# Classical study of an ensemble of electrons leading to a statistical interpretation for the electron radius

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Solutions of the relativistic Hamilton-Jacobi equation with repulsive Coulomb potential, and of the equation of continuity, lead to a statistical mixture of gamma distributions for an ensemble of electrons. The study leads to the concept of a characteristic distance in configuration space, which can be identified with the "electron radius".

Les solutions de l' équation de Hamilton-Jacobi relativiste dans un champ Coulombien, et celles de l' équation de continuité, nous donnent pour un ensemble des électrons une mixture des gamma densités. Il en résulte que il y a une distance charactériste qui peut s' indentifier avec le "rayon de l' électron".

**Key words**: Repulsive Coulomb potential, electron radius.

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## 1 Introduction

As it is known, the classical radius of the electron is a universal constant that is considered to give the limit of applicability of the classical field theory, and is defined as

$$r_e = \frac{e^2}{mc^2} = 2.818 \times 10^{-13}\text{cm}.$$

Theoretically, the "radius"results as follows: If we consider the electron as possessing a certain radius $r_e$, then its self-potential energy would be of order $e^2/r_e \simeq mc^2$, from which the dimension $r_e = e^2/mc^2$ results.

But if, as the theory of relativity requires, the electron is a point particle, then $r_e$ tends to zero, and consequently the potential self-energy and the mass of the electron become infinite. According to Landau [1], the occurence of these physically meaningless results, leads to the conclusion that the theory of classical electrodynamics has its limits when we go to sufficiently small distances, smaller than the "radius". In the litterature there have been attempts to resolve this deficiency of the theory; for instance Hautot [2] connects the concept of the "radius"with the nature and the structure of the electron, while Laserra et al. [3] interpret the "radius"as the minimal distance between particles in the case of rectilinear motion, and derive it from dynamical properties. In studying the statistical behavior of an ensemble of electrons under their own repulsive potential, we have seen that this fundamental distance of classical electrodynamics can be permitted to obtain a statistical physical significance.

In the present article we submit a method by means of which we find the probability density in configuration space of an ensemble of electrons described by the relativistic Hamilton-Jacobi equation with repulsive Coulomb potential. The method is outlined as follows: We solve the Hamilton-Jacobi equation and find the momenta as functions of the position coordinates. Then, by use of the canonical equations, we form the velocities also as functions of the position. These velocities are introduced as known functions in the equation of continuity, so that the equation of continuity can be solved in the unknown function of the probability density. In studying the statistical properties of the resulting density, we see that the theory leads to an interesting statistical interpretation for the radius of the electron.

## 2 Solutions of the Hamilton-Jacobi equation with repulsive Coulomb potential

The relativistic Hamilton-Jacobi equation with repulsive Coulomb potential expressed in polar coordinates, is

$$(\frac{\partial S}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial S}{\partial \theta})^2 - \frac{1}{c^2}(\frac{\partial S}{\partial t} + \frac{e^2}{r})^2 + m^2c^2 = 0 \qquad (1)$$

where the function $S(r, \theta, t)$ is the *Hamilton Principal* (*or characteristic*) *function*, $e$ is the electron charge, $m$ is the electron mass, and $c$ is the velocity of light.

The Hamiltonian associated with Eq.(1) is

$$\mathcal{H} = \pm\Big\{m^2c^4 + c^2P_r^2 + \frac{c^2}{r^2}P_\theta^2\Big\}^{1/2} + \frac{e^2}{r} \tag{2}$$

where $P_r = \partial S/\partial r$ is the orbital momentum, $P_\theta = \partial S/\partial\theta$ is the angular momentum and $\mathcal{H} = -\partial S/\partial t$ is the energy.

The canonical system of the Hamiltonian Eq. (2) gives the velocities as

$$\frac{dr}{dt} = \pm\frac{P_r c^2}{\mathcal{H} - (e^2/r)} \quad , \quad \frac{d\theta}{dt} = \pm\frac{P_\theta c^2}{r^2[\mathcal{H} - (e^2/r)]} \tag{3}$$

