Coincident Detection in Fourth Order Optical Interference Effects Part I

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ABSTRACT. Quantum theory predicts two-photon, fourth order interference which, unlike ordinary second order interference, cannot be viewed directly on a screen. Visibility requires coincident detection of distant photo-events. Implicit in the application of coincidence techniques is the existence of some definition of distant simultaneity. This, in turn, suggests a consideration of the principles of special relativity. Special relativity, as it applies to physical separability, conventions of synchronization and non-locality, is discussed from a set theoretic perspective. This approach begins with the information content of photo-correlation data and follows its reduction and transformation which culminate in a graph of quantum interference.

RÉSUMÉ. La théorie des quanta prédit l'interférence de quatrième ordre entre deux photons, interference qui, à la difference de l'interférence ordinaire de deuxième ordre, ne peut pas être vue à l'écran. La visibilité exige la détection coincidente d'événements-photo lontains. L'application de techniques de coincidence implique l'existence d'une définition de la simultanéité à distance. Ceci, à son tour, suggère qu'on devrait considerer les principles de la relativité spéciale. Nous proposons de discuter la relativité spéciale, telle qu'elle s'applique à la séparabilité physique anisi qu'aux conventions de synchronisation et de non-localité, dans une perspective théorique basée sur la théorie des ensembles mathématiques. Notre approche prend comme point de départ le contenu d'information des données de photo-correlation et suit la réduction et la transformation de ces données. Cette approche mène à la representation graphique de l'interférence de quatrième ordre.

1 Introduction

Many experiments may be said to exemplify the challenge posed to any locally realistic interpretation of non-local interaction. This discussion will be limited to some contemporary experiments in two-photon quantum optics that focus on joint detection probabilities in fourth order[1] interference. [1] That is, the experimental context of the following discussion is limited to the phenomenon of fourth order interference, focusing on the way in which experimental data can be used to support an interpretation of non-locality. The reason for choosing these particular experiments is that all evidence of fourth order interference is critically dependent upon *coincident* photo-detection. [2]

In selecting and applying coincident detection techniques, the desired effect is the identification of conjugate photon pairs via distant co-localization. The principle hypothesis being tested is the validity of quantum-theoretic calculations of joint detection probabilities as they apply to fourth order interference. An additional hypothesis under scrutiny is that of non-local interaction; i.e., that free choice regarding a precise localization of one photon of a conjugate pair can affect the distant localization properties of the other photon. [3]

The position taken here is that several practical considerations serve to restrict such localization measurements to a time-synchronous form.

Since, the subject quanta have no at-rest state, photon localization must be measured in terms of photo-detection *events*. The time argument of such events refers to the local arrival time of the photon at the detector and the event location argument refers to the spatial coordinates of the detector. Coincidence techniques attempt to identify simultaneously emitted conjugate pairs from among the many photons incident upon the detectors. This is accomplished by equalizing the time of flight for both photons and selecting for "simultaneous" distant photo-events. Photon coincidence techniques are thereby critically dependent upon a definition of distant simultaneity. [4]

Special relativity is the domain of definition for both distant simultaneity and separability.[2] Thus, by the preceding reasoning, fourth order interference experiments invite an approach based in relativity. Moreover, since classical theories of light are cited as inadequate in the case of fourth order

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[1] "Fourth order", i.e., generally involving two photons and two detectors; it denotes a description of interference involving terms that are fourth order in the electric field.
[2] And therefore, of locality and non-locality.

interference, fourth order phenomena appear to provide an undiluted interface between quantum theory and special relativity. [3, 5]

We begin by describing two equivalent idealized coincident detection apparatus, the first of which is in widespread use. The second description represents an experimental arrangement designed so that the condition of physical separability is maintained for distant photo-events. This apparatus permits the creation of two independent, local data sets. By systematically applying these hypothetical data sets to coincidence correlation, an attempt is made to discover precisely how an interpretation of non-locality can arise from physical conditions of local separability.

Within this general context, the argument follows an elementary set theoretic organization of the collection, reduction and interpretation of data obtained through the experimental device of coincident photo-detection. Particular attention will be paid to the *transformation of information* represented by that data as it progresses from apparatus design through final interpretation. For this purpose, a generalized hypothetical data set will be modeled after the type of experimental data in reference [1]. This model attempts to relate the number of coincident photo-detections n, to the setting of some experimental parameter, symbolized by the free variable θ. [6]

2 Two Photon Interference

The focus herein will be on the photo-detection apparatus itself and the organization of resultant data sets but generally without reference to any specific source of photons as a causative agency. Details of the relationship between the photon source and the synchronous detection apparatus will be reserved for Part II of this paper. However, a brief depiction of the entire apparatus, photon source and coincidence device, will help to place the following discussion in a broader context.

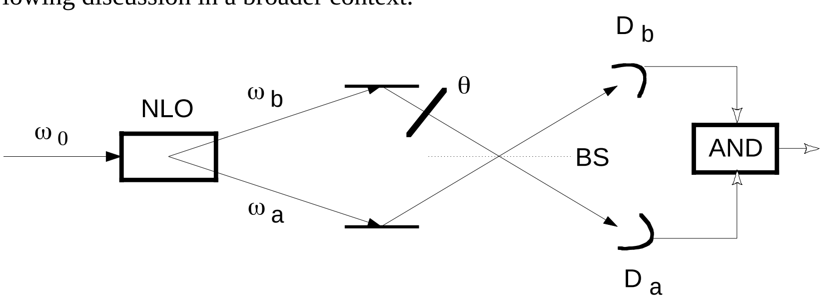

Figure 1
Idealized fourth order optical interference apparatus.

Figure 1 shows an idealized fourth order interference experiment. Photons at the pump frequency ω_0 are down-converted by the optically nonlinear properties of the crystal (NLO) into two simultaneously emitted photons ω_a and ω_b, where energy conservation requires $\omega_0 = \omega_a + \omega_b$. The photon pair is then allowed to interfere at the beam splitter (BS). There is no evidence of interference at either detector D_a or D_b separately; only after processing by the coincidence gate (logical AND) can interference be measured.

3 Coincident Detection

Figure 2 shows the basic components of the coincident photo-detection apparatus in greater detail; this is the type of apparatus in common laboratory use. Generally, via coaxial cables, electrical pulses from distant photo-detectors communicate separate photo-ionizations to a common AND gate. If the two electrical pulses overlap correctly at the AND gate, spatially separated photo-ionization events are recorded as having occurred simultaneously.

