Symmetry Properties and Construction of Relativistic Composite Fermion States

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ABSTRACT. Within a corresponding quantum field theoretic formalism composite fermion states are defined as solutions of generalized de Broglie-Bargmann-Wigner (BBW) equations. These equations are relativistically invariant quantum mechanical many body equations with nontrivial interaction, selfregularization and probability interpretation. Owing to these properties they are a suitable means for describing relativistic bound states of fermions. In accordance with de Broglie's fusion theory and modern assumptions about the partonic substructure of elementary fermions, i.e., leptons and quarks, or the quark structure of nucleons, the three-body generalized BBW-equations are investigated. In particular it is shown that the group theoretical constraints to be imposed on the wave functions and an integral equation for the mass eigenvalue are compatible with the antisymmetry of the wave function which is crucial for the consistency of the solution procedure of the generalized BBW-equations. This solution procedure is analyzed in detail and dual states are constructed which are required for the derivation of effective theories.

1 Introduction

Since the midst of the past century nucleons have been assumed to be composed of quarks, and in the last decades even quarks and leptons were assumed to possess a fermionic substructure, i.e., these particles are considered as bound states of various fermionic constituents.

For a quantitative description of these bound states their wave functions are required and although much nonrelativistic model building was applied, it is acknowledged that in principle such bound states are relativistic ones. However, in spite of numerous attempts to solve the relativistic bound state problem, the results which were obtained by conventional methods, e.g., Bethe-Salpeter equations, are considered as unsatisfactory. Without substantiating this assertion we refer to comments in literature,[1],[2],[3]. Hence the construction of relativistic two body and many body equations are active areas of current research,[4].

Recently, to solve the relativistic bound state problem, a new approach was developed which is radically different from the previous ones. It is based on the idea that de Broglie's spin fusion should be caused by direct interactions of fermions without the assistance of bosons. The corresponding theory which exclusively deals with spinorial interactions is based on a nonperturbatively regularized nonlinear spinor field with canonical quantization, relativistic invariance and probability interpretation. It can be considered as the quantum field theoretic generalization of de Broglie's fusion theory,[5] and as a mathematical realization and physical modification of Heisenberg's approach,[6] and is expounded in [7],[8].

Originally this model was intended to describe composite gauge bosons, quarks and leptons. But in a similar manner bound states of quarks can be treated by the same method, as the effective quark theory starts from a nonlinear spinor field model too,[9].

Then within such a spinor theory the bound state problem is formulated by generalized de Broglie-Bargmann-Wigner (BBW)-equations. And in this picture composite quarks, composite leptons or composite nucleons, respectively, are described as bound state solutions of BBW-equations for three fermions. Among other authors, the three fermion substructure of quarks and leptons was postulated by Harari,[10] and Shupe,[11]. But apart from this assumption our model has nothing in common with the Harari-Shupe model.

In the original theory of de Broglie,[5], and Bargmann and Wigner,[12], the three body problem was investigated in detail by Rarita and Schwinger,[13] who concentrated on spin 3/2 solutions owing to a symmetry postulate on the spin part of the wave functions. Such a symmetry postulate on the spin part narrows down the manifold of solutions and is neither necessary in the original version of de Broglie,, nor for the generalized BBW-equations, i.e., these equations also admit spin 1/2 solutions.

But in contrast to the exact solutions given by Rarita and Schwinger the generalized BBW-equations for the three-body case lead to integral equations of the Fredholm type. The latter equations are soluble in principle but hardly in practice. So for getting an information about the structure of the eigenvalue spectrum, the group theoretical analysis is the only means which allows to derive exact results concerning the corresponding spectrum. Such an investigation was started in a previous paper,[14], but owing to the rather complicated matter further analysis is needed which is given in this paper in order to complete the group theoretical discussion of the three-body problem and to provide the basis for quantitative calculations.

The fieldtheoretic background of our model was extensively discussed in [7],[8], so we refer for further information about the generalized BBW-equations and the corresponding model to these references.. The physical interpretation of the corresponding solutions was already given in a preliminary way in preceding papers,[8],[15],[16],[17]. But it is the intention to improve these statements by a more detailed group theoretical analysis as was initiated in the preceding paper,[14].

In particular in [14] the group theoretical constraints on the three-body solutions were derived, while it is the aim of this paper to show the compatibility of these constraints and the above mentioned integral equation with the antisymmetrization of the wave function. The latter requirement is crucial for any solution procedure as the antisymmetrization of the wave function stems from its field theoretic background and is responsible for the consistency of the definition of the generalized BBW-equations, i.e., without this consistency the treatment of such equations is impossible. In the following the common term “parton” is used either for the fermionic substructure of leptons and quarks or for the quarks themselves as the fermionic constituents of nucleons.

2 Relativistic three-parton equations

By means of the fieldtheoretic formalism wave equations for three-parton states can be derived,[7],[8]. For provisional guidance we assume that such equations and their states allow an appropriate description of leptons and quarks with partonic substructure or of nucleons with quark substructure, respectively. In this case the quantum numbers of those states must fit into the scheme of quantum numbers of the Standard model which was the topic of previous work, and will be the topic of forthcoming papers, while in this paper the general group theoretical constraints will be discussed.

It is a pecularity of the field theoretic formalism that from the beginning this formalism is not specialized to any definite parton number

n. And although we will exclusively deal with the parton number $n = 3$ in the following, the general field theoretic formulation is needed in order to be aware of the antisymmetry properties of the wave functions. Hence we start with the field theoretic version of the theory for hard core states which can be expressed by a single (covariant) functional equation. At this basic level of the theory it is convenient to use only symbolic general coordinate variables I which stand for the four dimensional space-time coordinate x and the algebraic indices Z. Then in this symbolic notation this hard core functional equation reads (using the summation convention), see [7],[8]:

$$K_{I_1 I}\partial_I|\mathcal{F}\rangle = U_{I_1 I_2 I_3 I_4}[F_{I_2 I} j_I \partial_{I_4}\partial_{I_3} + F_{I_3 I} j_I \partial_{I_2}\partial_{I_4} + F_{I_4 I} j_I \partial_{I_3}\partial_{I_2}]|\mathcal{F}\rangle \tag{1}$$

Definitions of the various quantities which are contained in this symbolic equation will be given below. At first we explain the states $|\mathcal{F}\rangle$. These states are defined by

$$|\mathcal{F}(j)\rangle = \varphi_n(I_1...I_n) j_{I_1}...j_{I_n}|0\rangle \tag{2}$$

where φ_n is a formally normal ordered matrix element of the parton dynamics for hard core states, while the set of base vectors $\{j_{I_1}...j_{I_n}|0\rangle\}$ is defined to be a fermionic Fock space with creation operators j_I and their duals ∂_K , which have not to be confused with ordinary particle creation and annihilation operators of quantum field theory as the former are elements of the generating functional space.

With regard to the application of equation (1) to the case $n = 3$, we choose in (2) the corresponding states and project (1) from the left hand side with $\langle 0|\partial_{N_1}\partial_{N_2}$. This yields

$$\sum_N K_{N_3 N}\mathcal{A}_{N_1 N_2 N}\varphi_{N_1 N_2 N} = \tag{3}$$

$$\sum_{I_2 I_3 I_4} U_{N_3 I_2 I_3 I_4}[-3F_{I_2 N_2}\mathcal{A}_{N_1 I_3 I_4}\varphi_{N_1 I_3 I_4} + 3F_{I_2 N_1}\mathcal{A}_{N_2 I_3 I_4}\varphi_{N_2 I_3 I_4}]$$

where the symbols $\mathcal{A}$ mean antisymmetrization in the corresponding indices. In all following calculations we omit the $\mathcal{A}$ symbols for brevity, but keep in mind that they are always present in the course of calculations. In order to perform such calculations one needs a more detailed representation of equations (3). In particular we define the following quantities: $\mathbf{r} \in R^3$, $x \in M^4$, and $Z = (i, \kappa, \alpha)$ where κ means superspin-isospin index, α = Dirac spinor index, i = auxiliary field index. The latter index characterizes the subfermion fields which are needed for the regulatization procedure.