Eq. (1) admits of the following pairs of particular solutions:

$$S = \pm 2\sqrt{2me^2r} \pm \frac{e^2\theta}{c} + mc^2t, \tag{4}$$

$$S = \pm 2i\sqrt{2me^2r} \pm \frac{e^2\theta}{c} - mc^2t. \tag{5}$$

In order to calculate the momenta and velocities as functions of the variables, we shall consider the real solutions Eq. (4)[1]. From Eq. (4), we get the momenta as

$$P_r = \frac{\partial S}{\partial r} = \pm\frac{\sqrt{2me^2}}{r} \quad , \quad P_\theta = \frac{\partial S}{\partial\theta} = \pm\frac{e^2}{c}. \tag{6}$$

These expressions, introduced in Eqs. (3), result the velocities

$$\frac{dr}{dt} = v_r = \mp\frac{c^2\sqrt{2me^2r}}{mc^2r + e^2} \quad , \quad \frac{d\theta}{dt} = v_\theta = \mp\frac{ce^2}{r(mc^2r + e^2)}. \tag{7}$$

By introducing now the constant with dimensions of length

$$r_0 = \frac{e^2}{mc^2}\ , \tag{8}$$

Eqs. (7) are written in the form

$$v_r = \mp c\frac{\sqrt{2r_0r}}{(r + r_0)} \quad , \quad v_\theta = \mp\frac{cr_0}{r(r + r_0)}\ , \tag{9}$$

where the sign ($\pm$) shows that the orbital and the angular velocities result in both directions.

[1] The fact that these solutions result negative energies does not affect the following theory, as we shall see.

## 3 Solutions of the equation of continuity

Now let us assume that we have an ensemble of electrons obeying the Hamilton-Jacobi equation Eq. (1). The ensemble is meant in the Boltzmann sence [4], which is understood as a very large number of electrons (of the order of $10^{23}$), practically non-interacting, and this ensemble is well described in configuration space by the equation of continuity

$$\vec{\nabla}(\rho \cdot \vec{v}) + \frac{\partial \rho}{\partial t} = 0. \tag{10}$$

where $\rho = \rho(\vec{r}, t)$ is the unknown probability density function and $\vec{v} = \vec{v}(\vec{r}, t)$ are the known functions of the velocities, as they result from the solutions of the Hamilton-Jacobi equation. The form of the velocities Eq. (9) leads us to consider here the equation of continuity in polar coordinates and time-independent, so that $\rho(\vec{r}, t) \rightarrow f(r, \theta)$. Then Eq. (10) becomes

$$f\left(\frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + v_r \frac{\partial f}{\partial r} + v_\theta \frac{\partial f}{\partial \theta} = 0, \tag{11}$$

where now $f = f(r, \theta)$ is the unknown probability density function and $v_r, v_\theta$ are the velocities given by Eqs. (9). By means of these functions and their derivatives, Eq.(11) becomes

$$\frac{r_0 - r}{r_0 + r} \frac{\sqrt{2r_0 r}}{2r} + \frac{\sqrt{2r_0 r}}{r} + \sqrt{2r_0 r}\, \frac{\partial \log[f]}{\partial r} + \frac{r_0}{r} \frac{\partial \log[f]}{\partial \theta} = 0 \tag{12}$$

Introducing in Eq. (12) the new positive dimensionless variable

$$s(r) = \sqrt{\frac{r_0}{r}}, \tag{13}$$

so that $f(r, \theta) \rightarrow u(s, \theta)$, we obtain the equation of continuity in the form

$$\frac{s^2 - 1}{s(s^2 + 1)} + \frac{2}{s} - \frac{\partial \log[u]}{\partial s} + \sqrt{2}\, \frac{\partial \log[u]}{\partial \theta} = 0. \tag{14}$$

This equation has the general solution

$$u(s, \theta) \sim (s + s^3) \exp\left\{-g\left(s + \frac{\theta}{\sqrt{2}}\right)\right\}, \tag{15}$$

where $g$ is an arbitary function of the argument. The position variables $s$ and $\theta$ are stochastically independent, so that the function $u$ will be of the form of the particular solutions

$$u(s,\theta;a) = C(s+s^3)e^{-a\left(s+\frac{\theta}{\sqrt{2}}\right)}, \tag{16}$$

where $C$ and $a$ are constant parameters.
This function, with $C$ and $a$ positive real numbers, is nonnegative and integrable in $s \in [0,\infty)$ and $\theta \in [0,2\pi]$. Consequently it is, up to a normalizing constant, a joint probability density [5] of the form

$$u(s,\theta) = u_s(s)u_\theta(\theta).$$

The coefficient $C = C_a$ can be easily calculated from the normalization condition

$$\int_0^\infty \int_0^{2\pi} u(s,\theta)dsd\theta = 1$$

and is

$$C_a = \frac{a}{\sqrt{2}\left[1-\exp\{-a\pi\sqrt{2}\}\right]}\,\frac{a^4}{a^2+6}. \tag{17}$$

