Remarks on momentum and energy flux of a non-radiating electromagnetic field

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ABSTRACT. This paper inspects more closely the problem of the momentum of a non-radiating electromagnetic (EM) field. It has been shown for a number of particular physical problems that the customarily defined momentum $P_{EM} = \varepsilon_0 \int_V \left(\vec{E} \times \vec{B}\right) dV$ for a non-radiating EM field does not, in general, represent the total momentum of such a field ($\vec{E}, \vec{B}$ are the electric and magnetic fields, respectively, and V denotes all space). Namely, for the non-radiating EM field, always attached to a system of particles with the charges q_i (sources of field), the momentum $\vec{P}_A = \sum_i q_i \vec{A}_i$ ($\vec{A}$ is the vector potential) should be also added. Then a transformation of mechanical to EM momentum and vice versa for a closed non-radiating system occurs in accordance with the requirement $\vec{P}_G$=const, where $\vec{P}_G = \vec{P}_M + \sum_i q_i \vec{A}_i$ is the generalized (canonical) momentum of the system ($\vec{P}_M$ is the mechanical momentum). Some physical inferences from the obtained results are discussed.

1 Introduction

It is well known that local validity of the energy conservation law requires the equality of the partial time derivative of electromagnetic (EM) energy in some spatial volume *V*, $\frac{\partial}{\partial t}\int_V u dV$, to the energy flux across the boundary of that volume and a transmission of energy to matter. Here

$$u = \varepsilon_0 \frac{\vec{E}^2}{2} + \varepsilon_0 c^2 \frac{\vec{B}^2}{2} \tag{1}$$

is the energy density of the EM field. One sees from Eq. (1) that

$$\frac{\partial u}{\partial t} = \varepsilon_0 \vec{E} \frac{\partial \vec{E}}{\partial t} + \varepsilon_0 c^2 \vec{B} \frac{\partial \vec{B}}{\partial t}. \tag{2}$$

Considering Eq. (2), Poynting proposed to use the Maxwell equations to evaluate the field partial time derivatives [1]:

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}, \tag{3}$$

$$\frac{\partial \vec{E}}{\partial t} = c^2 \left(\nabla \times \vec{B}\right) - \frac{\vec{j}}{\varepsilon_0}, \tag{4}$$

Then the substitution of Eqs. (3), (4) into Eq. (2) leads to the familiar equation

$$\frac{\partial u}{\partial t} + \nabla \vec{S} + \vec{E}\vec{j} = 0 \tag{5}$$

($\vec{j}$ is the current density), where

$$\vec{S} = \varepsilon_0 c^2 \left(\vec{E} \times \vec{B}\right) \tag{6}$$

is the Poynting vector, which defines the energy flux density of EM field.

Applying Eqs. (5), (6) to an EM radiation, one can see that the direction of $\vec{S}$ coincides with that of EM wave propagation, and the term $\vec{j}\vec{E}$ corresponds to an absorption of EM radiation by charged particles. The same Eqs. (5) and (6) are also applied to a non-radiating EM field, and according to a general theorem of classical mechanics, a momentum density $\vec{p}$ for both EM radiation and non-radiating EM field is defined as

$$\vec{p}_{EM} = \vec{S}/c^2 = \varepsilon_0\left(\vec{E} \times \vec{B}\right). \tag{7}$$

Then the total momentum of a non-radiating EM field is computed by integration of (7) over all space V:

$$\vec{P}_{EM} = \varepsilon_0 \int_V \left(\vec{E} \times \vec{B}\right) dV \ . \tag{8}$$

We should mention that Bessonov in a number of his papers (see, e.g. [2]) showed that the energy balance equation (5) meets a number of physical difficulties, when the point-like charged particles are involved. The problem becomes worse when the self-forces of electromagnetic fields of particles are also taken into account. However, an analysis of these problems and their resolution in [2] fall outside the scope of the present paper, because further we focus our attention on mutual transformation of an electromagnetic and mechanical momentum, but not on derivation of global law of conservation of energy.

