Dynamics of Spatially localized Fields Obeying Complex Hamiltonian Evolution Equations

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ABSTRACT. Several families of nonlinear field equations are known to possess spatially localized solutions which can serve as representations, on a very general level, of elementary particles. This paper presents a derivation of the equations describing the motion of a localized field when it interacts with given electromagnetic potentials A_μ which enter the field's equations through the covariant derivative forms $\partial/\partial x_\mu - igA_\mu$. The main result is: the motion of a spatially localized field, as a whole entity, is essentially identical to that of a point charge moving in the same electromagnetic potentials, if this field is a solution of any equation which is member of the above family and is gauge invariant. In the process of obtaining this result several unexpected links to quantum mechanics become apparent.

1. Introduction

Spatially localized fields (under various names) and their possible role in particle and field theories have been studied by many physicists. Among the most prominent are: L. de Broglie [1], W. Heisenberg [2], T. D. Lee [3, 4], N. Rosen [5,6]. . . The results in the present paper are new evidences that localized fields can be viable representations, on a very general level, of elementary particles. In earlier papers [12, 13] I have already shown that certain localized fields are naturally associated with waves which are characterized by de Broglie-type relations. These relations are derived rather than assumed.

Here we investigate the dynamics of spatially localized complex multi-component fields $\psi_\sigma = \psi_\sigma(x, t)$ which are solutions to *complex Hamiltonian evolution* (CHE) equations of order 2 or less, i.e., with the form

$$\frac{\partial \psi_\sigma}{\partial t} = -i\,\frac{\delta H}{\delta \psi^*_\sigma}\ , \qquad H = \int_{I\!R^3} \mathcal{H}(x,t,\psi^*,\psi,\partial\psi^*,\partial\psi)\, d^3x \tag{1.1}$$

where H is the *Hamiltonian functional* which must be *real-valued*, $\partial\psi$ denotes all space-derivatives $\partial\psi_\sigma/\partial x_i$, ψ^*_σ is the complex conjugate of ψ_σ and

$$\frac{\delta H}{\delta \psi^*_\sigma} = \frac{\partial \mathcal{H}}{\partial \psi^*_\sigma} - \frac{d}{dx_i}\frac{\partial \mathcal{H}}{\partial(\partial_i \psi^*_\sigma)} \tag{1.2}$$

is the *variational derivative* of H with respect to ψ^*_σ . ψ_σ may be a spinor, vector, scalar, or sets of coupled scalar fields defined on the entire Euclidian space $I\!R^3$ whose coordinates are $x = (x_1, x_2, x_3)$ and t is the time. The present paper is closely related to an earlier paper [7] by the same author which discusses the complex Hamiltonian evolution equations, some of their properties and associated implications for physics.

It should be observed that the field equations must be *nonlinear* in order to possess spatially localized solutions. On the other hand, when the Hamiltonian functional is bi-linear

$$H = \int \psi^*_\sigma \hat{H}_{\sigma\rho} \psi_\rho \, d^3x$$

where $\hat{H}_{\sigma\rho}$ is a self-adjoint matrix-differential operator, we obtain the family of *linear* CHE equations $i\,\partial\psi_\sigma/\partial t = \hat{H}_{\sigma\rho}\psi_\rho$ to which Schrödinger's and Dirac's equations are members.

All the results in this paper depend critically on the fact that certain nonlinear CHE field equations possess *spatially localized* (also called *soliton-like*) solutions. The existence of such solutions to a large family of *scalar* nonlinear field equations, including the nonlinear Schrödinger (NLS) equations, is proved and existence conditions are derived in Berestycki and Lions [8]. The existence of such solutions to *spinor* nonlinear field equations, including the nonlinear Dirac (NLD) equation is proved in Cazenave and Vazquez [9]. Also, there are many publications which demonstrate the existence and certain properties of localized solutions by numerical methods. Some additional references are: Bialynicki-Birula and Mycielski [10], Enz [11], Bodurov [12, 13]. It should be also pointed out that large number of works in Soliton theory investigate the *solitons* as one-dimensional models of elementary particles. A collection edited by Rebbi and Soliani [14] contains several such works.