Finally, the acceptable particular solutions of the equation of continuity Eq. (14) are

$$u(s,\theta;a) = \frac{a}{\sqrt{2}\left[1-\exp\{-a\pi\sqrt{2}\}\right]}\,e^{-\frac{a}{\sqrt{2}}\theta}\,\frac{a^4}{a^2+6}\,(s+s^3)\,e^{-as} \tag{18}$$

and depend on the real positive constant parameter $a$, which in the theory of probability is called *scale parameter* [6].

## 4 Properties of the marginal density

Of special interest is the marginal probability density

$$u_s(s;a) = \frac{a^4}{a^2+6}(s+s^3)\,e^{-as} \tag{19}$$

For $a \to 0$ the above function tends quickly to zero for every value of $s$. Small values of $a$ result smaller probabilities, while large values of $a$ result larger probabilities as $s = (r_0/r)^{1/2}$ increases. Increasing $s$ means that we go to smaller distances. Especially $s > 1$ means that we go to distances smaller than $r_0$ (Fig. 1).

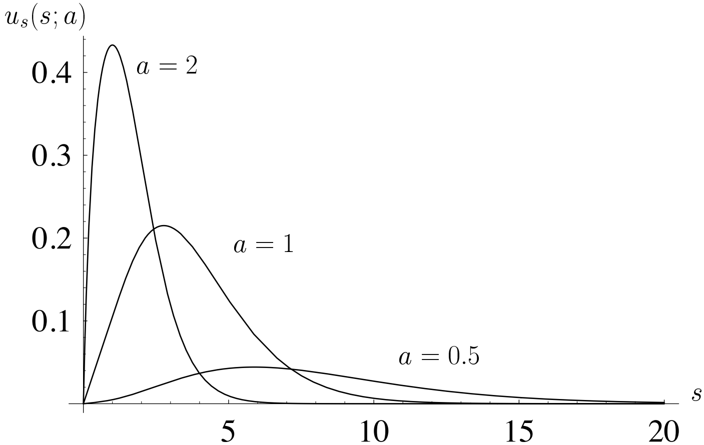


Figure 1: The function $u_s(s;a) = \frac{a^4}{6+a^2} e^{-as}(s + s^3)$ for three different values of the scale parameter $a$.

The function $u_s(s;a)$ yields the following statistical results:
The expected (average) value of $s$ is given in terms of $a$ by the relation

$$<s> = \frac{2(a^2 + 12)}{a^3 + 6a}. \tag{20}$$

The most probable value of $s$ is the one for which the density Eq. (19) becomes maximum, and it is given by the real and positive solution of the equation

$$as^3 - 3s^2 + as - 1 = 0 \tag{21}$$

The root mean square is

$$RMS[\,s\,] = \sqrt{<s^2>} = \frac{\sqrt{6}}{a} \sqrt{\frac{a^2 + 20}{a^2 + 6}} \tag{22}$$

and, the variance is

$$\sigma^2 = |<s^2> - <s>^2| = \frac{2(a^4 + 30a^2 + 72)}{a^2(a^2+6)^2}. \tag{23}$$

Finally, the probability of finding the position of an electron at sites equal or smaller than $r_0 = e^2/mc^2$, or for $s \geq 1$, is given by the distribution

$$P_{(s\geq 1)}(a) = \frac{a^4}{a^2+6}\int_1^\infty e^{as}(s+s^3\, ds = \frac{2e^{-a}(a^3+2a^2+3a+3)}{a^2+6} \tag{24}$$

This function is plotted in Fig. 4, in which it is shown that there exists a range of values of the parameter $a$ such that they give considerable probabilities to $r$ being much smaller than $r_0$.