It is customarily assumed that transformation of the momentum (8) into mechanical momentum of charged particles leads to a violation of Newton's third law in EM interactions. In the present paper we will show that such a physical interpretation of the momentum (8) is erroneous. Section 2 validates this assertion with a number of particular physical problems, dealing with transformation of EM momentum into mechanical momentum of a closed system of non-radiating charged particles. Section 3 shows that a transformation of mechanical to EM momentum and vice versa for a closed non-radiating system occurs in accordance with the requirement $\vec{P}_G$=const, where $\vec{P}_G = \vec{P}_M + \sum_i q_i \vec{A}_i$. This result fully explains the seeming paradoxes found in section 2. In this connection Section 4 analyses a possible physical meaning of the momentum $\vec{P}_{EM} = \varepsilon_0 \int\limits_V \left(\vec{E} \times \vec{B}\right) dV$ for a non-radiating EM field and a related problem of energy flux in such fields. Finally, Section 5 presents some conclusions.

2 The momentum of the non-radiating electromagnetic field: physical examples

It is well-known that the momentum of EM field is responsible for violation of Newton's third law in electromagnetic interaction. In this section we will consider the examples, which show that the customarily defined momentum (8), in general, does not define the total momentum of a non-radiating EM field. The opposite (and widespread) assertion leads to a number of paradoxes, to be considered below.

2.1. Current loop + charged particle

Consider the experiment depicted in Fig. 1. There is a conducting loop and a

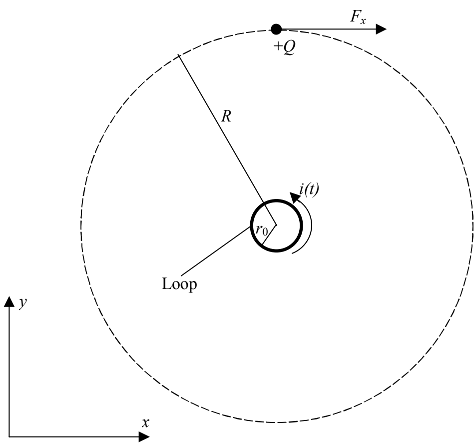

Fig. 1. Charged particle Q and the loop with the current $i(t)$.

particle with the charge Q, lying in the plane xy. Both the loop and particle rest in a laboratory. The axis of loop passes across the point x, y=0, and at the initial instant the charge has coordinates $\{0, R\}$. The radius of the loop is r_0<<R. Let initially the current in the loop be equal to zero. Then loop is connected to a battery, which produces a circulatory current $i(t)$, slowly increasing to its maximum stationary value I, so that the EM radiation is negligible. During the increase of current, the magnetic field also increases with time. It induces an azimuthal electric field along the circumference R, which can be found from the Maxwell equation and the Stokes theorem applied to the circumference with the radius R and area $\xi=\pi R^2$:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\text{ , or}$$

$$\int_\xi \left(\nabla \times \vec{E}\right) d\vec{\xi} = -\frac{d}{dt}\int_\xi \vec{B} d\vec{\xi}\text{ , or}$$

$$\int_\Gamma \vec{E} d\vec{l} = -\frac{d}{dt}\int_\xi \vec{B} d\vec{\xi} = -\frac{d}{dt}\int_\xi B_z d\xi\ .$$

Further, writing an explicit dependence of the electric and magnetic fields on space and time coordinates, we forbid an "action-at-a-distance"[1] and take into account a retardation effect, according to which the changing with time current $i(t)$ of loop produces the EM field at the instant ($t'=t-r/c$), where r is the distance between the designated point of loop and point of observation [6]. In the limit $R>>r_0$ we can simply write for the modulus of electric field:

$$E\left(t - R/c , R\right) = \frac{1}{2\pi R}\frac{d}{dt}\int_\xi B_z\left(t', \vec{r}\right) d\xi\ . \tag{9}$$