This paper poses and answers the question: What is the motion of a spatially localized field, as a whole entity, if its interaction with given scalar potential Φ and vector potential $\mathbf{A} = (A_1, A_2, A_3)$ is defined by the substitutions

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + ig\Phi \ , \qquad \frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_i} - igA_i$$

in its field equation (1.1). In answering the above question, this paper shows that there are a number of unexpected links between nonlinear field theory and quantum mechanics.

Notation and conventions: Here, all complex-valued fields will be denoted with the Greek letter ψ, all densities (the integrands of functionals) with script capital letters, all 3-vectors with bold letters. The complex conjugate of any field ψ will be written as ψ^*. The rest of the symbols will be defined at their first appearance. The summation convention of repeated indexes is assumed as usual. The domain of all space integrals is the entire 3-dimensional Euclidian space $\mathbb{R}^3$. The units are selected so that $c = 1$.

2. The position, velocity and momentum of a localized field

If we are to study the motion of a spatially localized field $\psi(x,t)$, as a whole entity, we should be able to calculate the position and velocity of the field's *localization region* and the field's total momentum from $\psi(x,t)$ alone. For this purpose we define the position of a spatially localized field, i.e., the coordinates of its center of localization, with the functionals

$$X_i = \frac{1}{N}\int_{\mathbb{R}^3} \psi_\sigma^* \psi_\sigma \, x_i \, d^3x \ , \qquad i = 1,2,3 \tag{2.1}$$

where

$$N = \int_{\mathbb{R}^3} \psi_\sigma^* \psi_\sigma \, d^3x \tag{2.2}$$

is the field's norm (squared). The field is not normalized since it is a solution of a nonlinear equation, hence, the appearance of the factor $1/N$ in (2.1).

Before we ask how the values of the functionals X_i depend on time, we need to address the simpler question: under what conditions is N constant in time? For this, we differentiate $\psi_\sigma^* \psi_\sigma$ with respect to time and use the field equations (1.1), accounting for (1.2), to obtain

$$\begin{aligned} &\frac{d}{dt}(\psi_\sigma^*\psi_\sigma) + i\,\frac{d}{dx_i}\left(\psi_\sigma \frac{\partial \mathcal{H}}{\partial(\partial_i\psi_\sigma)} - \psi_\sigma^* \frac{\partial \mathcal{H}}{\partial(\partial_i\psi_\sigma^*)}\right) \\ &= i\left(\psi_\sigma \frac{\partial \mathcal{H}}{\partial \psi_\sigma} - \psi_\sigma^* \frac{\partial \mathcal{H}}{\partial \psi_\sigma^*} + \frac{\partial \psi_\sigma}{\partial x_i}\frac{\partial \mathcal{H}}{\partial(\partial_i\psi_\sigma)} - \frac{\partial \psi_\sigma^*}{\partial x_i}\frac{\partial \mathcal{H}}{\partial(\partial_i\psi_\sigma^*)}\right) . \end{aligned} \tag{2.3}$$

It follows from the above that if the Hamiltonian density $\mathcal{H}$ satisfies the condition

$$\psi_\sigma \frac{\partial \mathcal{H}}{\partial \psi_\sigma} - \psi_\sigma^* \frac{\partial \mathcal{H}}{\partial \psi_\sigma^*} + \frac{\partial \psi_\sigma}{\partial x_i}\frac{\partial \mathcal{H}}{\partial(\partial_i\psi_\sigma)} - \frac{\partial \psi_\sigma^*}{\partial x_i}\frac{\partial \mathcal{H}}{\partial(\partial_i\psi_\sigma^*)} = 0 \tag{2.4}$$

the density $\psi_\sigma^*\psi_\sigma$ obeys a conservation law and, consequently, the norm N is constant in time. Furthermore, any function $\mathcal{H}$ of ψ, ψ^*, $\partial\psi$, $\partial\psi^*$ which is invariant under the *gauge type I transformations* [15]

$$\psi'_\sigma = \psi_\sigma\, e^{i\varepsilon} \ , \qquad \psi'^*_\sigma = \psi^*_\sigma\, e^{i\varepsilon} \tag{2.5}$$

where ε is the transformation parameter (independent of x and t), satisfies equation (2.4). One can verify this by inserting ψ'_σ and ψ'^*_σ into $\mathcal{H}$. Then, the condition for invariance

$$\frac{d}{d\varepsilon}\mathcal{H}(x,t,\psi'^*,\psi',\partial\psi'^*,\partial\psi')\Big|_{\varepsilon=0} = 0$$

immediately produces equation (2.4). Thus we have: *The norm N of a spatially localized solution to a CHE equation is constant in time if the Hamiltonian density $\mathcal{H}$ is gauge type I invariant.*