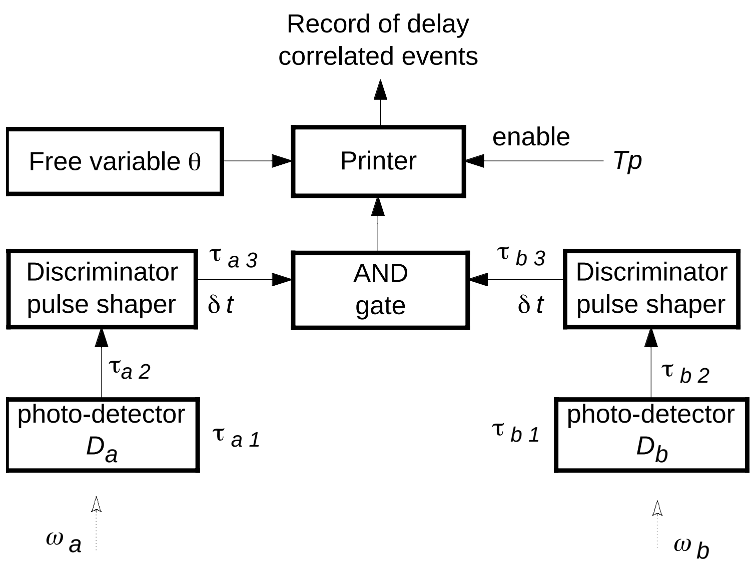

Figure 2
Coincident detection via delay equalization

Each arm of the apparatus can be modeled as a sequence of discrete delays, such as detector response time, cable propagation delay, etc. For the right and left arms, these delays are labeled τ_{bi} and τ_{ai} respectively. Ideally, the apparatus is designed so that the total delay in each arm will be identical, $\sum \tau_{ai} = \sum \tau_{bi}$. Delay equalization is one of several *conventions* for obtaining distant synchronization. [7, 8, 9]

4 Clock Transport Syncrhonization

When dealing with correlated observables in the context of the type of experiments exemplified by figure 1 [1], it has been recognized that non-local interpretations are in some way convolved with the post-processing of data. Paraphrasing Greenberger, Horne and Zeilinger [10], the non-local nature of correlated distant events can only be developed by bringing separate records of distant photo-ionizations together for comparison. In the delay-equalized method above, these “records” are brought together mechanically. However, it will be easier to articulate the relationship between distant correlations and non-locality if an alternative to delay equalized synchronization is employed.

The slow clock-transport method [11] of distant synchronization permits the single apparatus of delay synchronization to be replaced by a pair of identical, independent devices, each consisting of a local photo-detector, clock and data recorder. Figure 3 details one of two such identical devices. The salient feature of this apparatus is that it enables complete physical separability during the recording of distant photo-events. Figure 4 depicts an experimentally equivalent restatement of figure 1, employing the convention of clock transport synchronization in place of delay equalization.

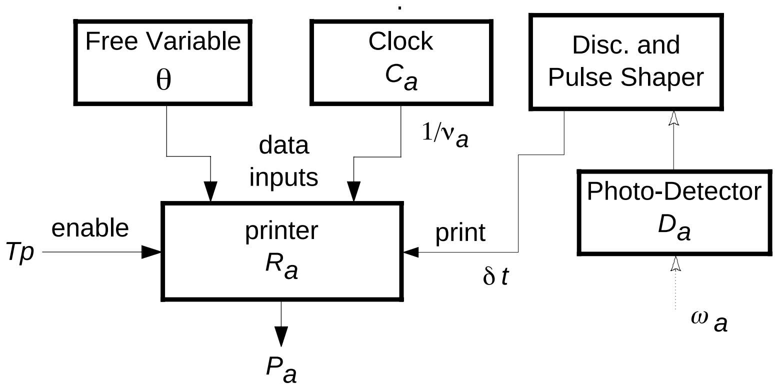

Figure 3
Detail of clock transport synchronization apparatus, see figure 4.

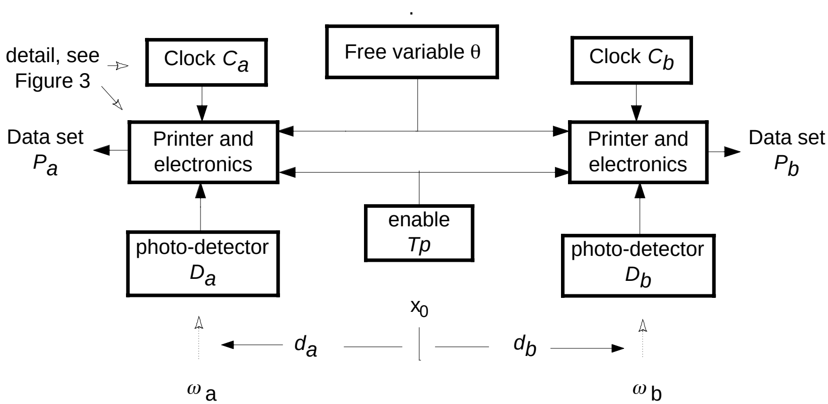

Figure 4
Coincident photo-detection via clock transport synchronization.

In figure 4, three clocks are brought to a common at-rest location x_o, for initial synchronization. [12] All three clocks are set to the same proper time and proper rest frequency $\upsilon_a = \upsilon_b = \upsilon_0$. Thereafter, one clock C_0 (not shown) remains stationary and two clocks, C_a and C_b, are separated by the distance $d_a + d_b$. Simply stated, assuming an Einstein world where the one-way speed of light is isotropic and the Lorentz transformations are appropriate, moving clocks run slow [13], so we are free to limit any change in initial synchronization due to transport by limiting the rate of that transport. [14]

However, unlike delay synchronization, clock transport synchronization allows the data collection procedure to be organized within a locally separable model. After the recording process is complete, the two physically independent sets of data will be brought together in an attempt to derive a third, correlated set of coincident photo-detections.

The following elementary set theoretic analysis will be most transparent if a formal introduction of the *effect* of clock transport synchronization is delayed until after the two sets of local-time-of-photo-detection data have been recorded. Therefore, we will assume that clocks C_a and C_b have been transport synchronized but will proceed naïve to that event until the appropriate point in the argument. [15] This approach is consistent with the conventional nature of distant synchronization: A priori, the adoption of a convention should have no effect upon the physics of a given experiment; i.e., we would not expect the phenomenon we wish to measure to be physically altered by our choice of conventions. That is, although the data may be transformed through the adoption of a convention, we would not expect the phenomenon itself to respond to an arbitrary agreement regarding the indicated time of otherwise identical physical clocks. Thus, the synchronization procedure itself has no effect on the physics of, light propagation, photo-detection nor upon physical separability of distant events.

5 Local Data Sets

The "printers" in figures 3 and 4 represent a generalized data recording system. Each printer has two data inputs and two control inputs; the first data input is the current setting of the observer controlled parameter θ, sent to both printers. The second data input is local to each printer; it is the local time from a local clock. Each local photo-detector initiates a print command by sending photo-electron pulses to the "print" control input. This creates a local-time stamp of each local photo-detection event. More specifically, the time t^i, from local clock C_i, of each local photo-event in detector D_i, is recorded on printer R_i, $i = a, b$. The superscript notation for time variables t^a and t^b is employed as a reminder that we are referring to times from spatially separated clocks. This issue is central to the analysis and will be addressed in detail shortly.

