by means of j_e and with $*F$ by means of j_m. In the formal generalization (11)-(13) we introduced 4 vector fields $\mathbf{a}^i$ and 4 functions a^i to describe the most general exchange process. So, we have to give the relativistic equivalent of equations (11)-(13). This is achieved through introducing two $\mathcal{R}^2$-valued differential 1-forms, denoted by $\Phi = \alpha^1 \otimes \varepsilon^1 + \alpha^2 \otimes \varepsilon^2$, and $\Psi = \alpha^3 \otimes \varepsilon^1 + \alpha^4 \otimes \varepsilon^2$, where $\alpha^i, i = 1,2,3,4$ are four 1-forms on M. The basic equation takes the form

$$\vee(\delta\Omega, *\Omega) = \vee(\Phi, *\pi_1\Omega) \; + \; \vee(\Psi, *\pi_2\Omega). \tag{32}$$

This equation is equivalent to (11)-(13) with $\alpha^i = (\mathbf{a}^i, a^i)$: α^1 describes the capability of the external field to exchange energy-momentum with F, α^4 describes the the capability of the external field to exchange energy-momentum with $*F$, and the couple (α^2, α^3) describes the capability of the external field to affect the intrafield energy-momentum redistribution. We'd like to mention once again that (32), or (11)-(13), describe the most general from formal point of view case, a concrete system may have, for example, some of the 1-forms α^i equal to zero, or dependent on one another.

We give now the relativistic equivalent of equation (24). In components and under the usual interpretation of $F_{\mu\nu}$ we have

$$\begin{aligned}\Big[*(\delta F \wedge F)\Big]_\mu &= (\delta F)^\sigma (*F)_{\sigma\mu} \\ &= \left[\left(\mathrm{rot}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\right) \times \mathbf{B} - \mathbf{E}\mathrm{div}\mathbf{B},\ -\mathbf{E}.\left(\mathrm{rot}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\right)\right].\end{aligned}$$

Recalling relations (20)-(22) we see that the right-hand side of the above relation is equal to

$$\Big[\vec{\mathcal{E}} \times \mathbf{B}, -\vec{\mathcal{E}}.\mathbf{E}\Big] = \Big[\vec{\mathcal{E}} \times \mathbf{B}, -\mathcal{H}\Big].$$

Now, equation (24) is equivalent to

$$\mathbf{d}\,(\delta F \wedge F) = 0. \tag{33}$$

If the components of F are finite functions with respect to the spatial coordinates (x, y, z), which is possible in the nonlinear sector of solutions of (32), then making use of Stokes' theorem, equation (33) leads to an integral conserved quantity, which after some normalization appears as a natural integral characteristic of the internal rotation (spin) properties of the solution as it was explained in the nonrelativistic consideration. We note two things: first, no isometries are needed here to build this conserved quantity since the energy-momentum tensor is not used; second, the local expression of this conserved quantity depends on the derivatives of the field functions while the energy-momentum densities do *not* depend on the derivatives of $F_{\mu\nu}$.

4 Conclusion

The main purpose of this paper was to show that the duality properties of the classical Maxwell equations together with the idea that the local energy-momentumn balance relations should be the starting point for basic theoretical assumptions, naturally lead to the nonlinear equations for the electromagnetic field (in vacuum and in presence of external fields) of Extended Electrodynamics. The full respect of duality we paid braught us to a full formal equivalence of the two vector-components of the field ω, or Ω, which was further recognised as a full $\mathcal{R}^2$-cavariance of the mathematical model object. We consider this approach for nonlinearization of the field equations as an appropriate extension of the second Newton's law of classical mechanics. Our view on the free microobjects as obeying the Planck's relation $W = h\nu$ necessarily resulted in favoring the soliton concept as the most appropriate working tool for now, because no point-like conception is able to explain the availability of spin-momentum of photons. This reflects our view, based on the conservation properties of the spin-momentum, that photons are real objects and NOT theoretical imagination. So, photons must carry also energy-momentum, their propagation in space must have some rotation-like component, and at every moment they must occupy finite 3-volumes of definite shape. Since we still don't know how this shape looks like we must keep the possibility to make use of *arbitrary* initial configurations, and our approach allows this through the allowed arbitrariness of the amplitude function $\Phi(x, y, \xi + \varepsilon z)$. This important moment allows the well known localizing functions from differential topology, used usually in the partition of unity construction, to be used for making the spatial dimensions of the solution FINITE, and NOT smoothly vanishing just at infinity, as is the case of the usual soliton solutions.

References

[1] S.Donev, LANL e-print, hep-th/0006208.

[2] S.Donev, M.Tashkova, Proc.Roy.Soc. of London A450, 281-291, 1995.
[3] B.Lehnert, Physica Scripta, vol.T82, 89-94, 1999.

(Manuscrit reçu le 21 novembre 2002)