To obtain the velocity of the region of localization $\mathbf{V} = (V_1, V_2, V_3)$, one differentiates the position functionals (2.1) with respect to time and then uses the field equations (1.1)

$$\begin{aligned} V_i &= \frac{1}{N}\int\left(\psi_\sigma^* \frac{\partial \psi_\sigma}{\partial t} + \frac{\partial \psi_\sigma^*}{\partial t}\psi_\sigma\right) x_i\, d^3x \\ &= \frac{i}{N}\int\left(\psi_\sigma \frac{\partial \mathcal{H}}{\partial(\partial_i\psi_\sigma)} - \psi_\sigma^* \frac{\partial \mathcal{H}}{\partial(\partial_i\psi_\sigma^*)}\right) d^3x \\ &+ \frac{i}{N}\int\left(\psi_\sigma \frac{\partial \mathcal{H}}{\partial \psi_\sigma} - \psi_\sigma^* \frac{\partial \mathcal{H}}{\partial \psi_\sigma^*} + \frac{\partial \psi_\sigma}{\partial x_j}\frac{\partial \mathcal{H}}{\partial(\partial_j\psi_\sigma)} - \frac{\partial \psi_\sigma^*}{\partial x_j}\frac{\partial \mathcal{H}}{\partial(\partial_j\psi_\sigma^*)}\right) x_i\, d^3x \ . \end{aligned}$$

Recognizing that the second integrant is zero, according to (2.4), we find that the velocity components of the localization region are given by the functionals

$$V_i = \frac{i}{N}\int_{\mathbb{R}^3}\left(\psi_\sigma \frac{\partial\mathcal{H}}{\partial(\partial_i\psi_\sigma)} - \psi_\sigma^* \frac{\partial\mathcal{H}}{\partial(\partial_i\psi_\sigma^*)}\right)d^3x \ . \tag{2.6}$$

In passing, we note that the integrant of the above functional is equal to the flux density associated with $\psi_\sigma^*\psi_\sigma$ as seen in equation (2.3). Accordingly, one can regard $\psi_\sigma^*\psi_\sigma$ as the density of some substance whose velocity at a point x is the integrant in (2.6) multiplied by i/N .

Next, following the common practice, we identify the densities of the field's linear momentum components $\mathcal{P}_i$ with the entries $-\mathcal{T}_{i0}$ of the *energy-momentum 4-tensor* for the complex field ψ

$$\mathcal{P}_i = -\mathcal{T}_{i0} = -\frac{\partial\psi_\sigma}{\partial x_i}\frac{\partial\mathcal{L}}{\partial(\partial_t\psi_\sigma)} - \frac{\partial\psi_\sigma^*}{\partial x_i}\frac{\partial\mathcal{L}}{\partial(\partial_t\psi_\sigma^*)} \tag{2.7}$$

where $\partial_t\psi_\sigma$ stands for $\partial\psi_\sigma/\partial t$ and

$$\mathcal{L} = \frac{i}{2}\left(\psi_\sigma^*\frac{\partial\psi_\sigma}{\partial t} - \frac{\partial\psi_\sigma^*}{\partial t}\psi_\sigma\right) - \mathcal{H} \tag{2.8}$$

is the Lagrangian density for the field equations (1.1). Inserting (2.8) into (2.7) produces for the linear momentum densities

$$\mathcal{P}_i = \frac{1}{2i}\left(\psi_\sigma^*\frac{\partial\psi_\sigma}{\partial x_i} - \frac{\partial\psi_\sigma^*}{\partial x_i}\psi_\sigma\right) \tag{2.9}$$

Consequently, the components of the field's linear momentum are given by the functionals

$$P_i = \frac{1}{2i}\int\left(\psi_\sigma^*\frac{\partial\psi_\sigma}{\partial x_i} - \frac{\partial\psi_\sigma^*}{\partial x_i}\psi_\sigma\right)d^3x = -i\int\psi_\sigma^*\frac{\partial\psi_\sigma}{\partial x_i}\,d^3x \tag{2.10}$$

It is remarkable that the last functional, obtained from purely classical arguments, differs from the *expectation value* of the linear momentum in quantum mechanics only by the multiplicative constant $\hbar/N$.