The resultant force acting on the charged particle is

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[1] However we have to note that many authors lately believe that "action-at-a-distance" is not forbidden by classical electrodynamics (see *e.g.* [3-5], and papers in the book mentioned in [3]).

$$F_x(t-R/c,R)=QE(t-R/c,R)=\frac{Q}{2\pi R}\frac{d}{dt}\int_{\xi}B_z(t',\vec{r})\,d\xi\,. \tag{10}$$

This force drives the particle along the axis *x*, and its momentum at the instant *t-R/c* is

$$P_{Mx}(t-R/c)=\int_0^{t'}F_x(t',R)\,dt'=\frac{Q}{2\pi R}\frac{d}{dt}\int_{\xi}B_z(t',\vec{r})\,d\xi=QA(t-R/c,R),\tag{11}$$

where

$$A(t-R/c,R)=\frac{1}{2\pi R}\int_{\xi}B_z(t',\vec{r})\,d\xi \tag{12}$$

is the value of the vector potential, directed counter-clockwise along the circumference *R*. Eq. (11) means that the mechanical momentum of particle begins to change after the time *R*/*c* since appearance of the current in loop, which is in full agreement with relativistic conceptions.

Further, we notice that an implementation of the law of conservation of total momentum for a closed system

$$\vec{P}_M(t)+\vec{P}_{EM}(t)=\text{const} \tag{13}$$

implies an identical time dependence of the mechanical $\vec{P}_M$ and electromagnetic $\vec{P}_{EM}$ momenta. (It means that in the absence of an action-at-a-distance, the momentum conservation law (13) should act locally). The expression for EM momentum is defined by the relationship

$$\vec{P}_{EM}=\varepsilon_0\int_V\left(\vec{E}(\vec{r})\times\vec{B}(t',\vec{r})\right)dV \tag{14}$$

in accordance with Eq. (8). It is essential that the electric field $\vec{E}(\vec{r})$ of resting (for $t<R/c$) charged particle is stationary and does not depend on time. In addition, in the limit $R>>r_0$ the retardation of spread of magnetic field near the loop ($r\approx r_0$) is negligible. Hence, the EM momentum (14), or at least its appreciable part, appears practically immediately after the appearance of current in loop, while the mechanical momentum of particle is equal to zero till the moment $t=R/c$ (see, Eq. (11)). Therefore, for the time $t<R/c$ we get a violation of the momentum conservation law (13), if we adopt the integral in Eq. (14) as the definition of total momentum of EM field.

2.2. Elongated solenoid and two oppositely charged particles

Next consider the experiment depicted in Fig. 2. There is a tall solenoid (S)

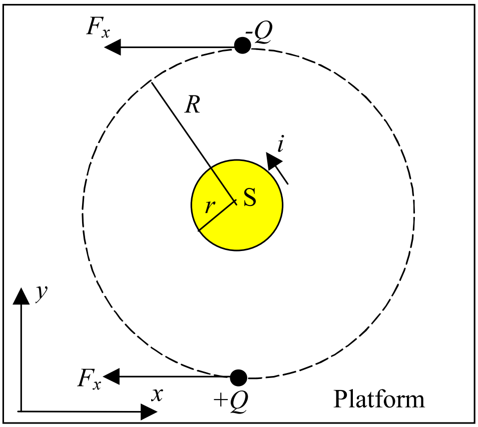

Fig. 2. Two charged particles $+Q$ and $-Q$ near the elongated solenoid S.