Let $\varphi_{Z_1Z_2Z_3}(x_1,x_2,x_3)$ be the covariant, antisymmetric state amplitude for the case $n=3$. Then from (3) the following equation can be derived for this state:

$$\begin{aligned}[D^{\mu}_{Z_3X_3}\partial_\mu(x_3)-m_{Z_3X_3}]\varphi_{Z_1Z_2X_3}(x_1,x_2,x_3) \\ =3U_{Z_3X_2X_3X_4}[-F_{X_2Z_2}(x_3-x_2)\varphi_{Z_1X_3X_4}(x_1,x_3,x_3) \\ +F_{X_2Z_1}(x_3-x_1)\varphi_{Z_2X_3X_4}(x_2,x_3,x_3)]\end{aligned} \tag{4}$$

Furthermore owing to the antisymmetrization in (3) one obtains two additional equations if the Dirac operator on the left hand side of (4) is applied to the coordinates x_1 and x_2. For brevity these two equations are not explicitly given, because apart from one exception, namely the derivation of the energy representation, these two equations are not needed if in every calculational step antisymmetrization is secured.

With respect to equation (4) the following definitions hold:

$$D^{\mu}_{Z_1Z_2} := i\gamma^{\mu}_{\alpha_1\alpha_2}\delta_{\kappa_1\kappa_2}\delta_{i_1i_2} \tag{5}$$

and

$$m_{Z_1Z_2} := m_{i_1}\delta_{\alpha_1\alpha_2}\delta_{\kappa_1\kappa_2}\delta_{i_1i_2} \tag{6}$$

and

$$F_{Z_1Z_2}(x_1-x_2) := -i\lambda_{i_1}\delta_{i_1i_2}\gamma^5_{\kappa_1\kappa_2}[(i\gamma^{\mu}\partial_\mu(x_1)+m_{i_1})C]_{\alpha_1\alpha_2}\Delta(x_1-x_2,m_{i_1}) \tag{7}$$

where $\Delta(x_1-x_2,m_{i_1})$ is the scalar Feynman propagator. The meaning of the index κ can be explained by decomposing it into two parts $\kappa := (\Lambda, A)$ with $\Lambda = 1,2$ superspin index of spinors and charge conjugated spinors and $A=1,2$ isospin index which can be equivalently expressed by $\kappa = 1,2,3,4$.

The vertex term in equation (4) is fixed by the following definitions:

$$U_{Z_1Z_2Z_3Z_4} := \lambda_{i_1}B_{i_2i_3i_4}V^{\kappa_1\kappa_2\kappa_3\kappa_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} \tag{8}$$

where $B_{i_2i_3i_4}$ indicates the summation over the auxiliary field indices and where the vertex is given by a scalar and a pseudoscalar coupling of the subfermion fields

$$V^{\kappa_1\kappa_2\kappa_3\kappa_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} := \frac{g}{2}\{[\delta_{\alpha_1\alpha_2}C_{\alpha_3\alpha_4}-\gamma^5_{\alpha_1\alpha_2}(\gamma^5C)_{\alpha_3\alpha_4}]\delta_{\kappa_1\kappa_2}[\gamma^5(1-\gamma^0)]_{\kappa_3\kappa_4}\}_{as[2,3,4]} \tag{9}$$

For vanishing coupling constant $g = 0$ de Broglie's original fusion equations for local three fermion states are obtained, and for a solution of the whole set of equations only equation (4) has to be used, as for antisymmetric wave functions the remaining equations can be derived from (4) by interchange of indices. In this context it should be emphasized that the antisymmetry of wave functions is not an additional postulate. Rather it is an outcome of the general functional formalism which is used to derive such equations, see equation (3).

Concerning the physical interpretation of the wave functions it is closely related to the role of the auxiliary fields (indices) which appear in the corresponding equations and their solutions.

The task of the auxiliary fields is twofold: on the one hand they are used for regularization, on the other hand due to their properties probability conservation can be deduced. As this topic was extensively treated for the two-parton case in [18] and the discussion of the three-parton case runs along the same lines we suppress the explicit deduction of these properties and describe only the corresponding results.

First we refer to the role of auxiliary fields in regularization, leading to the definition of the physical wave functions. We consider the wave functions of equation (4) with the full dependence on the auxiliary fields as unobservable, i.e., unphysical. In order to obtain the physical, singularity free wave functions in the case of three-parton states we decompose the index $Z := (\alpha, \kappa, i)$ into $Z := (z, i)$ and sum over i_1, i_2, i_3 . This gives

$$\hat{\varphi}_{z_1 z_2 z_3}(x_1, x_2, x_3) := \sum_{i_1 i_2 i_3} \varphi_{Z_1 Z_2 Z_3}(x_1, x_2, x_3). \tag{10}$$

These functions are by definition the physical states. One immediately realizes that the physical wave function $\hat{\varphi}$ has the same transformation properties as the original wave function φ .

In order to derive a probability interpretation for the physical parton wave functions the single time formulation of (10) has to be used, see [7],[8] and in addition the single time energy equation has to be derived from (4), see [18],[19]. Then with the single time density

$$\hat{\varphi}^\dagger \hat{\varphi} := \sum_{z_1 z_2 z_3} \hat{\varphi}_{z_1 z_2 z_3}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, t)^* \hat{\varphi}_{z_1 z_2 z_3}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, t) \tag{11}$$

for a general time dependent solution of the energy equation one obtains from this equation with $m_i = m + \delta m_i$, in the limit $\delta m_i \to 0$ current conservation,[18],[19]. This limit can be performed in the regularized wave functions without any difficulty after all calculations were done. Owing to current conservation the densities (11) are conserved positive quantities ,i.e., the physical state amplitudes $\hat{\varphi}$ are elements of a corresponding Hilbert space with the norm expression

$$\langle \hat{\varphi} | \hat{\varphi} \rangle = \int d^3r_1 d^3r_2 d^3r_3 \hat{\varphi}_{z_1 z_2 z_3}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, t)^* \hat{\varphi}_{z_1 z_2 z_3}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, t) \quad (12)$$

and they describe the states of the system with interaction. Hence one is able to extract all quantum mechanically meaningful information about this system from its given state space.

Finally it should be noted that in the latter limit the coupling constants $\lambda_i g$ in the vertex (8),(9) of the three-parton equation diverge. But the essential point is that the regularized solutions of these equations remain finite in the whole range $(0, \infty)$ of $\lambda_i g$. Hence as the auxiliary fields are unobservable and the whole physics depends on the regularized solutions this behavior of the coupling constants has no obsevable consequences. In addition if the effective field equations for the three parton states are derived, the quantities $\lambda_i g$ drop out.

3 Symmetry constraints

The transformation properties of the three parton wave functions are correlated to and determined by the transformation properties of the spinor field theory being the theoretical background for the derivation of the generalized BBW-equations.