From the mathematical point of view, all the values of $a \in [0,\infty)$ are permissible; but the physics of the problem gives us the possibility to select an appointed value. In fact, we remark that, in writing the function Eq. (19) as a sum

$$u_s(s;a) = u_1(s;a) + u_2(s;a) = \frac{a^4}{a^2+6}\, s\, e^{-as} + \frac{a^4}{a^2+6}\, s^3\, e^{-as} \tag{25}$$

we see that it is about a mixture [6] of two gamma densities. We remind that, in probability theory, by *mixture* we define a probability density of the form

$$w(x) = \sum_{j=1}^{k} n_j f_j(x) \qquad \text{with} \qquad \sum_{j=1}^{k} n_j = 1 \tag{26}$$

where the functions $f_j(x)$ are probability densities. The *gamma density*, concentrated on $[0,\infty)$ is defined by

$$\gamma(x; b+1, a) = \frac{a^{b+1}}{\Gamma(b+1)}\, x^b\, e^{-ax}, \tag{27}$$

where $\Gamma(b+1)$ is the well-known *gamma function*. Consequently we have

$$\gamma(s;2,a) = a^2 s e^{-as}, \qquad \gamma(s;4,a) = \frac{a^4}{6}\, s^3 e^{-as}. \tag{28}$$

From Eqs. (26) and (28) we see that the function Eq. (19) is the mixture

$$u(s;a) = n_1\, \gamma(s;2,a) + n_2\, \gamma(s;4,a) \tag{29}$$

with

$$n_1 = \frac{a^2}{a^2+6}, \qquad n_2 = \frac{6}{a^2+6}. \tag{30}$$

This mixture has the physical meaning that a fraction $n_1$ of the total number of electrons are distributed according to the probability density $\gamma(s;2,a) = a^2 s\, e^{-as}$, while the rest $n_2 = 1 - n_1$ follow the density $\gamma(s;4,a) = (a^4/6)s^3\, e^{-as}$. This implies that in the ensemble we have two subensembles of electrons with different statistical behavior[2]. Since we have assumed that the particles in the ensemble are identical, we have no reason to believe that the one or the other gamma density will prevail in the mixture; therefore we take the subensembles to be "equiprobable", i.e.

$$n_1 = n_2 = \frac{1}{2}\ . \tag{31}$$

From (30), (31), we see that this happens if

$$a = \sqrt{6} \simeq 2.44949$$

We may reasonably consider that this is a realistic value of the scale parameter $a$, so that the actual density functions for our ensemble, as given by Eqs. (18) and (19) for $a = \sqrt{6}$, are

$$u(s,\theta) = \frac{3\sqrt{3}}{1 - e^{-2\pi\sqrt{3}}}\ (s + s^3)\, e^{-\sqrt{6}\, s - \sqrt{3}\,\theta} \tag{32}$$

$$u_s(s) = 3\, e^{-\sqrt{6}\, s}(s + s^3) \tag{33}$$

In Figure 2 we show the function Eq. (32). In Figure 3 we show the function Eq. (33) as mixture of the two densities $\gamma(s;2,\sqrt{6})$ and $\gamma(s;4,\sqrt{6})$.

Inserting in Eqs. (20)-(24) the value $a = \sqrt{6}$ and by use of Eq. (13), we obtain the results
Expected value

$$< s >= \sqrt{\tfrac{3}{2}} = 1.22474, \text{ corresponding to } r_{<s>} = 0.66666\, r_0$$

[2]This might have some connection with the electron spin.

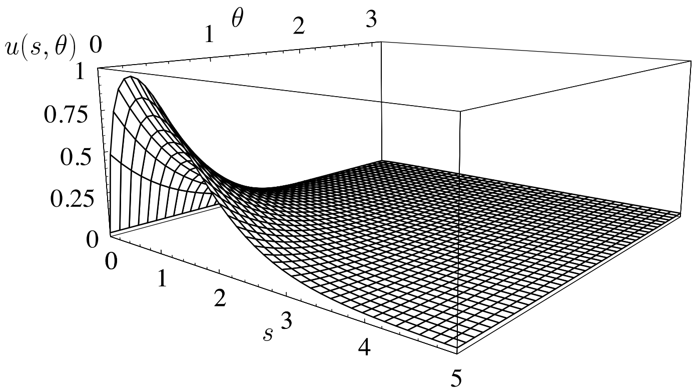


Figure 2: The function $u(s,\theta) = \frac{3\sqrt{3}}{1-e^{-2\pi\sqrt{3}}}(s+s^3)e^{-\sqrt{6}s-\sqrt{3}\theta}$ representing the joint probability density of electrons in the ensemble.