and two charged particles with opposite charges $+Q$ and $-Q$, fixed upon a circumference co-axial with solenoid, as shown. The radius of cross-section of the solenoid is r, the distance of the particles from the axis of solenoid is R. The length of solenoid is sufficient to allow neglect of the magnetic field outside it. The solenoid and both particles are mechanically fixed upon a platform P, which is free to move in the xy plane without friction. Let initially the current in solenoid be equal to zero. Then the solenoid is connected to a battery, which produces a current $i(t)$. We assume that the current increases from 0 to its maximum value I, and it changes slowly enough to allow neglect of any radiative processes. (In fact, this problem represents a modification of the Shockley-James paradox [7]). During this process the magnetic field inside the solenoid also increases from zero to some maximum stationary value B_z. Therefore, increase of magnetic field in solenoid induces an azimuthal electric field

$$2\pi RE(t) = -\frac{d}{dt}\int_{\pi R^2} B_z(t)\,d\xi = -\frac{d}{dt}\int_{\pi r^2} B_z(t)\,d\xi = -\frac{dB_z(t)}{dt}\pi r^2,$$

$$E(t) = -\frac{dB_z(t)}{dt}\frac{r^2}{2R},$$

and the total force acting on the charged particles is

$$F_x(t) = 2QE(t) = -Q\frac{dB_z(t)}{dt}\frac{r^2}{R}. \quad (15)$$

Designating the total time of increase of current in solenoid as T, we obtain a total mechanical momentum, acquired by the platform P along the axis x during this time:

$$P_{Mx} = \int_0^T F_x(t)\,dt = -\frac{QB_z r^2}{R} = -2QA(R), \quad (16)$$

where $A(R)=B_z r^2/2R$ is the value of vector potential of solenoid acting at the circumference of radius R.

If we now decrease the current I to zero in the reverse order, the force in Eq. (15) changes its sign, and the mechanical momentum of the platform decreases its value from P_{Mx} to 0. So, when we return an electric system (solenoid + charged particles) to its initial state, the platform P also returns to the state $\vec{P}_M$ =0. This result reflects the law of conservation of total momentum, where the sum of mechanical momentum of P and the momentum of the electromagnetic field maintains a constant value.

However, the momentum of EM field, in its conventional definition (8), is close to zero for the problem considered. Indeed, for a long enough solenoid, the magnetic field in its rest frame exists only in the inner volume of the solenoid. The electric field of the two charged particles penetrates inside the conducting solenoid only in its face regions, and a penetrating depth is comparable with the diameter of solenoid. For a very long solenoid, when its faces are very far from the charges, the electric field of these charges is negligible, because it falls as r^{-2}. The electric field exists outside the solenoid, where the magnetic field goes to zero with increase of the length of solenoid. Hence, we find that the EM momentum given by Eq. (8) tends to zero for the system of solenoid + charged particles with increase of the length of solenoid. On the other hand, the mechanical momentum is defined by Eq. (16), which does not depend on the length of a tall solenoid. As a result, we again get a contradiction with the momentum conservation law.

The problems considered in sub-sections 2.1 and 2.2 demonstrate incorrectness of a widespread opinion that the EM momentum of a non-radiating EM field, when defined via the Poynting vector, is transformed into mechanical momentum of charged particles. Instead, as we will show in the next section, a mutual transformation of mechanical and EM momenta occurs in accordance with the requirement of constancy of generalized (canonical) momentum for a closed system of charged particles.

3 About a mutual transformation of the electromagnetic and mechanical momenta

Let us consider the interaction of two non-radiating charged particles q_1 and q_2, moving at the velocities $\vec{v}_1$ and $\vec{v}_2$ at t=0. One wants to determine the change with time of the total mechanical momentum of this closed system.