With the momentum and velocity functionals well defined one can define the mass m of a spatially localized field as the ratio

$$m = \frac{P_i}{V_i} \qquad \text{(no summation)} \tag{2.11}$$

which definition is, clearly, independent of any relativistic arguments (i.e. $m = \text{energy}/c^2$). In general, m will be a function of V since the functionals P_i and V_i are not proportional. However, when the field equations are of the form

$$i\,\frac{\partial\psi_\sigma}{\partial t} = -\,\mu\nabla^2\psi_\sigma + G(\psi_\tau^*\psi_\tau)\,\psi_\sigma \tag{2.12}$$

where μ is a positive constant, the functionals P_i and V_i are proportional. Indeed, (2.12) is the family of scalar and multi-component nonlinear Schrödinger equations. One verifies this by inserting the Hamiltonian density for equations (2.12)

$$\mathcal{H} = \mu\nabla\,\psi_\sigma^*\cdot\nabla\psi_\sigma + \mathcal{G}(\psi_\sigma^*\psi_\sigma)$$

where $G(\rho) = d\mathcal{G}(\rho)/d\rho$, into the velocity functional (2.6) to obtain $P_i = (N/2\mu)\,V_i$. Hence the mass of a spatially localized field, obeying a NLS equation, is $m = N/2\mu$.

A fundamental property of the total linear momentum of any system is that it is constant if the Hamiltonian does not explicitely depend on the coordinates. To show that this is true for the linear momentum functional (2.10) when ψ is a spatially localized solution of any CHE equation of the form (1.1) we can either directly calculate the total time-derivative of (2.9) or we can use the following property of the energy-momentum tensor $\mathcal{T}_{\mu\nu}$:

$$\frac{d\,\mathcal{T}_{i0}}{dt} + \frac{d\,\mathcal{T}_{ij}}{dx_j} = -\,\frac{\partial\mathcal{L}}{\partial x_i}\,, \qquad i,j = 1,2,3$$

where the tensor's entries $\mathcal{T}_{i0}$ are given by (2.7) and $\mathcal{T}_{ij}$ by

$$\mathcal{T}_{ij} = \frac{\partial\mathcal{L}}{\partial(\partial_j\psi_\sigma)}\,\frac{\partial\psi_\sigma}{\partial x_i} + \frac{\partial\mathcal{L}}{\partial(\partial_j\psi_\sigma^*)}\,\frac{\partial\psi_\sigma^*}{\partial x_i} - \mathcal{L}\delta_{ij}\ .$$

In either case the result is

$$\frac{d\mathcal{P}_i}{dt} + \frac{d}{dx_j}\left(\frac{\partial\mathcal{H}}{\partial(\partial_j\psi_\sigma)}\,\frac{\partial\psi_\sigma}{\partial x_i} + \frac{\partial\mathcal{H}}{\partial(\partial_j\psi_\sigma^*)}\,\frac{\partial\psi_\sigma^*}{\partial x_i} + \mathcal{L}\delta_{ij}\right) = -\,\frac{\partial\mathcal{H}}{\partial x_i}\ . \tag{2.13}$$

This becomes a *conservation law* if $\partial\mathcal{H}/\partial x_i = 0$, that is, when the Hamiltonian density $\mathcal{H}$ is translation invariant along the x_i-coordinate. A consequence of the continuity law (2.13) is the equation

$$\frac{dP_i}{dt} = -\int_{I\!R^3}\frac{\partial\mathcal{H}}{\partial x_i}\,d^3x \tag{2.14}$$

which is obtained by integrating (2.13) over all space. From it follows that, indeed, the linear momentum component P_i of a localized field is constant if $\partial\mathcal{H}/\partial x_i = 0$. Relation (2.14) is very general since it holds on any spatially localized solution of any CHE equation (1.1). Its counterpart in quantum mechanics is Ehrenfest theorem (second part)

$$\frac{d\bar{P}_i}{dt} = -\int \psi_\sigma^* \, \frac{\partial \hat{H}}{\partial x_i} \, \psi_\sigma \, d^3x$$

written in our notation, where $\bar{P}_i$ is the linear momentum expectation value and $\hat{H}$ is the Hamiltonian operator of the quantum system (see Messiah [16]).