The latter theory is formulated in terms of spinor fields $\psi_{\alpha A i}(x)$ and formally charge conjugated spinor fields $\psi^c_{\alpha A i}(x)$. For the definition of the indices see section 2. In particular A is the index of a $SU(2)$ spinor basis. We first treat the symmetry constraints resulting from the transformation properties of the three parton wave functions under these $SU(2)$ transformations and an additional $U(1)$ transformation. If one combines spinors and charge conjugated spinors into a superspinor field by introducing the index κ, see section 2, then this superspinor field $\psi_{\alpha\kappa i}(x)$ transforms under $SU(2)$ transformations in the following way:

$$\psi'_{\alpha\kappa i}(x) = U_{\kappa\kappa'} \psi_{\alpha\kappa' i}(x) \quad (13)$$

with

$$U = exp[-i \sum_{k=1}^{3} \varepsilon_k G^k] \tag{14}$$

where the superspin-isospin generators are given by

$$G^k_{Z_1 Z_2} = \frac{1}{2} \begin{pmatrix} \sigma^k & 0 \\ 0 & (-)^k \sigma^k \end{pmatrix}_{\kappa_1 \kappa_2} \delta_{\alpha_1 \alpha_2} \tag{15}$$

In addition the superspinors admit a $U(1)$ global gauge group with

$$U = exp[-i\varepsilon F] \tag{16}$$

and

$$F_{Z_1 Z_2} = \frac{1}{3} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}_{\kappa_1 \kappa_2} \delta_{\alpha_1 \alpha_2} \tag{17}$$

Concerning the transformation properties of the three parton wave functions we consider for simplicity the physical wave functions $\hat{\varphi}$ in order to avoid the explicit dependence of the index set on the auxiliary field index i. The corresponding transformation properties are not changed by the transition from φ to $\hat{\varphi}$.

The transformation properties of the wave functions φ or $\hat{\varphi}$, respectively, must be compatible with the transformation properties of the spinor field theory in the background. For the global gauge groups this is the case if $\hat{\varphi}$ is transformed by

$$\hat{\varphi}^{\kappa_1 \kappa_2 \kappa_3}_{\alpha_1 \alpha_2 \alpha_3}(x_1, x_2, x_3)' = U_{\kappa_1 \kappa_1'} U_{\kappa_2 \kappa_2'} U_{\kappa_3 \kappa_3'} \hat{\varphi}^{\kappa_1' \kappa_2' \kappa_3'}_{\alpha_1 \alpha_2 \alpha_3}(x_1, x_2, x_3) \tag{18}$$

On the other hand if owing to (18) the three- body wave functions are elements of a representation space of this group, then these wave functions must satisfy the group theoretical constraints

$$\begin{aligned} \frac{9}{4}\varphi_{I_1 I_2 I_3} + 2[G^k_{I_1 K_1} G^k_{I_2 K_2} \varphi_{K_1 K_2 I_3} + G^k_{I_1 K_1} G^k_{I_3 K_2} \varphi_{K_1 I_2 K_2} \\ + G^k_{I_2 K_1} G^k_{I_3 K_2} \varphi_{I_1 K_1 K_2}] = t(t+1)\varphi_{I_1 I_2 I_3} \end{aligned} \tag{19}$$

$$G^3_{I_1 K} \varphi_{K I_2 I_3} + G^3_{I_2 K} \varphi_{I_1 K I_3} + G^3_{I_3 K} \varphi_{I_1 I_2 K} = t_3 \varphi_{I_1 I_2 I_3} \tag{20}$$

if for brevity we introduce the general index $I = Z, x$. The relation of the quantum numbers t and t_3 to the phenomenological quantum numbers is given in [8],[14].

Concerning the compatibility of these transformations with the generalized BBW-equations, it is convenient to treat this problem by replacing equation (4) by the corresponding homogenous integral equation for bound states which reads:

$$\varphi_{Z_1Z_2Z_3}(x_1,x_2,x_3)=\int d^4x G_{Z_3X_1}(x_3-x)U_{X_1X_2X_3X_4}\times \quad (21)$$
$$3[-F_{X_2Z_2}(x-x_2)\varphi_{Z_1X_3X_4}(x_1,x,x)$$
$$+F_{X_2Z_1}(x-x_1)\varphi_{Z_2X_3X_4}(x_2,x,x)]$$

Furthermore to simplify matters we sum in this equation over i_1,i_2,i_3 , as the summation over auxiliary fields does not change the transformation properties of the wave function. Then one obtains with notation at full length

$$\hat{\varphi}^{\kappa_1\kappa_2\kappa_3}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3)=\frac{g}{2}\int d^4x\sum_i \lambda_i G_{\alpha_3\alpha_1'}(x_3-x,m_i)\delta_{\kappa_3\kappa_1'}\times \quad (22)$$
$$\sum_h\{v^h_{\alpha_1'\beta}\delta_{\kappa_1'\rho}[(v^hC)_{\beta'\beta''}[\gamma^5(1-\gamma^0)]_{\rho'\rho''}-(v^hC)_{\beta''\beta'}[\gamma^5(1-\gamma^0)]_{\rho''\rho'}]$$
$$-v^h_{\alpha_1'\beta'}\delta_{\kappa_1'\rho'}[(v^hC)_{\beta\beta''}[\gamma^5(1-\gamma^0)]_{\rho\rho''}-(v^hC)_{\beta''\beta}[\gamma^5(1-\gamma^0)]_{\rho''\rho}]$$
$$-v^h_{\alpha_1'\beta''}\delta_{\kappa_1'\rho''}[(v^hC)_{\beta'\beta}[\gamma^5(1-\gamma^0)]_{\rho'\rho}-(v^hC)_{\beta\beta'}[\gamma^5(1-\gamma^0)]_{\rho\rho'}]\}\times$$
$$3[-\sum_j\lambda_j(-i)\gamma^5_{\rho\kappa_2}F_{\beta\alpha_2}(x-x_2,m_j)\hat{\varphi}^{\kappa_1\rho'\rho''}_{\alpha_1\beta'\beta''}(x_1,x,x)$$
$$+\sum_j\lambda_j(-i)\gamma^5_{\rho\kappa_1}F_{\beta\alpha_1}(x-x_1,m_j)\hat{\varphi}^{\kappa_2\rho'\rho''}_{\alpha_2\beta'\beta''}(x_2,x,x)]$$

The investigation of the invariance properties of the generalized BBW-equations with respect to these transformations was performed in the preceding paper,[14], and the result can be summarized by

Proposition1: The three parton generalized BBW-equations (22) are invariant under the global gauge group transformations (14), (16), i.e., with $\hat{\varphi}$ also $U\otimes U\otimes U\hat{\varphi}$ for any group element U are solutions of (22).

Owing to this invariance the constraints (19) and (20) are compatible with the generalized BBW-equations (22) and any solution of these equations must satisfy these constraints. The most simple ansatz is given by

$$\hat{\varphi}^{\kappa_1\kappa_2\kappa_3}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3)=\Theta^l_{\kappa_1\kappa_2\kappa_3}\hat{\varphi}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) \quad (23)$$

and the requirement of Θ^l being symmetric in all indices under permutations. This ansatz only then leads to a simplified calculation if Θ^l can be eliminated from (22) and such a separation is only possible if Θ^l has no γ^5 contribution. Hence on Θ^l the additional condition

$$\gamma^5_{\kappa_1\kappa_2}\Theta^l_{\kappa_1\kappa_2\kappa_3} = 0 \tag{24}$$

must be imposed. Owing to this condition only sixteen of the twenty symmetric states Θ^l are admitted. These states are explicitly tabulated in [16] and a table with the physical interpretation of the quantum numbers is contained in [8]. But from this table it follows that the ansatz (23) is too simple to obtain a complete agreement with phenomenology. Such an agreement can be only achieved without the separation (23) and if (24) is replaced by the weaker condition

$$\gamma^5_{\kappa_1\kappa_2}\hat{\varphi}^{\kappa_1\kappa_2\kappa_3}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) = 0 \tag{25}$$

The effect of this modification was discussed and tabulated in [17] by solving the corresponding energy equation in the strong coupling limit. The following investigations do not depend on the choice of assumption (24) or (25), respectively. Hence for the sake of brevity we eliminate the superspin-isospin indices from the further calculations by using the assumption (23) and (24).