It is known that the Lagrangian for a particle q_1 with the proper mass m_1 in the EM field of particle q_2 is

$$L_1 = -m_1 c^2 \sqrt{1 - v_1^{\,2}/c^2} - q_1 \varphi_{12} + q_1 \left(\vec{v}_1 \vec{A}_{12}\right), \tag{17}$$

where $\varphi_{12}, \vec{A}_{12}$ are the scalar and vector potentials of the particle q_2 at the location of particle q_1. Then the motional equation of the particle q_1 is

$$\frac{d}{dt}\frac{\partial L_1}{\partial \vec{v}_1} = \frac{\partial L_1}{\partial \vec{r}_1}\text{, or}$$

$$\frac{d\vec{p}_{M1}}{dt} + q_1 \frac{d\vec{A}_{12}}{dt} = -q_1 \frac{\partial \varphi_{12}}{\partial \vec{r}_1} + q_1 \frac{\partial}{\partial \vec{r}_1}\left(\vec{v}_1 \vec{A}_{12}\right), \tag{18}$$

where $\vec{r}_1$ is the spatial co-ordinate of particle q_1, and $\vec{p}_{M1} = m_1 \vec{v}_1 \Big/ \sqrt{1 - v_1^{\,2}/c^2}$ is its mechanical momentum. In a similar way we write the Lagrangian for the particle q_2 with the mass m_2 in the field of the first particle:

$$L_2 = -m_2 c^2 \sqrt{1 - v_2{}^2/c^2} - q_2 \varphi_{21} + q_2 \left(\vec{v}_2 \vec{A}_{21}\right), \tag{19}$$

where $\varphi_{21}, \vec{A}_{21}$ are the scalar and vector potentials of particle q_1 at the location of particle q_2. The motional equation is

$$\frac{d\vec{p}_{M2}}{dt} + q_2 \frac{d\vec{A}_{21}}{dt} = -q_2 \frac{\partial \varphi_{21}}{\partial \vec{r}_2} + q_2 \frac{\partial}{\partial \vec{r}_2}\left(\vec{v}_2 \vec{A}_{21}\right). \tag{20}$$

Summing up Eqs. (18) and (20), we obtain

$$\frac{d\left(\vec{p}_{M1} + \vec{p}_{M2}\right)}{dt} + q_1 \frac{d\left(\vec{A}_{12} + \vec{A}_{21}\right)}{dt} = -q_1 \frac{\partial \varphi_{12}}{\partial \vec{r}_1} - q_2 \frac{\partial \varphi_{21}}{\partial \vec{r}_2} + q_1 \frac{\partial}{\partial \vec{r}_1}\left(\vec{v}_1 \vec{A}_{12}\right) + q_2 \frac{\partial}{\partial \vec{r}_2}\left(\vec{v}_2 \vec{A}_{21}\right). \tag{21}$$

The scalar and vector potentials produced by the particle q_2 at the location of particle q_1, and vice versa, are (to the accuracy of the order c^{-2}, [8]):

$$\varphi_{12} = \frac{q_2}{4\pi\varepsilon_0 r_{12}},\ \vec{A}_{12} = \frac{q_2\left[\vec{v}_2 + \left(\vec{v}_2 \vec{n}_2\right)\vec{n}_2\right]}{8\pi\varepsilon_0 c^2 r_{12}},\ \varphi_{21} = \frac{q_1}{4\pi\varepsilon_0 r_{21}},\ \vec{A}_{21} = \frac{q_1\left[\vec{v}_1 + \left(\vec{v}_1 \vec{n}_1\right)\vec{n}_1\right]}{8\pi\varepsilon_0 c^2 r_{21}}, \tag{22}$$

where $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$, $\vec{r}_{21} = \vec{r}_1 - \vec{r}_2$, $\vec{n}_2$ is the unit vector at the direction from q_2 to q_1, and $\vec{n}_1$ is the unit vector from q_1 to q_2 ($\vec{n}_2 = -\vec{n}_1$). Substituting the values of scalar and vector potentials from Eqs. (22) into Eq. (21), one gets:

$$\frac{d\left(\vec{p}_{M1} + \vec{p}_{M2}\right)}{dt} + q_1 \frac{d\left(\vec{A}_{12} + \vec{A}_{21}\right)}{dt} = -\frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\partial}{\partial \vec{r}_1}\left(\frac{1}{r_{12}}\right) - \frac{q_2 q_1}{4\pi\varepsilon_0} \frac{\partial}{\partial \vec{r}_2}\left(\frac{1}{r_{21}}\right) +$$