The two-part, apparatus design of figures 3 and 4 is based on the principle of separability; i.e., we expect that photon interactions local to detector D_a will be time stamped via clock C_a only and similarly for D_b and C_b. This application of separability results in two data sets, P_a and P_b, representing physically independent, time stamped photo-detection events. [16] The symbol P, will be used to represent a set of photo-detection event times recorded under these conditions of separability, which, in turn, can be stated as,

$$t^a \in P_a\,, \quad t^b \in P_b\,, \quad t^a \notin P_b\,, \quad t^b \notin P_a. \tag{1}$$

Whenever the experimental parameter θ is incremented to a new value, both printers are "enabled" for a fixed interval of time Tp, via the remaining control input in figures 3 and 4. [17] Thereby, each P_a and P_b is naturally partioned into N subsets according to the range of the free variable θ_j, $j = 1, ..., N$. Thus, the complete data set, is contained in a collection of $2N$ proper subsets, $P_{a\,j}$ and $P_{b\,j}$.

$$P_a = \bigcup_j P_{a\,j} \quad \text{and} \quad P_b = \bigcup_j P_{b\,j} \tag{2}$$

With no loss of generality but with some helpful notational simplicity, focus will be maintained on the collected subsets $P_{a\,j}$ and $P_{b\,j}$.

Define, $$P_{a\,j} = \{ t_l^a : \quad l = 0, ..., n'_j \} \text{ and } P_{b\,j} = \{ t_m^b : \quad m = 0, ..., n''_j \}. \tag{3}$$

The total number of time-of-photo-detection elements t^i, in each subset is n'_j and n''_j respectively, where these upper bounds are, in general, unrelated $n'_j \neq n''_j$. [18] The magnitudes of n'_j and n''_j remain relatively constant over θ and will be on the order of $\eta\Phi/Tp$, where η represents the effective photo-detector efficiency and Φ is the mean photon flux. [19, 20] The dependent variables we wish to track as a function of θ_j, are the number of t^i elements with time values that are equivalent in $P_{a\,j}$ and $P_{b\,j}$.

The referents of t^i : Next, for reasons that will hopefully become clear, it is necessary to acknowledge and treat in an explicit manner the full range of information represented by each t^i element: Local time values, symbolized by t^i, can signify six types of information. First, t^i signifies a physical photo-detection interaction at D_i; second, it signifies a time-dimensioned physical quantity where, "physical quantity" is itself a compound information referent [3, 4] consisting of a pure number, a series product of exponentiated physical "dimensions"[5] and a physical units convention, e.g., "seconds". Fifth, t^i, can signify a unique position on a local time coordinate. Sixth, through the superscript, it signifies one of two distinct locations in the 3-space coordinate system of the experiment, i.e., the location of the photo-detector.

Regarding the *local-time* uniqueness postulate above: A priori, no circumstance can arise where two time values from a given clock will have identical values, $t_l^a \neq t_{l+k}^a$ and $t_m^b \neq t_{m+k}^b$, $k =$ a non-zero integer representing k discrete clock oscillation cycles. This, combined with the principle of unidirectional physical clock time yields,

$$t_l^a < t_{l+1}^a, \quad l = 0,\dots,n'_j \text{ and } t_m^b < t_{m+1}^b, \quad m = 0,\dots,n''_j. \tag{4}$$

Thus, $P_{a\,j}$ and $P_{b\,j}$ are linearly ordered which justifies having written their elements in vector form, equation (3) above.

Hereafter, we will attempt to track these information referents throughout the operations leading to a graphical representation of fourth order interference. Although t^i carries the dimensional grouping and physical units of time, the superscript expands the actual information referent to that of a spacetime event.

Assuming the "raw" data has been recorded and according to θ_j organized into subsets $P_{a\,j}$ and $P_{b\,j}$, the next step is to identify those distant photo-events that were produced by conjugate photon pairs. For the experimental designs in reference [1], this means photons that were detected "simultaneously". It was for this operation that the local times of distant photo-detection events were recorded.

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[3] Referent, meaning that which is signified by the signifier, t^i.
[4] Regarding the information referents signified by "physical quantity", see Appendix.
[5] E.g., for energy, the "series product of …" is commonly written ML^2T^{-2}. See Appendix.

6 Photon Correlation via the Set Product Operator

The apparatus of clock transport synchronization, figure 4, provides conditions of local separability during the recording of all photo-events. In addition, as noted earlier, clock transport synchronization itself does not affect the physics of local photo-event records. Thus, being naive to any prior synchronization, we can state that the elements from each subset $P_{a\,j}$ and $P_{b\,j}$ were created by distant events which are unrelated. That is, absent any knowledge of synchronization, there is no means by which we may relate local events $t_l^a \in P_{a\,j}$ to distant local events $t_m^b \in P_{b\,j}$. Sets of unrelated quantities are disjoint by definition so the intersection of such sets is simply the null set.

$$P_{a\,j} \cap P_{b\,j} = \phi \tag{5}$$

Without some alternate, overriding principle, the analysis will halt here. If we wish to attribute meaning to time-correlations of physically separable events, some modification is necessary.

We have previously defined a convention, clock transport synchronization, whereby distant events may be treated as if they had occurred simultaneously. By this means, the physical condition of local separability can be replaced by a conventional definition wherein some time elements from each of $P_{a\,j}$ and $P_{b\,j}$ may be regarded as having a basis for correlation. This synchronization convention employs the principle of locality and a technique based in physical law [6] to predict what information would be available *if* time was universally absolute. [21]

Since the elements of non-intersecting sets $P_{a\,j}$ and $P_{b\,j}$ represent separable local-time quantities, it will be convenient to view the simultaneity convention as a special function that maps those elements onto synchronous-time images. This means that prior to any meaningful application of the set intersection (set product operator), the information referents of t^i must be mapped from the physical universe of separability onto a conventional world of distant simultaneity; see figure 5. Under this mapping, some of the information intrinsic to the physical circumstances of the creation of elements $t_l^a \in P_{a\,j}$ and $t_m^b \in P_{b\,j}$ will be transformed.

[6] The clock postulate as it is deduced from the time Lorentz transform.

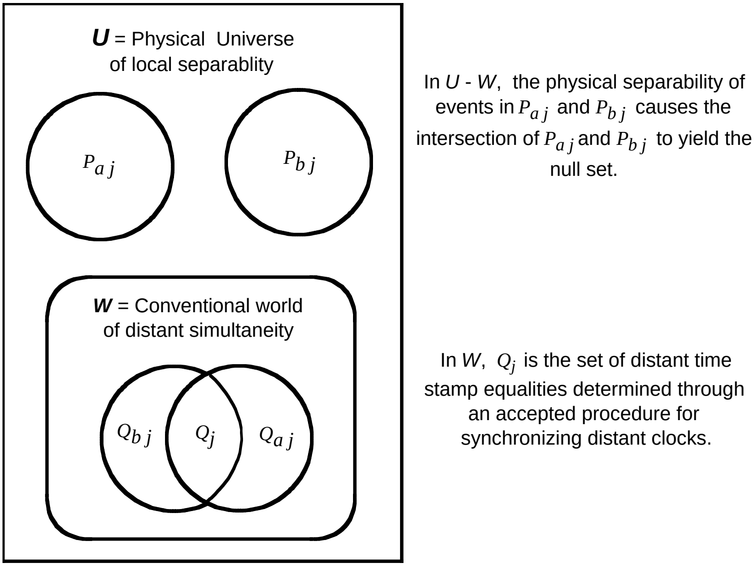

Figure 5
Venn diagram of distant photo-detection data sets before and after applying the time-synchronization operation.