Next we turn to the space-time transformations which lead to additional constraints represented by the Casimir opretators of the Poincare group. We directly discuss this problem by means of the covariant equation (21), because this equation is the basis for the whole formalism. As in the discussion of space-time transformations the gauge group indices κ are only spectator indices the ansatz (23) implies no loss of generality. After some rearrangements this leads to the following covariant equation

$$\begin{aligned}\hat{\varphi}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) = 6g\int d^4x \sum_i \lambda_i G_{\alpha_3\alpha_1'}(x_3-x,m_i)\times \\ \sum_h [v^h_{\alpha_1'\beta'}\hat{\varphi}_{\alpha_1\beta'\beta''}(x_1,x,x)(v^hC)_{\beta''\beta}\sum_j \lambda_j(i)F_{\beta\alpha_2}(x-x_2,m_j) \\ -v^h_{\alpha_1'\beta'}\hat{\varphi}_{\alpha_2\beta'\beta''}(x_2,x,x)(v^hC)_{\beta''\beta}\sum_j \lambda_j(i)F_{\beta\alpha_1}(x-x_1,m_j)]\end{aligned} \tag{26}$$

The behavior of the solutions of equation (26) under space-time transformations can be characterized by means of the Pauli-Lubanski spin vector W_μ. Owing to the relativistic invariance of (26) its solutions can be classified by the values of $W_\mu W^\mu$ and $P_\mu P^\mu$ as representations of the Poincare group. It is this property which constitutes the link to the quantum numbers of the energy equation. Owing to their transformation properties these solutions can be treated in the rest frame without loss of generality. In this case the Pauli-Lubanski spin vector reads

$$W_\mu = \frac{1}{2p_0}\varepsilon_{\mu\nu\rho 0} M^{\nu\mu} P^0 \tag{27}$$

and in this expression the representation of the generators P^μ and $M^{\mu\nu}$ depends on the dimension of the coordinate space. In the rest system W_0 vanishes and one obtains from (27)

$$W_i = \frac{1}{2}\varepsilon_{ijk} M^{jk} = -J^i \tag{28}$$

where the J^i are the angular momentum operators of the little group. Thus in the rest frame the quantum numbers of the solutions of equation (26) should be given by the eigenvalues of the Casimir operators of the little group $\mathbf{J}^2$ and J^3 . In the following it will be demonstrated that equation (26) indeed is compatible with these group theoretic constraints, i.e., that the solutions can be classified as representations of this group.

Proposition2: In the rest system of an eigenstate of equations (26) the eigenvalues of $\mathbf{J}^2$ and J^3 are good quantum numbers and their values are determined by those of the reduced solution $\hat{\varphi}(x_1, x, x)$.

The proof of this assertion depends crucially upon the transformation properties of the wavefunctions and was given in [14]. In the spinor-charge conjugated spinor representation the wave functions transform as the direct product of Dirac spinors. This was shown in [20] and without further explanation we refer to [20]. Therefore in accordance with this transformation property in the three-fermion space the generators J^i are to be defined by

$$J^i = L^i + S^i = \sum_{\alpha=1,2,3} (L^i_\alpha + S^i_\alpha) = \sum_{\alpha=1,2,3} i\varepsilon_{ijk}[x^\alpha_j \partial^\alpha_k - x^\alpha_k \partial^\alpha_j] \tag{29}$$
$$+\frac{1}{2}[\Sigma^i_{\alpha_1\alpha_1'}\delta_{\alpha_2\alpha_2'}\delta\alpha_3\alpha_3' + \delta_{\alpha_1\alpha_1'}\Sigma^i_{\alpha_2\alpha_2'}\delta_{\alpha_3\alpha_3'} + \delta_{\alpha_1\alpha_1'}\delta_{\alpha_2\alpha_2'}\Sigma^i_{\alpha_3\alpha_3'}]$$

with $\partial^\alpha_k := \partial/\partial x^{\alpha,k}$. For brevity we apply only the J^3 constraint because already by means of this condition the representations can be classified.

In accordance with proposition 2 we then have to satisfy the following two constraints

$$\begin{aligned}[S^3_{\varrho_h\varphi\varphi',\alpha_h\beta'\beta''} + \delta_{\varrho_h\alpha_h}\delta_{\varphi\beta'}\delta_{\varphi'\beta''}(L^3_h + L^3_x)]\hat{\varphi}_{\alpha_h\beta'\beta''}(x_h,x,x) \\ = j_3\hat{\varphi}_{\varrho_h\varphi\varphi'}(x_h,x,x)\end{aligned} \quad (30)$$

and

$$J^3_{\varrho_1\varrho_2\varrho_3,\alpha_1\alpha_2\alpha_3}\hat{\varphi}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3)_{as} = j^3\hat{\varphi}_{\varrho_1\varrho_2\varrho_3}(x_1,x_2,x_3)_{as} \quad (31)$$

which are compatible with the generalized BBW-equations (26).

We assume that an analogous result holds for the $\mathbf{J}^2$ condition without doing the rather complicated calculations explicitly. Then the existence of these constraints in addition to the energy eigenvalue equation means: The state space can be decomposed into a set of irreducible representation spaces of the little group. In analogy to the two-parton case we assume that only the lowest dimensional representations lead to stable bound states.

The most simple lowest dimensional representation is a spin *1/2* representation with orbital angular momentum zero. This representation describes a composite spin *1/2* fermion which in combination with the superspin-isospin quantum numbers we identify with the members of the lepton generations. The next higher dimensional representation contains an orbital angular momentum *1* and a spin angular momentum *1/2*. The former leads to a triplett which is energetically degenerate. This triplett forms a representation of the little group rotations, i.e., an *O(3)* representation and, in combination with the other quantum numbers, should be identified with the quark generations. With respect to the corresponding quantum numbers of three-quark states which constitute the nucleons, we refer to the literature.

4 Eigenstates of energy and angular momentum

Owing to the translational invariance of the three-parton equation their solutions admit a representation where the total four momentum is diagonalized. This leads to the ansatz

$$\hat{\varphi}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) = exp[-ik(x_1+x_2+x_3)\frac{1}{3}]\hat{\chi}_{\alpha_1\alpha_2\alpha_3}(x_2-x_1,x_3-x_2) \quad (32)$$

In the rest system $k = (k_0, 0, 0, 0)$ the total angular momentum commutes with the translational part of (32), i.e.

$$[exp[-ik(x_1 + x_2 + x_3)]_{/k=k_0}, J_3]_- = 0 \tag{33}$$

and therefore in this system the energy eigenvalue and the angular momentum eigenvalue can be simultaneously calculated.