$$+ \frac{q_1 q_2 \left(\vec{v}_1 \vec{v}_2\right)}{8\pi\varepsilon_0 c^2} \frac{\partial}{\partial \vec{r}_1}\left(\frac{1}{r_{12}}\right) + \frac{q_2 q_1 \left(\vec{v}_2 \vec{v}_1\right)}{8\pi\varepsilon_0 c^2} \frac{\partial}{\partial \vec{r}_2}\left(\frac{1}{r_{21}}\right) +$$

$$+\frac{q_1 q_2}{8\pi\varepsilon_0 c^2}\frac{\partial}{\partial \vec{r}_1}\left(\frac{(\vec{v}_2\vec{n}_2)(\vec{v}_1\vec{n}_2)}{r_{12}}\right)+\frac{q_2 q_1}{8\pi\varepsilon_0 c^2}\frac{\partial}{\partial \vec{r}_2}\left(\frac{(\vec{v}_1\vec{n}_1)(\vec{v}_2\vec{n}_1)}{r_{21}}\right). \qquad (23)$$

Taking into account the equalities:

$$r_{12}=r_{21}\,,\ \vec{r}_{12}=-\vec{r}_{21}\,,\ \vec{n}_2=-\vec{n}_1\,, \qquad (24)$$

we get the equations

$$\frac{\partial}{\partial \vec{r}_1}\left(\frac{1}{r_{12}}\right)=-\frac{\partial}{\partial \vec{r}_2}\left(\frac{1}{r_{21}}\right),\ (\vec{v}_2\vec{n}_2)(\vec{v}_1\vec{n}_2)=(\vec{v}_1\vec{n}_1)(\vec{v}_2\vec{n}_1),$$

$$\frac{\partial}{\partial \vec{r}_1}(\vec{v}_2\vec{n}_2)(\vec{v}_1\vec{n}_2)=-\frac{\partial}{\partial \vec{r}_2}(\vec{v}_1\vec{n}_1)(\vec{v}_2\vec{n}_1). \qquad (25)$$

The obtained Eqs. (25) allow to derive that *rhs* of Eq.(23) is equal to zero, and

$$\frac{d(\vec{p}_{M1}+\vec{p}_{M2})}{dt}+q_1\frac{d(\vec{A}_{12}+\vec{A}_{21})}{dt}=0\,.$$

We can rewrite this equation as

$$\frac{d\vec{P}_G}{dt}=\frac{d\vec{P}_M}{dt}+\frac{d\vec{P}_A}{dt}=0\,, \qquad (26)$$

where $\vec{P}_G=\vec{P}_M+\vec{P}_A$ is the generalized momentum, $\vec{P}_M=\vec{p}_{1M}+\vec{p}_{1M}$ is the total mechanical momentum for the closed system of two particles, and $\vec{P}_A=q_1\vec{A}_{12}+q_2\vec{A}_{21}$. In the adopted approximation Eq. (26) is extended to the case of arbitrary number *i* of charged particles due to the principle of superposition:

$$\frac{d}{dt}\left(\sum_i \vec{P}_{Mi} + \sum_i q_i \vec{A}_i\right) = 0 \text{ , or}$$

$$\frac{d}{dt}\sum_i \vec{P}_{Mi} = -\frac{d}{dt}\sum_i q_i \vec{A}_i \ . \qquad (27)$$