Relation (2.14) will play a key role in the next section.

3. Complex spatially localized fields interacting with given electromagnetic fields

As it is well known, quantum mechanics accounts for the interaction of a point-like particle with given electric and magnetic fields by replacing the space and time derivatives, in the wave-function equations, with covariant derivatives according to

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + ig\Phi \ , \qquad \frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_i} - igA_i \tag{3.1}$$

where Φ and $\mathbf{A} = (A_1, A_2, A_3)$ are the scalar and vector potentials of the electromagnetic field, $g = q/\hbar$ and q is the particle's charge.

Here, we define the interaction of a spatially localized field, obeying a CHE equation of the form (1.1), by making the same substitutions (3.1) in its equation except for the value of the constant g which, initially, is left unspecified. In this section we will show that the main consequence of this definition is: *If the Hamiltonian density H is gauge type I invariant and the electromagnetic field intensities do not exceed certain limits, then the center of the localization region moves as a classical point charge (with mass m) would move in the same electromagnetic field.*

First, we need to discuss the existence of localized solutions of CHE equations which have been modified by the substitutions (3.1). If a CHE equation possesses localized solutions, after the substitution (3.1) it may not have such solutions. However, if the potentials Φ and A_i are constant in time and space, at least within the localization region, then the localized solutions will certainly exist. This can be shown as follows: The modified CHE equation is

$$i\,\frac{\partial\psi_\sigma}{\partial t} - g\Phi\,\psi_\sigma = \frac{\delta}{\delta\psi^*_\sigma}\int \mathcal{H}(\psi^*, \psi,\, \partial\psi^* + igA\psi^*,\, \partial\psi - igA\psi)\, d^3x \quad (3.2)$$

where $\mathcal{H}(\psi^*, \psi, \partial\psi^*, \partial\psi)$ is the Hamiltonian density for the original equation, which is assumed to be gauge type I invariant (see (2.5)). If we transform ψ according to

$$\psi'_\sigma = \psi_\sigma\, \exp\big(-\, ig(\Phi t + A_i x_i)\big) \quad (3.3)$$

(not a gauge I transformation) we see that ψ' satisfies the original equation. Then, according to (3.3) we have $\psi'^*_\sigma\,\psi'_\sigma = \psi^*_\sigma\,\psi_\sigma$ from which follows that the modified equation (3.2) has localized solutions if the original equation has such. Next, *continuity* arguments tell us that localized solutions will exist even when the potentials Φ and A_i are not constant provided the latter vary sufficiently slow within the region of localization.

Now, we are ready to derive the main result of this paper. If the original equation is CHE with Hamiltonian functional $H = \int\mathcal{H}(\psi^*, \psi, \partial\psi^*, \partial\psi)\, d^3x$ then the field equation (3.2) is also CHE whose Hamiltonian functional is

$$H' = \int \mathcal{H}'\, d^3x = \int \Big(\mathcal{H}(\psi^*, \psi,\, \partial\psi^* + igA\psi^*, \partial\psi - igA\psi) + g\,\Phi\,\psi^*_\sigma\psi_\sigma\Big)\, d^3x\,. \quad (3.4)$$

When the substitution (3.1) is applied to the momentum functional (2.10) we have

$$\Pi_i = \frac{1}{i}\int \psi^*_\sigma \left(\frac{\partial\psi_\sigma}{\partial x_i} - igA_i\,\psi_\sigma\right) d^3x = P_i - g\int A_i\,\psi^*_\sigma\psi_\sigma\, d^3x \quad (3.5)$$

where P_i, given by (2.10), is the total linear momentum. The functional Π_i will be called the "mechanical" linear momentum, i-component. The relation (3.5) is in complete correspondence with a similar relation in the classical mechanics of a point mass/charge (see Goldstein [17]). To find the total time-derivative of Π_i we use the relation (2.14)