First we discuss the energy (mass) eigenvalue equation. As in any solution procedure of equation (26) the antisymmetry of the wave function has to be secured,(and is required by the general formalism), we antisymmetrize equation (26) explicitly and in addition substitute the corresponding vertex matrices in the resulting equation. In order to obtain a compact formulation we introduce the following definitions

$$F_1(x)_{\alpha\beta} = \sum_i \lambda_i \int \frac{d^4p_1}{(2\pi)^4} f_i(\gamma^\mu p^1_\mu + m_i)_{\alpha\beta} exp[-ip^1_\kappa x^\kappa] \tag{34}$$

and

$$F_2(x)_{\alpha\beta} = \sum_j \lambda_j(i) \int \frac{d^4p_2}{2\pi)^4} f_j(-\gamma^\nu p^2_\nu + m_j)_{\alpha\beta} exp[-ip^2_\chi x^\chi] \tag{35}$$

and

$$V_{\alpha\beta\gamma\delta} = [-\delta_{\alpha\beta}\delta_{\gamma\delta} + \gamma^5_{\alpha\beta}\gamma^5_{\gamma\delta}] \tag{36}$$

Using these definitions equation (26) can be rewritten in the following form:

$$\begin{aligned} \hat{\varphi}_{\alpha_1\alpha_2\alpha_3}(x_1, x_2, x_3)_{as} = \quad (37) \\ 2g\{\int d^4x \hat{F}_1(x_1 - x)_{\alpha_1\nu} V_{\nu\beta\nu'\beta'} \hat{\varphi}_{\alpha_2\beta\beta'}(x_2, x, x) \hat{F}_2(x - x_3)_{\alpha_3\nu'} \\ - \int d^4x \hat{F}_1(x_2 - x)_{\alpha_2\nu} V_{\nu\beta\nu'\beta'} \hat{\varphi}_{\alpha_1\beta\beta'}(x_1, x, x) \hat{F}_2(x - x_3)_{\alpha_3\nu'} \\ + \int d^4x \hat{F}_1(x_2 - x)_{\alpha_2\nu} V_{\nu\beta\nu'\beta'} \hat{\varphi}_{\alpha_3\beta\beta'}(x_3, x, x) \hat{F}_2(x - x_1)_{\alpha_1\nu'} \\ - \int d^4x \hat{F}_1(x_3 - x)_{\alpha_3\nu} V_{\nu\beta\nu'\beta'} \hat{\varphi}_{\alpha_2\beta\beta'}(x_2, x, x) \hat{F}_2(x - x_1)_{\alpha_1\nu'} \\ + \int d^4x \hat{F}_1(x_3 - x)_{\alpha_3\nu} V_{\nu\beta\nu'\beta'} \hat{\varphi}_{\alpha_1\beta\beta'}(x_2, x, x) \hat{F}_2(x - x_2)_{\alpha_2\nu'} \\ - \int d^4x \hat{F}_1(x_1 - x)_{\alpha_1\nu} V_{\nu\beta\nu'\beta'} \hat{\varphi}_{\alpha_3\beta\beta'}(x_3, x, x) \hat{F}_2(x - x_2)_{\alpha_2\nu'}\} \end{aligned}$$

Substitution of (32) into (37) and transformation to the center of mass coordinates

$$z = \frac{1}{3}(x_1 + x_2 + x_3); \quad u = (x_2 - x_1); \quad v = (x_3 - x_2) \qquad (38)$$

then leads to the following equation

$$\begin{aligned}
& exp(-ikz)\hat{\chi}_{\alpha_1\alpha_2\alpha_3}(u,v) = \qquad (39)\\
& 2g\int d^4x V_{\nu\beta\nu'\beta'}\{\hat{F}_1(z - \frac{1}{3}(2u+v) - x)_{\alpha_1\nu} exp[-ik(z + \frac{1}{3}(u-v) + 2x)\frac{1}{3}] \times \\
& \hat{\chi}_{\alpha_2\beta\beta'}(x - z - \frac{1}{3}(u-v), 0)\hat{F}_2(x - z - \frac{1}{3}(u+2v))_{\alpha_3\nu'} \\
& -\hat{F}_1(z + \frac{1}{3}(u-v) - x)_{\alpha_2\nu} exp[-ik(z - \frac{2}{3}(2u+v) + 2x)\frac{1}{3}] \times \\
& \hat{\chi}_{\alpha_1\beta\beta'}(x - z + \frac{1}{3}(2u+v), 0)\hat{F}_2(x - z - \frac{1}{3}(u+2v))_{\alpha_3\nu'} \\
& +\hat{F}_1(z + \frac{1}{3}(u-v) - x)_{\alpha_2\nu} exp[-ik(z + \frac{1}{3}(u+2v) + 2x)\frac{1}{3}] \times \\
& \hat{\chi}_{\alpha_3\beta\beta'}(x - z - \frac{1}{3}(u+2v), 0)\hat{F}_2(x - z + \frac{1}{3}(2u+v))_{\alpha_1\nu'} \\
& -\hat{F}_1(z + \frac{1}{3}(u+2v) - x)_{\alpha_3\nu} exp[-ik(z + \frac{1}{3}(u-v) + 2x)\frac{1}{3}] \times \\
& \hat{\chi}_{\alpha_2\beta\beta'}(x - z - \frac{1}{3}(u-v), 0)\hat{F}_2(x - z + \frac{1}{3}(2u+v))_{\alpha_1\nu'} \\
& +\hat{F}_1(z + \frac{1}{3}(u+2v) - x)_{\alpha_3\nu} exp[-ik(z - \frac{1}{3}(2u+v) + 2x)\frac{1}{3}] \times \\
& \hat{\chi}_{\alpha_1\beta\beta'}(x - z + \frac{1}{3}(2u+v), 0)\hat{F}_2(x - z - \frac{1}{3}(u-v))_{\alpha_2\nu'} \\
& -\hat{F}_1(z - \frac{1}{3}(2u+v) - x)_{\alpha_1\nu} exp[-ik(z + \frac{1}{3}(u+2v) + 2x)\frac{1}{3}] \times \\
& \hat{\chi}_{\alpha_3\beta\beta'}(x - z - \frac{1}{3}(u+2v), 0)\hat{F}_2(x - z - \frac{1}{3}(u-v))_{\alpha_2\nu'}\}
\end{aligned}$$

With the transition to the new variable $x' = x - z - \frac{1}{3}(u-v)$, or $x' = x - z + \frac{1}{3}(2u+v)$, or $x' = x - z + \frac{1}{3}(u+2v)$, respectively, one can eliminate the center of mass part $exp(-ikz)$ as well as the center of mass coordinate z from equation (39) and obtains the final form of the equation for the energy or mass eigenvalue, respectively.

$$\hat{\chi}_{\alpha_1\alpha_2\alpha_3}(u,v) = \tag{40}$$
$$2g\int d^4x' V_{\nu\beta\nu'\beta'}\{\hat{F}_1(-u-x')_{\alpha_1\nu}exp[-ik(u-v+2x')\frac{1}{3}]\hat{F}_2(x'-v)_{\alpha_3\nu'}\hat{\chi}_{\alpha_2\beta\beta'}(x',0)$$
$$-\hat{F}_1(u-x')_{\alpha_2\nu}exp[-ik(-2u-v+2x')\frac{1}{3}]\hat{F}_2(x'-u-v)_{\alpha_3\nu'}\hat{\chi}_{\alpha_1\beta\beta'}(x',0)$$
$$+\hat{F}_1(-x'-v)_{\alpha_2\nu}exp[-ik(u+2v+2x')\frac{1}{3}]\hat{F}_2(x'+u+v)_{\alpha_3\nu'}\hat{\chi}_{\alpha_3\beta\beta'}(x',0)$$
$$-\hat{F}_1(-x'+v)_{\alpha_3\nu}exp[-ik(u-v+2x')\frac{1}{3}]\hat{F}_2(x'+u)_{\alpha_1\nu'}\hat{\chi}_{\alpha_2\beta\beta'}(x',0)$$
$$+\hat{F}_1(u+v-x')_{\alpha_3\nu}exp[-ik(-2u-v+2x')\frac{1}{3}]\hat{F}_2(x'-u)_{\alpha_2\nu'}\hat{\chi}_{\alpha_1\beta\beta'}(x',0)$$
$$-\hat{F}_1(-u-v-x')_{\alpha_1\nu}exp[-ik(u+2v+2x')\frac{1}{3}\hat{F}_2(x'+v)_{\alpha_2\nu'}\hat{\chi}_{\alpha_3\beta\beta'}(x',0)\}$$

Putting $v = 0$ in equation (40), the resulting equation is obviously a selfconsistent integral equation for the reduced function $\hat{\chi}(u,0)$.