Let s_T be the clock transport synchronization convention mapping function such that for,

$$P_{a\,j} = \{\, t_l^a : l = 0, \ldots, n'_j \,\} \text{ and } P_{b\,j} = \{\, t_m^b : m = 0, \ldots, n''_j \,\} \qquad \text{(Eqn. 3)}$$

$$s_T : P_{a\,j} \to Q_{a\,j}, \;\; s_T : P_{b\,j} \to Q_{b\,j},$$

where,

$$s_T\left(t_l^a \right) = \tau_l, \qquad\qquad s_T\left(t_m^b \right) = \tau_m .$$

Since s_T is an isomorphism,

$$Q_{a\,j} = \{\, \tau_l : l = 0, \ldots, n'_j \,\} \quad \text{and} \quad Q_{b\,j} = \{\, \tau_m : m = 0, \ldots, n''_j \,\}.$$

The elements of $P_{a\,j}$ and $P_{b\,j}$ are mapped onto images in $Q_{a\,j}$ and $Q_{b\,j}$, respectively. However, since clocks C_a and C_b are already synchronous, s_T has no mathematical function; it only operates on the information referents of elements t_l^a and t_m^b; i.e., in this example the numerical values are unchanged by s_T mapping. [22] For example, $s_T\left(t_l^a\right) = \tau_l$ is a mapping of the local-time element $t_l^a \in P_{a\,j}$ onto a synchronous-time element $\tau_l \in Q_{a\,j}$. Since s_T is both "one-to-one" and "onto", the elements of each image Q and preimage P are in one-to-one correspondence so they are of the same cardinality, $\left|P_{a\,j}\right| = \left|Q_{a\,j}\right|$ and $\left|P_{b\,j}\right| = \left|Q_{b\,j}\right|$.[7] In addition, the ordering of elements t_l^a and t_m^b within each subset remains unchanged under s_T. Therefore, s_T is a bijective mapping that preserves order relations; i.e., s_T is an isomorphism.

The referents of τ: Six information referents of t^i are listed in the previous section. Under s_T the fifth and sixth referents are transformed: The fifth referent of t^i, now symbolized by τ, is transformed to signify a unique[8] time location within a *synchronous-time* coordinate system; see figure 6. In addition, the sixth referent of t^i , unique spatial location, is suppressed and replaced by the following: Within that synchronous-time system, each τ_l element signifies a potential time correlation with another τ_m element in its parallel data set. Here, "potential" means the superposition of the binary conditions (*1, 0*), *1* = correlated and *0* = not correlated.

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[7] The "| |" operator, in |*P*| for example, returns the number of elements in set *P*.
[8] Unique in that, although the same time-axis position can occur at more than one spatial location, at any given spatial location it occurs at only one position on the time-axis at that location. See figure 5.

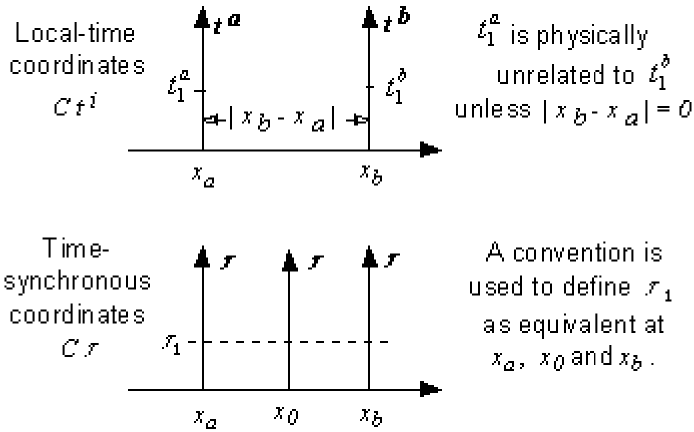

Figure 6
Local versus time-synchronous coordinate systems.

In contrast with the local-time elements of $P_{a\ j}$ and $P_{b\ j}$, the synchronous-time elements of $Q_{a\ j}$ and $Q_{b\ j}$ are potentially correlatable, so we may now state,

$$Q_{a\ j} \cap Q_{b\ j} = Q_j\ , \quad j = 1, ..., N \qquad (7)$$

Here each $Q_{\ j}$ is not constrained to the null set as in (5). In terms of information, s_T operates on each $P_{i\ j}$ such that it is restated in terms of a non-numerated potential for correlatable observations, symbolized by $Q_{i\ \ j}$. Thereafter, the set product operator acts on the $Q_{i\ j}$'s to reduce corresponding[9] vector potentials to a countable entity, subset Q_j in equation (7).

The fact that each t^i represented a photo-detection event, identified by a time-dimensioned physical quantity,[10] has been retained under the s_T mapping onto τ. However, the separable nature of local-times t^a and t^b, was

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[9] Corresponding, i.e., subsets, $Q_{i\ j}$ with equal j values.

[10] originally the local time of the photo-event

sacrificed under that mapping. This is represented symbolically by the absence of a, b spatial location superscripts in τ_l and τ_m.

7 Time Correlation and Non-Locality

The purpose of the mapping procedure is to establish a basis for the identification of conjugate photons from data obtained under conditions of physical separability. The mapping procedure provides access to the correlational properties of time-synchronization. Synchronization calibrates distant local times in order to emulate a world of absolute time and therefore, a world wherein certain information propagates at an infinite rate of speed. Under this convention, knowing the time at one location, means instantly knowing the time at a distance; it is an attempt to simulate the Newtonian ideal.

A key element of locality [23] is spatial separation. Simultaneity conventions necessarily suppress the relevance of distance in favor of time synchrony. This therefore, describes at least one form of non-locality. For example, events (x_a, τ_1), (x_0, τ_1) and (x_b, τ_1) in figure 6 lie outside each other's light cone and are therefore causally unrelated. Yet by s_T they are defined as time-synchronous and therefore informationally co-related. Similarly, in an effort to find a basis of correlation for distant photo-electron events, the physical separability of local times is purposefully supplanted by a form of non-locality that is intrinsic to distant "simultaneity".

In coincident detection, the "simultaneity" of distant measurements is critical to the process of correlation; i.e., compare equations (5) and (7). As previously mentioned, distant "simultaneity" is only definable in terms of a generally accepted procedure, i.e., a convention; it does not induce distant causality. [24] Therefore, it should be noted that any interpretation of non-locality that results from the use of coincident detection may be conventional rather than physical in nature.

8 Concluding the Data Reduction Procedure - Irreversibility

Now, returning the data reduction procedure: First, each element of Q_j denotes the existence of a (numeric) equality. The remaining information referents of Q_j are, two photo-detections, a pure number, a time based "dimensional" grouping, a physical units convention and two temporally contiguous[11] locations on a spatially distributed synchronous-time coordinate system.

It may be worthwhile to note that, again in terms of information, the issue of irreversibility seems to enter at this point, since from Q_j it is not possible to factor either $Q_{a\,j}$ and $Q_{b\,j}$ or the original $P_{a\,j}$ and $P_{b\,j}$. [25] While the mapping operation may appear to have the most significant effect on the data sets, s_T mapping is reversible between synchronous and local time. This follows from the fact that s_T is bijective; i.e., since s_T is both one-to-one and onto, it is invertible; $s_T^{-1} : Q_{a\,j} \to P_{a\,j}$. This is also consistent with the conventional nature of its function. However, the set product operation discards non-synchronous elements from $Q_{a\,j}$ and $Q_{b\,j}$; this property makes it non-reversible.