$$\frac{dP_i}{dt} = -\int \frac{\partial\mathcal{H}'}{\partial x_i}\, d^3x$$

with the Hamiltonian density $\mathcal{H}'$ defined by (3.4) and the definition (3.5) as follows:

$$\frac{d\Pi_i}{dt} = -\int \frac{\partial \mathcal{H}}{\partial x_i}\, d^3x - g\int \frac{\partial \Phi}{\partial x_i}\, \psi^*_\sigma \psi_\sigma\, d^3x - g\,\frac{d}{dt}\int A_i\, \psi^*_\sigma\psi_\sigma\, d^3x\ . \quad (3.6)$$

Taking in account the form of $\mathcal{H}$, as shown in (3.4), the first term in (3.6) becomes

$$\begin{aligned}\int \frac{\partial \mathcal{H}}{\partial x_i}\, d^3x &= -\,ig\int \frac{\partial A_j}{\partial x_i}\Big(\psi_\sigma\, \frac{\partial \mathcal{H}}{\partial(\partial_j\psi_\sigma)} - \psi^*_\sigma\, \frac{\partial \mathcal{H}}{\partial(\partial_j\psi^*_\sigma)}\Big) d^3x \\ &= -\,gN\int \frac{\partial A_j}{\partial x_i}\, \mathcal{V}_j\, d^3x\end{aligned} \quad (3.7)$$

where

$$\mathcal{V}_j = \frac{i}{N}\Big(\psi_\sigma\, \frac{\partial \mathcal{H}}{\partial(\partial_j\psi_\sigma)} - \psi^*_\sigma\, \frac{\partial \mathcal{H}}{\partial(\partial_j\psi^*_\sigma)}\Big) \quad (3.8)$$

is the integrant of the velocity functional according to (2.6). The third term in (3.6) is calculated using the CHE equations for ψ

$$\begin{aligned}&\frac{d}{dt}\int A_i\, \psi^*_\sigma\psi_\sigma\, d^3x = \int \frac{\partial A_i}{\partial t}\, \psi^*_\sigma\psi_\sigma\, d^3x \\ &+ i\int A_i\Big(\Big(\frac{\partial \mathcal{H}}{\partial \psi_\sigma} - \frac{d}{dx_j}\frac{\partial \mathcal{H}}{\partial(\partial_j\psi_\sigma)}\Big)\psi_\sigma - \psi^*_\sigma\Big(\frac{\partial \mathcal{H}}{\partial \psi^*_\sigma} - \frac{d}{dx_j}\frac{\partial \mathcal{H}}{\partial(\partial_j\psi^*_\sigma)}\Big)\Big) d^3x\ .\end{aligned}$$

After integration by parts and regrouping terms the above becomes

$$\begin{aligned}&\frac{d}{dt}\int A_i\, \psi^*_\sigma\psi_\sigma\, d^3x \\ &= \int \frac{\partial A_i}{\partial t}\, \psi^*_\sigma\psi_\sigma\, d^3x + i\int \frac{\partial A_i}{\partial x_j}\Big(\psi_\sigma\, \frac{\partial \mathcal{H}}{\partial(\partial_j\psi_\sigma)} - \psi^*_\sigma\, \frac{\partial \mathcal{H}}{\partial(\partial_j\psi^*_\sigma)}\Big) d^3x \\ &+ i\int A_i\Big(\psi_\sigma\, \frac{\partial \mathcal{H}}{\partial \psi_\sigma} - \psi^*_\sigma\, \frac{\partial \mathcal{H}}{\partial \psi^*_\sigma} + \partial_j\psi_\sigma\, \frac{\partial \mathcal{H}}{\partial(\partial_j\psi_\sigma)} - \partial_j\psi^*_\sigma\, \frac{\partial \mathcal{H}}{\partial(\partial_j\psi^*_\sigma)}\Big) d^3x \\ &= \int \frac{\partial A_i}{\partial t}\, \psi^*_\sigma\psi_\sigma\, d^3x + N\int \frac{\partial A_i}{\partial x_j}\, \mathcal{V}_j\, d^3x\end{aligned} \quad (3.9)$$

where the integrant proportional to A_i vanishes because $\mathcal{H}$ is gauge type I invariant and hence (2.4) holds. Inserting (3.7) and (3.9) into (3.6) produces the desired result

$$\frac{d\Pi_i}{dt} = -\,g\int\Big(\frac{\partial \Phi}{\partial x_i} + \frac{\partial A_i}{\partial t}\Big)\psi^*_\sigma\psi_\sigma\, d^3x + gN\int\Big(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\Big)\mathcal{V}_j\, d^3x\ . \quad (3.10)$$