Suppose now that $\hat{\chi}(u,0)$ is a solution of this resulting equation. Then using this solution one can calculate $\hat{\varphi}(x_l,x,x)$, $l = 1,2,3$ by means of equation (32). In particular one obtains from (32)

$$\hat{\varphi}_{\alpha_1\alpha_2\alpha_3}(x_l,x,x) = exp[-ik(x_l+2x)\frac{1}{3}]\hat{\chi}_{\alpha_1\alpha_2\alpha_3}(x-x_l,0) \tag{41}$$

and substituting these functions into (37) only by integrations the full solutions can be generated.

The equation resulting from (40) by putting $v = 0$ is a rather complicated integral equation. Thus one cannot expect to derive exact solutions although in principle such solutions must exist according to the Fredholm theory. On the other hand we know that the full equation (37) is compatible with the angular momentum constraint (31). Hence any solution of (37) must fulfil (31). This in its turn is guaranteed if the angular momentum condition (30) is satisfied. Hence for a first classification of the energy spectrum of equation (37) we analyze condition (30) in the rest system. Substitution of (32) into (30) then gives after elimination of the center of mass term

$$[S^3_{\rho\varphi\varphi',\alpha\beta\beta'} + \delta_{\rho\alpha}\delta_{\varphi\beta}\delta_{\varphi'\beta'}(L^3_l + L^3_x)]\hat{\chi}_{\alpha\beta\beta'}(x-x_l,0) = j_3\hat{\chi}_{\rho\varphi\varphi'}(x-x_l,0) \tag{42}$$

We now apply the transformation $z=(x_l+2x)\frac{1}{3}$, $\quad u_l=(x-x_l)$ and this transformation leads to $L_l^3+L_x^3=L_z^3+L_{u_l}^3$ which yields for (42) the final formula

$$[S^3_{\rho\varphi\varphi',\alpha\beta\beta'}+\delta_{\rho\alpha}\delta_{\varphi\beta}\delta_{\varphi'\beta'}L^3_{u_l}]\hat{\chi}_{\alpha\beta\beta'}(u_l,0)=j_3\hat{\chi}_{\rho\varphi\varphi'}(u_l,0) \tag{43}$$

Of course the argument u_l can be replaced by u as for all u_l equation (43) is referred to the same function $\hat{\chi}$ and the same angular quantum number j_3.

5 Consistency of the antisymmetry condition

On the one hand the three parton wave functions are assumed to be solutions of the corresponding generalized BBW-equations, on the other hand these wave functions must fit into the field theoretic formalism. The latter requirement includes that the wave functions are to be interpreted as formally normal ordered matrix elements stemming from time ordered matrix elements. And this means that these wave functions must possess the transformation properties which characterize such matrix elements. In the preceding papers [20],[21] the transformation properties with respect to the global gauge groups and the Lorentz group were analyzed and shown to be compatible with the required interpretation. So we concentrate now on the question whether the antisymmetry of the wave functions - which is required by their field theoretic interpretation - is compatible with the solution procedure of the generalized BBW-equations. The treatment of this question is nontrivial and crucial for the selfconsistency of the generalized BBW-equations themselves as the whole formalism depends on this assumption.

In any case antisymmetry of the full physical wave function (23) can be enforced by direct antisymmetrization, namely

$$\hat{\varphi}_{Z_1Z_2Z_3}(x_1,x_2,x_3)_{as}\equiv\frac{1}{6}\sum_{h_1h_2h_3}(-1)^P\hat{\varphi}_{Z_{h_1}Z_{h_2}Z_{h_3}}(x_{h_1},x_{h_2},x_{h_3}) \tag{44}$$

leading to the representation of the internal part of the wave function

$$\hat{\chi}_{Z_1Z_2Z_3}(x_2-x_1,x_3-x_2)_{as}\equiv\frac{1}{6}\sum_{h_1h_2h_3}(-1)^P\hat{\chi}_{Z_1Z_2Z_3}(x_{h_2}-x_{h_1},x_{h_3}-x_{h_2}) \tag{45}$$

The six permutations in the fermion coordinates x_1, x_2, x_3 can be expressed by the internal coordinates u, v in the following way

$$\begin{aligned}
&x_2 - x_1, x_3 - x_2 =: u, v =: \bar{u}_1, \bar{v}_1 \qquad (46)\\
&x_1 - x_2, x_3 - x_1 =: -u, u + v =: \bar{u}_2, \bar{v}_2\\
&x_3 - x_2, x_1 - x_3 =: v, -u - v =: \bar{u}_3, \bar{v}_3\\
&x_2 - x_3, x_1 - x_2 =: -v, -u =: \bar{u}_4, \bar{v}_4\\
&x_1 - x_3, x_2 - x_1 =: -u - v, u =: \bar{u}_5, \bar{v}_5\\
&x_3 - x_1, x_2 - x_3 =: u + v, -v =: \bar{u}_6, \bar{v}_6
\end{aligned}$$

Using these definitions one obtains with the permutations $P\{1,2,3\} \equiv \sum_{l=1}^{6}(h_1^l, h_2^l, h_3^l)$

$$\hat{\chi}_{Z_1 Z_2 Z_3}(x_2 - x_1, x_3 - x_2)_{as} \equiv \frac{1}{6}\sum_{l=1}^{6}(-1)^P \hat{\chi}_{Z_{h_1^l} Z_{h_2^l} Z_{h_3^l}}(\bar{u}_l, \bar{v}_l) \qquad (47)$$

We now consider the action of orbital angular momentum on the antisymmetrized wave function. In the rest system the application of the orbital angular momentum operators yields

$$(L_1^3 + L_2^3 + L_3^3)\hat{\varphi}_{Z_1 Z_2 Z_3}(x_1, x_2, x_3)_{as} = exp(-ik_0 z_0)(L_u^3 + L_v^3)\hat{\chi}_{Z_1 Z_2 Z_3}(u, v)_{as} \qquad (48)$$

or explicitly with the symbolic notation: $Z_n^l := Z_{h_n^l}$

$$(L_u^3 + L_v^3)\hat{\chi}_{Z_1 Z_2 Z_3}(u, v)_{as} = \frac{1}{6}\sum_l (L_u^3 + L_v^3)\hat{\chi}_{Z_1^l Z_2^l Z_3^l}(\bar{u}_l, \bar{v}_l) \qquad (49)$$