Most probable value

$$s_m = 0.651549, \text{ corresponding to } r_{s_m} = 1.53457\, r_0$$

Root mean square

$$RMS[\,s\,] = \sqrt{\tfrac{13}{6}} = 1.47196, \text{ corresponding to } r_{[RMS]} = 0.461538\, r_0$$

Variance

$$\sigma^2 = \frac{2}{3} = 0.666666$$

Probability to find the position of an electron at sites equal or smaller than $r_0$

$$P_{(s\geq 1)} = \frac{1}{6}(15 + 9\sqrt{6})\, e^{-\sqrt{6}} = 0,533069.$$

## 5 Statistical interpretation of the electron radius

In our model the constant $r_0 = e^2/mc^2$ results as a characteristic distance in configuration space and coincides with the radius of the electron $r_e$. The electron is considered as pointlike, and as representative of an ensemble of electrons. The probability density characterizing the

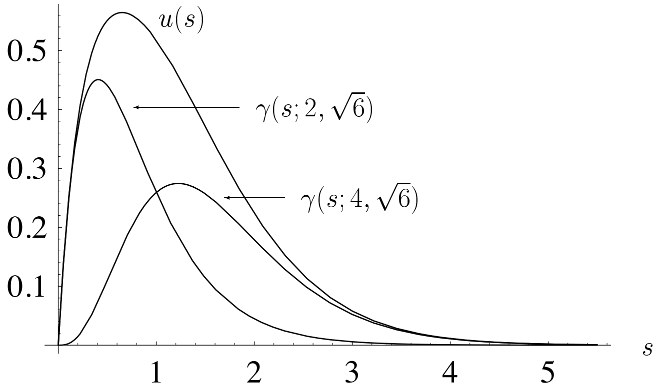


Figure 3: The function $u_s(s) = 3e^{-\sqrt{6}s}(s + s^3)$ as a mixture of the densities $\gamma(s; 2, \sqrt{6})$ and $\gamma(s; 4, \sqrt{6})$.

state of the ensemble has been derived in a straightforward way. We can see from the results given in the previous section that the characteristic statistical values of the magnitude $s = (r_0/r)^{1/2}$ give distances of the same order of magnitude and even close to the electron radius.

Furthermore, we note that the radius lies in the mean between the *expected* and the *most probable* distance where an electron is found. In fact, from the results of the previous section, we see that

$$r_e \simeq \frac{r_{<s>} + r_{s_m}}{2} = \frac{(0.66666 + 1.53457)}{2}\, r_0 = 1.10062\, r_0.$$

We may say that, by this model, the radius obtains a statistical interpretation, as *the average distance between the expected and the most probable position of an electron in configuration space.*

Another interestig remark is, that the expected and the RMS values of the variable $s$ yield distances smaller than $r_e$, and that actually the probability to find an electron at distances smaller than the radius equals 0.533069. Thus, although the electrons *are still considered as poinlike particles*, the presented theory allows us to move in distances much smaller than the radius in the frame of classical electrodynamics, and in this sense we can say that there is no limit in the application of classical electrodynamics.

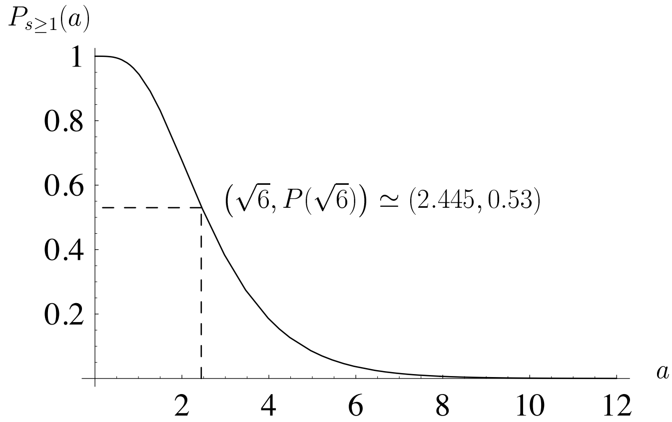


Figure 4: Distribution for $s \geq 1$ or the probability of finding an electron at $r \leq r_0$ in terms of $a$. For $a = \sqrt{6} \simeq 2.445 \quad , \quad P(\sqrt{6}) \simeq 0.53$.

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