Now, to complete the data reduction procedure: Denumerate the elements $q_{k\,j} \in Q_j$, $Q_j = \left\{ q_{k\,j} : \quad k = 1, ..., n_j \right\}$, where n_j represents the cardinal number of coincident photo-detections for each $\theta = \theta_j$; i.e., $\left| Q_j \right| = n_j$, $j = 1, ..., N$. It is interesting to note that the only *locally* recorded information referent that has survived to this stage is that each element of Q_j signifies one "count"; i.e., two physically separable photo-detection events regarded as simultaneous under the synchronization convention.

Finally, the set $Q = \{ \left(\theta_j , n_j\right) \; : j = 1,...,N \}$ is graphed as N orthogonal data pairs in order to make sensible any interference-like pattern.

The previous sections have been an attempt to demonstrate that, within a set theoretic organization, the s_T mapping function followed by the set product operator are together capable of developing correlations between certain, otherwise separable events. These operations change space (a, b) and time (t) information about the locality of separable photo-detections into correlations based in a synchronization convention.

9 Discussion

Coincidence techniques are employed in reference [1], in an effort to measure conjugate photon spatial distributions in the form of co-localization at a distance. However, the fact that photons have no at-rest state forces us to accept time dependent co-localization. Time dependent co-localization

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[11] I.e., at least temporally contiguous, perhaps simultaneous.

simply means that the local times of distant photo-detections are assigned a relationship based upon distant synchronization. Therefore, implicit in the selection of the experimental stratagem of coincident detection is the necessity to adopt some convention for defining that distant synchronization. The practical effect of such a convention is to reassign priority to the time values of distant events over the physical separability of their creation. Thus, while distant photo-events themselves may be physically separable, the sought for correlations are based in a synchronization convention. The structure and characteristics of that convention will naturally effect the subsequent interpretation of any correlation it enables. For example, when we say detector D_a measures a photo-event at the “same time” as distant detector D_b, this correlation is based in a commonly agreed upon method for treating the information content of data; it is not a physical relationship. This extends to the correlation of distant physical states as well; i.e., implicit synchronization can enter a calculation by simply formulating distant field magnitudes in terms of a common time. [26]

The principle of special relativity, as applied here serves two roles. On one hand, it postulates a fundamental physical condition of locality, i.e., the physical separability of distant events. On the other hand, it prescribes a method by which we may view those events as if they were simultaneous and thus temporally co-related. However, the former is a physical principle while the latter is a device we employ to predict the effects of a hypothetical absolute time.

The device of clock transport synchronization is based in the principle of special relativity and critical use is made of locality in the application of that principle. However, with regard to the physics of photo-detection, the moving clock is not a causal agent, it only carries information about what distant local times would be if causal propagation delays were zero. Synchronization, in effect, specifies a coordinate system that is temporally unified but spatially distributed. The superimposition of such a coordinate convention upon the local times of physically separable events naturally invites a non-local interpretation of time-based relations between those events.

The well-known example from classical probability, i.e., where uncertainty, expressed as a spatial probability distribution, collapses instantaneously to a specific location upon observation, demonstrates that the proper application of physical locality should be the constraint of physical relationships not information about those relationships. By comparison, synchronization conventions can modify the way we use information about distant physical events; however, such conventions neither constrain nor modify the physical circumstances of those events.

Perhaps due to a long history of use, coincident detection techniques are commonly employed without critical examination of spacetime properties. From a spacetime perspective, these techniques are in fact richly complex in their implications for physical measurement. This is especially so within the context of non-locality.

In isolation, as presented here, without reference to a source of photons, the apparatus of figures 2 and 4 are only capable of producing non-local correlations. However, the experiments in reference [1], involve light sources that produce simultaneously created photon pairs, which separate spatially then annihilate "simultaneously" in distant photo-detectors. The interpretations expressed in reference [1] are that joint or coincident detection reveals evidence of non-locality; i.e., coincident detection does not itself induce non-locality. This interpretation will be examined in Part II where the photon source and interference apparatus will be taken into account.

10 Appendix

A) The U W Relationship

This section contains a digression from the immediate context of experimental data sets into a brief inquiry regarding the general relationship between *U*, the physical universe of separability and *W*, the conventional world of distant simultaneity, as shown in figure 4.

a) The mapping function s_T, operates on the information referents of t^i where $t^i \in P_{i\,j}$ and $P_{i\,j} \subseteq U$. Therefore, U and W will be treated in terms of the organization of those referents. By this, *U* represents a locally separable organization of informational referents while *W* represents a reorganization of those referents into a time-synchronous form. We will begin by inquiring into the fundamental set relationships, $W \cap U$ and $W \cup U$: If *U* is the "universe" then *W* must be a subset of *U* and trivially, $W \cap U = W$ and $W \cup U = U$.

b) However, in one sense *U* and *W* may be considered intrinsically disjoint. W is created by a mapping of U; i.e., W is the image of s_T acting on

U.[12] By this action, some preimage in U is fundamentally changed so that after mapping the preimage no longer exists; thus, U and W might be seen as disjoint. Stated in physical terms, local separability and distant simultaneity are mutually exclusive.

This perspective implies $W \cap U = \phi$ so that contrary to (**a**), we have $W \cap U \neq W$ and therefore $W \cup U > U$. From this it seems that "universe", defined as the physical universe of separability, is incomplete. The complete universe must be $\mathbf{U} > U$, that is, $\mathbf{U} = W \cup U$.

c) On the other hand, it's possible to adopt the following perspective. Let $I = W \cap U$ and write, $W \cap U = W \cap \left(W \cap U\right)$ but $W \cap U = \left(U - W^{C}\right)$ so that $I = W \cap \left(U - W^{C}\right)$. This suggests that the image of s_T in W and its preimage in

(U - W^{C}) may be related through information referents that are invariant under s_T.

Following this line of thought, recall that the argument of the s_T mapping function is a spacetime event, t^{i}. From equations (5) of the main text, the spatial referent, superscript i of the elements $t^{i} \in P_{i\,j}$ is suppressed under s_T, leaving the time referent, signified by the elements $\tau \in Q$. [27] Time itself $\mathcal{T}$, has three well-known informational referents: A fundamental dimension by which we organize the universe D_T, time in the form of a geometrical coordinate C_T and time as a physical quantity Pq_T,[13]

$$\mathcal{T} = D_T \cup C_T \cup Pq_T \tag{A1}$$

Note: Time, the physical quantity Pq_T, signifies other more primitive referents; its compound structure will be treated later in the Appendix. Time as a coordinate C_T generally signifies a geometric line with an associated basis vector. However, D_T appears to be conceptually primitive and thus irreducible.

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[12] Or some yet unknown subset of U, or some cell or collection of cells within a partition of U.