By transformation of the orbital angular momentum operators from the variables u,v to the variables $\bar{u}_l$, $\bar{v}_l$ the following relation results

$$(L_u^3 + L_v^3)\hat{\chi}_{Z_1^l Z_2^l Z_3^l}(\bar{u}_l, \bar{v}_l) = (L_{\bar{u}_l}^3 + L_{\bar{v}_l}^3)\hat{\chi}_{Z_1^l Z_2^l Z_3^l}(\bar{u}_l, \bar{v}_l) \qquad (50)$$

and as $\hat{\chi}(\bar{u}_l, \bar{v}_l)$ is for all l = 1,..,6 the same function, the eigenvalue of the orbital part of the angular momentum does not depend on the permutations. The spin operator in (30) is invariant under permutations of the spin indices. Hence one obtains

$$\begin{aligned}
&(S^3_{Z_1 Z_2 Z_3, Z_1' Z_2' Z_3'} + L_u^3 + L_v^3)\hat{\chi}_{Z_1 Z_2 Z_3}(u, v)_{as} = j_3 \hat{\chi}_{Z_1 Z_2 Z_3}(u, v)_{as} \qquad (51)\\
&\equiv \frac{1}{6}\sum_l (S^3_{Z_1^l Z_2^l Z_3^l, Z_1^{l\prime} Z_2^{l\prime} Z_3^{l\prime}} + L_{\bar{u}_l}^3 + L_{\bar{v}_l}^3)\hat{\chi}_{Z_1^{l\prime} Z_2^{l\prime} Z_3^{l\prime}}(\bar{u}_l, \bar{v}_l) = j_3 \sum_l \hat{\chi}_{Z_1^l Z_2^l Z_3^l}(\bar{u}_l, \bar{v}_l)
\end{aligned}$$

Furthermore from equation (46) one can conclude: Each permutation with respect to the general coordinates $(x_1 Z_1), (x_2 Z_2), (x_3 Z_3)$ induces a corresponding permutation in the general coordinates $\bar{u}_l, \bar{v}_l, Z_1^l, Z_2^l, Z_3^l$, l = 1,..,6 . Then, if a permutation operation is applied to the second line of equation (51) it follows that in the rest frame the angular momentum operator J^3 is mapped onto itself, i.e., it is invariant under permutations.

Therefore suppose that one has an unsymmetrical wave function with good quantum number j_3, then antisymmetrization conserves this quantum number as the permutation operator and the angular momentum operator commute. And this means, that if one is forced to apply approximations one can start the calculations with unsymmetrical eigenfunctions of the angular momentum operator.

Similar considerations hold for the other components of the angular momentum operator. This can be summarized by

Proposition3: In the rest system of an eigenstate of equation (37) the angular momentum operators $\mathbf{J}^2$ and J^3 are invariant under permutations, i.e., these operators commute with the permutation operator.

Finally we consider the behavior of the mass eigenvalue equation (40) under operations of the permutation group. With the symbolic notation $\alpha_n^l := \alpha_{h_n^l}$ we represent equation (40) in terms of the variables $\bar{u}_l, \bar{v}_l$. This gives the equivalent expression for equation (40) in the following form:

$$\hat{\chi}_{\alpha_1\alpha_2\alpha_3}(u,v)_{as} = \frac{2}{3}\sum_l (-1)^P \int d^4x' V_{\nu\beta\nu'\beta'} \times \tag{52}$$

$$\{\hat{F}_1(-x'-\bar{u}_l)_{\alpha_1^l\nu} exp[-ik(\frac{2}{3}x' + \frac{1}{3}(\bar{u}_l - \bar{v}_l)]\hat{F}_2(x'-\bar{v}_l)_{\alpha_3^l\nu'}\hat{\chi}_{\alpha_2^l\beta\beta'}(x',0)\}$$

In combination with the remarks made above from this equation it directly follows

Proposition4: In an arbitrary frame of reference the mass eigenvalue equation for an associated eigenstate is invariant under operations of the permutation group.

The assertion that in a general frame of reference the eigenvalue is defined by the invariant mass value is justified by the invariance of the generalized BBW-equations under Poincare transformations.

6 Construction of eigenstates and dual states

In section 4 an integral equation (40) for the reduced eigenstate $\chi(u,0)$ was derived. The solution of this equation is a necessary but not sufficient condition for the construction of the full eigenstates (32). As the latter are required for physical applications their derivation has to be discussed in detail, in particular with respect to regularization and the construction of dual states.

In the following the decomposition of the index $Z=(z,i)$ of section 1 is used. In this notation equations (21) read

$$\varphi^{z_1z_2z_3}_{i_1i_2i_3}(x_1,x_2,x_3)=\int d^4xG^{z_3a_1}_{i_3l_1}(x_3-x)\lambda_{l_1}V_{a_1a_2a_3a_4}\times \tag{53}$$

$$3\sum_{l_2}[-F^{a_2z_2}_{l_2i_2}(x-x_2)\Omega^{z_1a_3a_4}_{i_1}(x_1,x,x)+F^{a_2z_1}_{l_2i_1}(x-x_1)\Omega^{z_2a_3a_4}_{i_2}(x_2,x,x)]$$

with

$$\sum_{i_2i_3}\varphi^{z_1z_2z_3}_{i_1i_2i_3}(x_1,x_2,x_3)=:\Omega^{z_1z_2z_3}_{i_1}(x_1,x_2,x_3) \tag{54}$$

and by summation over i_2,i_3 the equations

$$\Omega^{z_1z_2z_3}_{i_1}(x_1,x_2,x_3)=\int d^4x\hat{G}^{z_3a_1}(x_3-x)V_{a_1a_2a_3a_4}\times \tag{55}$$

$$3[-\hat{F}^{a_2z_2}(x-x_2)\Omega^{z_1a_3a_4}_{i_1}(x_1,x,x)+\sum_{l_2}F^{a_2z_1}_{l_2i_1}(x-x_1)\hat{\varphi}^{z_2a_3a_4}(x_2,x,x)]$$

result with

$$\sum_{i_3}\lambda_{l_1}G^{z_3a_1}_{i_3l_1}(x)=:\hat{G}^{z_3a_1}(x) \tag{56}$$

Defining the inhomogenous term

$$I^{z_1z_2z_3}_{i_1}(x_1,x):=\int d^4x'\hat{G}^{z_3a_1}(x-x')V_{a_1a_2a_3a_4}\times \tag{57}$$

$$\sum_{l_2}F^{a_2z_1}_{l_2i_1}(x'-x_1)\hat{\varphi}^{z_2a_3a_4}(x,x',x')$$

from (55) one obtains the selfconsistent equations

$$\Omega^{z_1z_2z_3}_{i_1}(x_1,x,x)=-\int d^4x'\hat{G}^{z_3a_1}(x-x')V_{a_1a_2a_3a_4}\times \tag{58}$$

$$3\hat{F}^{a_2z_2}(x'-x)\Omega^{z_1a_3a_4}(x_1,x',x')+I^{z_1z_2z_3}_{i_1}(x_1,x)$$

If one assumes now that the eigenvalue problem (40) has been solved, then (41) is a well known function and an explicit solution of equation (58) can be obtained by applying a corresponding resolvent Γ which leads to

$$\Omega_{i_1}^{z_1 z_2 z_3}(x_1, x, x) = \int d^4x' \Gamma_{b_2 b_3}^{z_2 z_3}(x, x') I_{i_1}^{z_1 b_2 b_3}(x_1, x') \tag{59}$$

Thus having explicitly calculated $\hat{\varphi}$ as well as Ω, the full eigenstate φ can be generated by substitution of Ω into (53) leading to φ solely by integration.