The second integrant can be written, using vector notation, as

$$\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\mathcal{V}_j = \big(\mathcal{V}\times(\nabla\times\mathbf{A})\big)_i$$

where $\mathbf{A} = (A_1, A_2, A_3)$, $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3)$ denote vectors and $(\mathcal{V}\times(\nabla\times\mathbf{A}))_i$ stands for the i-component of $\mathcal{V}\times(\nabla\times\mathbf{A})$. Finally, recognizing that the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are given in terms of the potentials as

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}\ , \qquad \mathbf{B} = \nabla\times\mathbf{A} \tag{3.11}$$

equation (3.10), written as a vector equation, becomes

$$\frac{d\mathbf{\Pi}}{dt} = g\int\big(\mathbf{E}\,\psi^*_\sigma\psi_\sigma + N\mathcal{V}\times\mathbf{B}\big)\,d^3x\ . \tag{3.12}$$

where $\mathbf{\Pi} = (\Pi_1, \Pi_2, \Pi_3)$ is the vector-valued functional of the "mechanical" momentum. It should be observed that the above equation is very general since it holds for all spatially localized fields obeying all CHE equations of the form (3.2). This equation becomes particularly simple, when the electromagnetic fields change sufficiently slow, so that they can be assumed to be constant within the region of localization. Then we obtain from (3.12) the equation which describes the motion of a spatially localized field as a whole entity

$$\frac{d\mathbf{\Pi}}{dt} = g\mathbf{E}\int\psi^*_\sigma\psi_\sigma\,d^3x - gN\mathbf{B}\times\int\mathcal{V}\,d^3x = gN\big(\mathbf{E}+\mathbf{V}\times\mathbf{B}\big)\ . \tag{3.13}$$

where $\mathbf{V} = \int\mathcal{V}\,d^3x$ is the velocity vector of the localization region. Equation (3.13) is exactly the same as the equation which describes either the classical or the relativistic motion of a point mass with charge $q = gN$ in the electromagnetic field $\mathbf{E}, \mathbf{B}$ and with $\mathbf{\Pi}$ being the corresponding "mechanical" momentum of this charge (see Goldstein [17] or Jackson [18]).

The above requirement that $\mathbf{E}$ and $\mathbf{B}$ change sufficiently slow may be superfluous, since the potentials Φ and A_i were already assumed to change sufficiently slow in order to assure the existence of localized fields (see Section 3).

4. Concluding discussion

While deriving the main results, i.e., that relations (2.14) and (3.12) hold for spatially localized fields which are solutions of a large family of CHE equations, we obtained another result which has not been discussed so far. This will be done now.

In Section 2 the mass m of a localized field was defined as the ratio of the value of the linear momentum functional to that of the velocity functional, i.e. by (2.11). Then, for all NLS equations with the form (2.12) we get that the mass is $m = N/2\mu$ from which we find

$$\mu = \frac{N}{2m} \ . \tag{4.1}$$

Comparing this with the value $\mu = \hbar/2m$ of the same constant in Schrödinger's equation we see that the field's norm N appears in place of Planck's constant $\hbar$.

The same correspondence is found by comparing equation (3.13) with the equation for the motion of a point-charge in an electromagnetic field. Accordingly, we have to set $q = gN$, where q is the charge of the localized field and hence

$$g = \frac{q}{N} \ . \tag{4.2}$$

However, when the substitutions (3.1) are used in quantum mechanics $g = q/\hbar$. Thus, again we find the norm N in place of $\hbar$. It is not likely that this correspondence is a result of some peculiar coincidence because the arguments leading to (4.1) and to (4.2) are not related. Furthermore, the same correspondence is obtained in Bodurov [12, 13] from entirely different arguments.

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