[13] And symmetrically, space is, $\mathcal{S} = D_L \cup C_L \cup Pq_L$

In contrast with the local-time coordinate system of *U*, distant, synchronous-time in *W* is characterized by a coordinate system having a single, unified-time-axis that is spatially distributed. Thus, s_T can be thought of as a mapping from a set of local time coordinates $C\,t^i$ onto a synchronous-time coordinate system $C\,\tau$.

$$s_T : C\,t^i \to C\,\tau \qquad \text{(A2)}$$

$C\,\tau$ is a mapping of $C\,t^i$ that defines distant time axes to be formally (but not physically) coincident in time, see figure 6, main text.

This leaves two invariant informational referents in U ; time as a fundamental dimension D_T and time as a physical quantity Pq_T. Since these referents are required in both *U* and *W*, they remain invariant under s_T and thus comprise the desired intersection,

$s_T : D_T \to D_T$, $s_T : Pq_T \to Pq_T$ so that, $I = W \cap U = D_T \cup Pq_T$.

$$\text{i.e.,} \quad I \subset W \quad \text{and} \quad I \subset U \quad \text{so that} \quad I \cup U = U. \qquad \text{(A3)}$$

But from (A2), there are elements of *W*, not contained in *I*, $I \neq W$ so we still have $W \cup U > U$.

Thus, *U* and *W* are not disjoint so $I \neq \phi$ as suggested in (**b**). [28] However, by either (**b**) or (**c**) we arrive at the same conclusion regarding the incompleteness of the definition of "universe" as physically separable. Thus, we may conclude that the conventional, time-synchronous organization of information in W is necessary for a complete definition of $\mathbf{U}$. [29]

$$\mathrm{U} = U \cup \left(s_T : U \to W\right) \qquad \text{(A4)}$$

B) "<u>Physical Quantity</u>"

The term "physical quantity" has been used several times in this paper. This section outlines a set theoretic organization of the information referents that comprise this common term. We will use the example of energy rather than time simply because it provides a richer informational basis. Time, as it is used in coincident detection will be treated in the closing discussion.

P1) The term “physical quantity” *Pq*, is defined at two levels. In the example of energy Pq_E, the minimum definition refers to energy in and of itself, while the full definition refers to a specific amount of energy expressed in the units of some convention, e.g., SI. *Pq* is informationally compound, i.e., in contrast to primitive referents such as a pure number *N* or time as a dimension D_T .

P2) All measurable physical quantities *Pq*, are constrained to descriptions based in four presumptive, primitive properties or elements of physical existence. They are the basis properties from which we construct our concept of physical reality. They are commonly labeled “dimensions” of measurement.[14] Let *E* be the set of those primitive elements, $E = \{ e_i : i = 1,2,3,4 \}$, where e_1=mass, e_2=length, e_3=time, e_4=charge

P3) *Pq* is a property of physical existence. Any specific *Pq* is defined through a unique grouping of the elements of *E*. In the example of energy, $Pq_E \equiv e_1\, e_2^2\, e_3^{-2}\, e_4^0$, which is commonly written ML^2T^{-2} .[15] We will attempt to organize a construction of the energy grouping G_E , which forms the basis of Pq_E.

P3a) Let O be the set of exponential operators,

$$O = \{ x^{-3}, x^{-2}, x^{-1}, x^0, x^1, x^2, x^3 \} \text{ or more generally,}$$

$$O = \{ x^Z : Z = \textit{the set of integers} \}.$$

P3b) In order to create a rich set of ordered pairs, take the Cartesian product,

$$O \times E = \{ ..., (x^{-3}, e_1), (x^{-3}, e_2), (x^{-3}, e_3), (x^{-3}, e_4), ...$$

[14] They are also called “physical quantities”. However, term “physical quantity” is overdetermined through common use, referring in this case to qualities of physical existence that are not quantified. E.g., the expression, “a mass of 3”, has no meaning; see (A4), “physical units”.

[15] Assume $e_i^0 \equiv 1$.

$$\ldots,\left(x^{-2},e_3\right),\ldots,\left(x^{1},e_1\right),\ldots,\left(x^{0},e_4\right),\ldots,\left(x^{2},e_2\right),\ldots\ \}$$

In this sample of $O\times E$, the four rightmost ordered pairs have been preferentially selected for display in order to facilitate the energy example. [30]

P3c) Define a general bijective mapping function on special ordered pairs,

$$\alpha:\left(x^{Z},a_i\right)\rightarrow a_i^{Z}\ ,$$

such that an α mapping of (O x E) onto H is, [16]

$$\alpha:\left(O\times E\right)\rightarrow H\ ,\ H=\left\{\ \ldots,e_1^{-3},e_2^{-3},e_3^{-3},e_4^{-3},\ldots,e_3^{-2},\ldots,e_1^{1},\ldots,e_4^{0},\ldots,e_2^{2},\ldots\right\}$$

P3d) Any physical quantity can be defined in terms of a proper subset of H. Let $G\subset H$, where the e_i subscript selection rule excludes identities for $\left|G\right|\le 4$. Since $\left|E\right|=4$, the class of subsets of H represented by $\left|G\right|=4$ is sufficient for full physical generality; for this case, the subscript rule becomes, $\{e_i^Z:\ i=1,2,3,4\}$. The fourth order energy grouping G_E, is one such subset of H,

$$G_E=\left\{\ e_1,e_2^{2},e_3^{-2},e_4^{0}\ \right\}$$

Note that the ordering of elements within G_E is not relevant to the function or definition of G_E but has been arranged here for clarity. Also, G the collected subsets of H may have intersection sets that are non-null so they do not form a partition of H.

P3e) Define another general mapping function β where, $\beta:a_i\in A\rightarrow\prod a_i$, so that, $\beta:G_E\rightarrow Pq_E$, that

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[16] Since the set of integers ranges over $-\infty\le Z\le+\infty$, in principle H can be infinite; however, in common practice, $-3\le Z\le+3$.

is, $\beta:\left\{ e_1, e_2^2, e_3^{-2}, e_4^0 \right\} \rightarrow e_1\, e_2^2\, e_3^{-2}\, e_4^0$ so, $Pq_E = e_1\, e_2^2\, e_3^{-2}$ = mass x (length)^2 x (time)$^{-2}$ (=ML^2T^{-2}), Pq_E = energy, per se.

Thus, we arrive at the minimum definition of Pq_E as a physical "quantity" in and of itself (a property of physical existence) without reference an amount of energy or a units convention.

P4) In order to undertake a measurement of a physical quantity we need to employ a physical units convention *Pu*. By combining *Pu* and *G*, then β mapping the result, we can create "derived physical units" *Du*. Let,

$$Pu = \left\{ \varepsilon_i : i = 1, 2, 3, 4 \right\},$$

Arbitrarily select the SI convention where, ε_1 = kilograms, ε_2 = meters, ε_3 = seconds and ε_4 = coulombs.

$E \cap Pu = \phi$, $\varepsilon_i \notin E$: For example, the primitive concept of mass and any conventional unit of its expression are disjoint; $e_1 \cap \varepsilon_1 = \phi$. Conceptually, "mass" is the property of an object that reacts to accelerated motion whereas "kilogram", for example, is an element of a measurement convention. Primitive, inertial mass is not comprised of units of mass. Primitive mass combined with a units convention yields a derived physical quantity which specifies a particular convention for the measure of mass. That is, mass $\cap$ kilograms = ϕ, mass $\cup$ kilograms = derived units, Du_M (kilograms of mass); $e_1 \cup \varepsilon_1 = Du_M$.