The crucial point in this construction is the closely associated regularization of vertex functions which is an automatic consequence of the solution procedure itself. In this way the integral kernel of equation (40) exclusively contains only regularized functions which allows the application of Fredholm's theory to its solution.

Although the classical Fredholm theory does not explicitly comprise Feynman integrals, in any case due to the regularization all Feynman integrals of the kernel and their iteration give finite results and thus the Fredholm series can be calculated without producing infinities.

The same argument applies to the kernel of equation (58). Hence its resolvent can be constructed by means of the Fredholm theory too.

The systematic effect of this automatic regularization can be illustrated by counting the remaining factors λ_i. Each single regularization of a function eliminates one λ_i. And, referring to the definitions (7),(34),(35),(36) and (56), one easily realizes that the kernel of equation (40) contains no more λ_i at all. The same holds for the kernel of equation (58) and its kernel Γ, whereas the inhomogenous term (57) exactly contains one λ_i.

From the final integral representation (53) of φ one therefore deduces the general formula

$$\varphi_{i_1 i_2 i_3}^{z_1 z_2 z_3}(x_1, x_2, x_3) = \lambda_{i_1}\lambda_{i_2}\lambda_{i_3}\varphi^{z_1 z_2 z_3}(x_1, x_2, x_3, m_{i_1}, m_{i_2}, m_{i_3}) \tag{60}$$

of the dependence of φ on the λ_i.

This formula is a suitable starting point for the construction of dual states which are required in physical applications, e.g., derivation of effective theories by weak mapping.

For the case under consideration, i.e., the three body states, the set of duals $\{R^k\}$ is defined by the orthogonality relation

$$R_{I_1 I_2 I_3}^k \varphi_{k'}^{I_1 I_2 I_3} = \delta_{k'}^k \tag{61}$$

for single time wave functions where $I := \{\mathbf{r}, t = 0, Z\}$ and k and k' run through all state quantum numbers of the three parton problem.

Provided the state functions φ_k are explicitly given, the construction of duals is a purely mathematical problem. Hence there exist no rigorous physical arguments for the derivation of duals. Physical considerations in this respect are thus only of heuristic value to support a mathematical deduction.

Based on such heuristic arguments the construction of dual states has been treated at full length in [7] and [8]. But we will show that by means of such considerations one really arrives at a rigorous solution of (61).

From [7] and [8] one obtains for the duals

$$R^k_{I'_1 I'_2 I'_3} = g^{-1}_{kk} (\varphi^{I_1 I_2 I_3}_k)^\dagger G^{-1}_{I_1 I_2 I_3, I'_1 I'_2 I'_3} \tag{62}$$

where g_{kk} is the norm of φ_k in Krein space and G is the metric tensor of the auxiliary spinor fields in Krein space. The latter tensor describes the properties of the auxiliary spinor field vacuum.

Concerning this vacuum, in the eigenvalue equations (4) its representation is fixed by the propagator F. According to (7) this propagator is assumed to be the free auxiliary field propagator. However, after regularization, for physical states a modified field propagator results which describes a nontrivial vacuum. Hence the choice of a trivial vacuum in Krein space induces by regularization a nontrivial vacuum in physical state space. In order to be compatible with these preconditions G must be referred to free auxiliary fields too.

The corresponding proof has been given in [7] and leads to

$$G^{-1}_{3,3} = G^{-1}_{1,1} \otimes G^{-1}_{1,1} \otimes G^{-1}_{1,1} = \lambda^{-1}_{i_1} \lambda^{-1}_{i_2} \lambda^{-1}_{i_3} \delta_{I_1 I'_1} \delta_{I_2 I'_2} \delta_{I_3 I'_3} \tag{63}$$

where $G_{1,1}$ is defined by

$$G_{1,1} =: \langle 0 | \psi^+_I | \psi_{I'} | 0 \rangle \tag{64}$$

for free auxiliary fields ψ_I without Dirac vacuum.

As already mentioned this construction is only heuristic and must be justified by the fulfillment of (61). Substitution of (62) and (60) into (61) leads to

$$\langle R | \varphi \rangle = \sum_{i_1 i_2 i_3} \sum_{z_1 z_2 z_3} \int d^3 r_1 d^3 r_2 d^3 r_3 \tag{65}$$

$$\lambda^{-1}_{i_1} \lambda^{-1}_{i_2} \lambda^{-1}_{i_3} \varphi^{z_1 z_2 z_3}_{i_1 i_2 i_3} (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)^\dagger \varphi^{z_1 z_2 z_3}_{i_1 i_2 i_3} (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$$

whereas the physical scalarproduct is given by (12).

In spite of the difference between (65) and (12) the following proposition holds for the scalar products (65) and (12) of two different states φ_k and $\varphi_{k'}$: **Proposition5** : Let R^k or $R^{k'}$, respectively, be the duals of φ_k or $\varphi_{k'}$ constructed by (62). Then the corresponding scalar products defined by (65) and (12) fulfill identical orthogonality relations, provided that the set of quantum numbers given by k and k' are referred to the same mass eigenvalue.

Proof: The orthogonality relations can be deduced by means of the maximal set of commuting observables. For BBW-equations this set was defined in [14]. Owing to the Lorentz invariance this set can be considered in the rest frame. It consists of angular momentum operators and operators of the algebraic groups. The former operators are selfadjoint, while the latter operators are symmetric. In both types of operators no reference to the auxiliary field numbers i_1, i_2, i_3 is made. Hence the general form of the eigenvalue equation for an infinitesimal generator $\mathcal{G}$ reads

$$\mathcal{G}\varphi_{i_1 i_2 i_3} = g\varphi_{i_1 i_2 i_3} \tag{66}$$

From this equation for the states with numbers k and k' either the relation

$$(g^k - g^{k'})\langle\hat{\varphi}^k|\hat{\varphi}^{k'}\rangle = 0 \tag{67}$$

or the relation

$$(g^k - g^{k'})\sum_{i_1 i_2 i_3}\langle(\varphi^k_{i_1 i_2 i_3})^\dagger \lambda_{i_1}^{-1}\lambda_{i_2}^{-1}\lambda_{i_3}^{-1}\varphi^{k'}_{i_1 i_2 i_3}\rangle = 0 \tag{68}$$

can be deduced which show that (65) and (12) satisfy identical orthogonality relations. ♢

With respect to the scalar produkt for $k \equiv k'$ (12) is positiv definite while (65) is indefinite, i.e. it cannot be excluded that this scalar product vanishes for nonvanishing φ. Leaving aside this extraordinary case, from (65) for $k \equiv k'$ it follows that the sum over i_1, i_2, i_3 regularizes $|\varphi|^2$ which defines the metric in Krein space of the auxiliary fields owing to the varying signs of $\lambda_{i_1}, \lambda_{i_2}, \lambda_{i_3}$. But independently of the sign of this scalar product, the dual can be defined in such a way that the definition (61) can be satisfied.

Finally it remains the problem of the scalar products for different masses of the states k and k'. This problem concerns (65) as well as

(12), because no selfadjoint operator for the mass eigenvalues can be derived from the BBW-equations. Thus the conventional arguments for showing orthogonality cannot be applied. Without speculating about the physical meaning of this exception of the usual treatment one can solve this problem by forming linear combinations of states with equal quantum numbers but different masses and apply Schmidt's orthogonalization procedure [22].

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(Manuscrit reçu le 7 mars 2003)