Now, for a general approach, we must return to (P2) taking the diagonal relation R_D on $(E \times Pu)$, $R_D = \left\{ (e_1,\varepsilon_1),(e_2,\varepsilon_2),(e_3,\varepsilon_3),(e_4,\varepsilon_4) \right\}$. Next, define a general bijective mapping function on ordered pairs σ, such that

$$\sigma:(e_i,\varepsilon_i) \rightarrow (e_i \cup \varepsilon_i) \text{ and } \sigma: R_D \rightarrow Em \text{, where,}$$

$$Em = \left\{ (e_1 \cup \varepsilon_1),(e_2 \cup \varepsilon_2),(e_3 \cup \varepsilon_3),(e_4 \cup \varepsilon_4) \right\}$$

Returning to (P3b), substitute Em for E, so that $\alpha:(O \times Em) \rightarrow Hm$, (P3c). In the example of energy, $Gm_E \subset Hm$, so that as in (P3d),

$$Gm_E = \left\{ \left(e_1 \cup \varepsilon_1\right), \left(e_2 \cup \varepsilon_2\right)^2, \left(e_3 \cup \varepsilon_3\right)^{-2}, \left(e_4 \cup \varepsilon_4\right)^0 \right\}$$

and as in (P3e),

$$\beta : Gm_E \to Du_E \text{ where, } Du_E = \left(e_1 \cup \varepsilon_1\right)\left(e_2 \cup \varepsilon_2\right)^2\left(e_3 \cup \varepsilon_3\right)^{-2}$$

Du_E = Energy in Joules (for example), a measurement convention.

P5) The final component in the full definition of “physical quantity” is the quantifier itself, i.e., a pure number *N* such that, $N \cup Du_E = N\ Joules$.

11 Appendix discussion

Summarizing the process above for the example of energy: Note, the first two steps are very general,

a) Given *E* and *O*, (P2) and (P3a).

b) α **:** $(O \text{ x } E)\ \_\ H$, (P3b) and (P3c).

c) Select the energy grouping, $G_E \subset H$, $G_E = \left\{ e_1, e_2^2, e_3^{-2}, e_4^0 \right\}$**,** (P3d).

d) β **:** $G_E\ \_\ Pq_E$ (*energy per se, a physical “quantity”*), (P3e).

e) Substitute *Em* for *E* then β **:** $Gm_E\ \_\ Du_E$ (*energy in Joules*), (P4).

f) $N\ \_\ Du_E = N$ *Joules* (*a specific amount of energy*), (P5).

In coincident photo-detection, we are concerned with the full definition of time, i.e., “time” per (P5) : $N\ \_\ Du_T = N$ *seconds of time*. The information content of this form of time is comprised of the following collected properties,

1 Primitive physical elements or “dimensions” of measurement, $e_i \in E$, (P2).

2 The exponential operator *O*, acting on e_i, via the Cartesian product and α mapping (P3a-c).

3 [All combinations of the above two items are contained in *H*, (P3c).]

4 The selection rules governing G_T, a subset of *H*, (P3d).

5 The elements of G_T, expressed as a series product via β mapping, (P3e).

6 [The five items above might be summarized as, a series product of exponentiated physical “dimensions”.]

7 A derived units convention Du_T, (P4).

8 A quantifier, i.e., an associated number *N*, (P5).

References

[1] A sample of papers on this topic: L. Mandel, *Phys Rev A*, **28**, 929 (1983); Z. Y. Ou, Phys. Rev., **A37**, 1607, (1988); H. Paul, *Rev. Mod. Phys.*, **58**, 209, (1986); R. Gosh, C.K. Hong, Z.Y. Ou and L. Mandel, *Phys Rev A*, **34**, 3962 (1986); R. Gosh and L. Mandel, *Phys. Rev. Lett.*, **59**, 1903, (1987); Z.Y. Ou and L. Mandel, *Phys. Rev. Lett.*, **61**, 50, (1988); Z.Y. Ou and L. Mandel, *Phys. Rev. Lett.*, **61**, 54, (1988); C. K. Hong, Z. Y. Ou and L. Mandel, *Phys. Rev. Lett.*, **59**, 2044, (1987); Z.Y. Ou and L. Mandel, *Phys. Rev. Lett.*, **62**, 2941, (1989); A. M. Steinberg, P. G. Kwait, and R. Y. Chiao, *Phys. Rev. Lett.*, **68**, 2421, (1992); P. G. Kwait, A. M. Steinberg and R. Y. Chiao, *Phys. Rev.*, **A45**, 7729, (1992); J.G. Rarity and P.R. Tapster, *Phys. Rev. Lett.*, **64**, 2495, (1990); J.D. Franson, *Phys. Rev. Lett.*, **62**, 2205, (1989); P.G. Kwiat, A.M. Stienberg, R.Y. Ciao, *Phys. Rev. A*, **47**, 2472, (1993); Z.Y. Ou, X.Y. Zou, L.J. Wang and L. Mandel, *Phys. Rev. Lett.*, **65**, 321, (1990); J. D. Franson, *Phys. Rev. Lett.*, **67**, 290, (1991).

[2] A more Detailed discussion can be found in, *Quantum Optics*, M. O.Scully and M. S. Zubiary, Cambridge Univ. Press, 1997 and R. Loudon, *The Quantum Theory of Light*, 2nd. ed., Oxford University Press, Clarendon, 1983.

[3] See, e.g., R. Gosh and L. Mandel, p. 1904

[4] “Coincidence” here is defined as temporal coincidence between distant events; i.e., distant “simultaneity”, a conventional form of simultaneity. Contrast with collisions, for example, that are both spatially and temporally coincident and therefore physically simultaneous.

[5] C. K. Hong, Z. Y. Ou and L. Mandel, *Phys. Rev. Lett.*, **59**, 2044, (1987)

[6] Typically, in order to display the envelope of any interference-like effects, the sampled frequency of coincident detections n, as a function of some free variable θ, is plotted orthogonally with respect to that variable.

[7] The conventionality of distant synchronization occurs on two levels: First, in any agreed upon method for synchronizing distant clocks, such as Einstein’s light-pulse time of flight delay, the effect is informational rather than physical. That is, the method only alters indicated time and has no effect upon the local physics. Second, there exists a somewhat more philosophical conjecture, first raised by Einstein, concerning the physical undecidability of the *one-way* speed of light; i.e., as opposed to the well-established round-trip measurements. The standard synchronization convention simply *stipulates* that the one-way speed shall be defined as equal to the round trip speed. Having acknowledged this conjecture, this paper will adopt the standard synchronization throughout.

[8] Einstein’s own view on the role of light signals can be found in, Einstein, *The Meaning of Realtivity*, 3rd Ed., p. 27, Methuen, London, 1946. Paraphrasing that material, the type of process used to establish distant time relationships is immaterial; hence, assuming proper attention to details, electronic pulses may be substituted for light pulses.

[9] Regarding the conventionality of distant simultaneity see, H. Reichenbach, *The Philosophy of Space and Time*, Dover, NY, 1958; A. Grunbaum, *Philosophical Problems of Space and Time*, D. Reidel, Boston, 1973; A. A. Ungar, *Found. Phys.*, **21**, 691, (1991). For an historical background and contemporary analysis of the conventionality of distant simultaneity, see B. van Fraassen, *An Introduction to the Philosophy of Space and Time*, Columbia Univ. Press, NY, 1970. Chapter IV, section 3, "Einstein's Critique of Simultaneity" outlines the fundamental issues.

[10] This view is expressed in an overview of topics in quantum optics that are of concern in this paper: D. M. Greenberger, M. A. Horne and A. Zeilinger, *Phys. Today*, Aug., 1993, p. 25.

[11] A more detailed perspective can be found in, R. Mansouri and R. U. Sexl, *Gen. Relativ. Gravit.*, **8**, 497, 515, 809, (1977). See "internal" synchronization therein. Also, ref 8 therein provides a brief discussion of the problem associated with this approach including a bibliography.

[12] Although only two clocks are required, the three-clock method is chosen for its geometric symmetry and for the symbolic preservation of a reference time on the stationary clock C_0.

[13] More precisely, given two identical clocks in an inertial frame: Let one be stationary in that frame and let the other be in relative motion. Compared to the stationary clock, the proper period of the clock in motion will be longer (dialated).

[14] In a real situation, there are many additional effects to be taken into account; however, for the purpose of this analysis we will assume they are not important. See for example, J. C. Hafle and R. Keating, *Science*, **177**, 166, 168 (1971).

[15] Given the identical clock $\upsilon_a = \upsilon_b = \upsilon_0$, and steady clock assumptions herein, numerical synchronization may be done either before or after data collection since the synchronization transform will be linear. From the perspective of the transformation of information (beginning in "Local Data Sets"), this actually becomes a detail, immaterial to the central thrust of the discussion.

[16] Regarding, the issue of space-like separation: Generally, we would like to insure that time stamps which appear to be simultaneous are genuine distant photo-detections, not caused by an experimental artifact propagating within the apparatus. To that end, photo-detector pulses must have some finite time duration δt. For conditions of standard synchronization, if the experiment is

designed so that $\delta t < (d_a + d_b)/c$, conditions of space-like separation can be created for near-simultaneous event regions at D_a and D_b.

[17] Issues of simultaneous enabling and disabling of the data recorders are disregarded here with the thought that a regular procedure will yield a reproducible overlap between C_a and C_b and that an exact normalization of photo-counts is not critical to this discussion.

[18] Environmental effects local to each detector and its associated apparatus such as natural radioactivity, electromagnetic interference and overall (effective) collection and detection inefficiencies, combine so that, in general, $n'_j \neq n''_j$. That is, in spite of the fact that the type of source employed in reference [1] emits photons in pairs.

[19] The experiments under consideration generally operate in the single photon region so that, on average, only one photon (pair) at a time is within the apparatus. If the mean photon flux is Φ and the photon counting gate time window is τ then conditions will generally be optimal when the flux incident on either detector is adjusted so that $\tau\,\Phi < 1$. See P. Grainger, G. Roger and A. Aspect, *Europhys. Lett.*, **1**, (4), 173, (1986) or A. Aspect in *Sixty-Two Years of Uncertainty*, A.I. Miller, ed., Plenum Press, NY, 1990.

[20] "Effective" means that detector D_a with quantum efficiency η_{qa}, and all relevant optical efficiencies that result from various mean absorptions, reflections, etc. are combined. For example, $\eta_a = \eta_{qa} \cdot \eta_{1a} \cdot \eta_{2a} \cdot \ldots \cdot \eta_{na}$ and similarly for η_b.

[21] In Einstein's view, "There is no such thing as simultaneity of distant events; consequently there is also no such thing as immediate action at a distance [distant physical simultaneity] …." , *Albert Einstein: Philosopher-Scientist*, P. A. Schlipp, ed., Cambridge Univ. Press, third ed. 1969, p. 61.

[22] The reason for beginning with the assumption clocks C_a and C_b were synchronized prior to data collection was to emphasize the conventional nature of synchronization and its lack of influence upon physical conditions of separability. In addition, this approach focuses the argument on non-mathematical information transformations. Otherwise, S_T *would* now have an additional mathematical function, the linear time transform that synchronizes C_a and C_b with C_0.

[23] Locality is often defined in terms of its compliment, a mechanical form of spatial separability. If two quanta can be spatially separated to such a degree that it becomes possible to disregard any mutual interaction arising out of fields local to either particle, then those quanta are said to be separable. If we then add to this state of spatial separation, the condition that any further interaction is limited to causal disturbances, which travel no faster than the speed of light, the quanta are said to be Einstein separable.

[24] This is Einsten's initial stipulation and forms the basis of the Reichenbach-Grunbaum hypothesis

[25] Only trivially, when $Q_{a\,j} = Q_{b\,j}$ can the complete $Q_{a\,j}$ and $Q_{b\,j}$ be factored from Q_j. The symmetric difference $Q_{a\,j} \oplus Q_{b\,j}$ presumably contains those time-stamped photo-detections that result from noise and less than ideal apparatus efficiencies.

[26] References that employ equations describing distant fields in terms of a common time (implicit synchronization) are too numerous to list in full. However, a random sampling is offered: L. Mandel, *Phys. Rev.*, **A28**, 929, (1983) p. 929-931; Z. Y. Ou, Phys. Rev., **A37**, 1607, (1988), p. 1608; J. D. Franson, *Phys. Rev. Lett.*, **67**, 290, (1991); R. Louden, The Quantum theory of Light, Oxford university press, 2nd. ed., 1997 p. 96, 106 & 216; M. O. Scully and M. S. Zubiary, Quantum Optics, Cambridge University Press, 1997, p. 113 (and pp. in index under, "correlation function, second order" and "two photon", etc.); Born and Wolf, Principles of Optics, 6th ed., Pergamon Press, NY, 1983, p. 257, 500. H. Paul, *Rev. Mod. Phys.*, **58**, 209, (1986) by employing the Heisenberg Picture, Paul seems to recognize the issue but attempts to dismiss it via notation, p. 214.

[27] In equations (6) of the main text, only two of six possible information referents were mapped under s_T, implying that the remaining four are invariant under s_T. See bold headings "The referents of t^i" and "The referents of τ".

[28] Applying this line of reasoning to equation (5) does not result in a meaningful result since there, we are concerned with the value of particular physical quantities, not "physical quantity" itself. See Appendix, "Physical Quantity".

[29] This evokes a much more general question, are conventions *necessary* for a complete understanding of the physical universe and if so, where do they occur, how are they constructed and what is their precise role? Of course, a satisfactory resolution lies well beyond the scope of this paper.

[30] It is possible to adopt the view that the next step should be to define a relation on $O \times E$. For example, let R_E be an "energy" relation on $O \times E$ that selects for ordered pairs such that the combination of exponential operators x^Z and elements $e_i \in E$ are appropriate to the example of energy, $R_E = \left\{ \left(x^{-2}, e_3\right), \left(x^1, e_1\right), \left(x^0, e_4\right), \left(x^2, e_2\right) \right\}$. From this, $\alpha \circ \beta : R_E \to G_E \to Pq_E$. However, this still involves some form of incompletely defined selection rule(s). The route through H was taken for its compactness of expression.

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