Eq. (27) shows that the total time derivative of resultant mechanical momentum (total mechanical force, acting on the closed non-radiating system of charged particles due to violation of Newton's third law for EM interaction) is equal with the opposite sign to the total time derivative of "momentum" $\vec{P}_A = \sum_i q_i \vec{A}_i$. Hence, under change of non-radiating EM fields in the points of location of moving non-radiating particles, just the momentum $\vec{P}_A$ transforms to the mechanical momentum of the non-radiating system, but not the momentum $\vec{P}_{EM}$, defined by Eq. (8). Here we mention that the present proof of theorem (27) is not unique. Another way to prove the theorem is described in [9]. In addition, we notice that the Lagrangian (17), being used in our theorem, describes an interaction of charged particle with the external electromagnetic field, and does not include inertial reaction (the equivalent mass of the field) and the radiation reaction. The latter effect is taken negligible by supposition (the accelerations of particle are small). The remarks about the mass of field are given below in section 4.

Eq. (27) naturally resolves the paradoxes in subsections 2.1-2.2. For the problem in Fig. 1 (sub-section 2.1) the change of mechanical momentum of particle (Eq. (11)) is defined by the change of vector potential $\vec{A}$ at the point of charge location. Such a local character of the momentum conservation law excludes any "action-at-a-distance" and provides an identical dependence of $\vec{P}_M(t)$ and $\vec{P}_{EM}(t)$ on time in Eq. (13).

For the problem in Fig. 2 (sub-section 2.2) the mechanical momentum P_{Mx} is equal with the reverse sign to the sum of "momenta" QA_x of both particles (Eq. (16)). The results just obtained allow us to deduce a number of physical consequences:

1. We notice that the momentum $\vec{P}_A$ of a considered system is not associated with an energy flux across the boundary of that system. This is explained by the fact that energy flux in empty space is a property of the EM field solely,

whereas the momentum $\vec{P}_A$ belongs to the whole system "EM field + its sources". At the same time this means a qualitative difference between the conventional momentum of a particle and the momentum $\vec{P}_A$. It is known that the mechanical momentum represents the components of the energy four-vector $\{E, \vec{P}\}$. One can easily see that $\vec{P}_A$ forms a four-vector with the potential electric energy of charged particles $\{U_e, \vec{P}_A\}$, where $U_e = \sum_i q_i\varphi_i$, φ being the electric potential. For this reason we propose to name $\vec{P}_A$ as "potential" momentum. The term of "potential momentum" was also introduced in [10].

2. Eq. (27) loses its physical meaning in the case of EM radiation, when the sources of the EM field, in general, may be absent in an arbitrary space volume. Hence, for that (source-free) kind of EM fields, the momentum density is defined by the conventional expression through the Poynting vector $\varepsilon_0\left(\vec{E}\times\vec{B}\right)$. This shows that non-radiating EM field and EM radiation represent two different physical entities. It simultaneously means that there should be a physical mechanism, allowing to distinguish the non-radiating EM field and EM radiation in their mixture. In the author's opinion, such a physical mechanism is based on the fact that EM radiation is absorbed, at least in principle, by a charged particle, while a non-radiating field is not (the change of kinetic energy of particle is equal to change of its electric potential energy with the reverse sign).

Further, it is necessary to stress that for the system of N charged particles, the momentum of a single particle $\vec{p}_{iA} = q_i\vec{A}_i$ cannot be attributed to its proper total momentum; rather it represents a contribution of the particle i to the total potential momentum of the whole system; it is only $\vec{P}_A = \sum_i q_i\vec{A}_i$, which has a physical meaning. Even in case, where occasionally the total potential momentum of a system under consideration coincides with the value $\vec{p}_{iA} = q_i\vec{A}_i$ for a single charged particle i, the total time derivative $-\frac{d\vec{p}_{iA}}{dt}$ is not equal to the force, acting on the particle i, but it defines the force, acting on the whole system (the charged particle i+ the sources of the field $\vec{A}_i$). Consider, for example, a motion of charged particle inside a mechanically free elongated solenoid. Let at the initial time moment the velocity of the particle $\vec{v}$ lie in the plane *xy*, while the magnetic field of solenoid $\vec{B}$ lies in the negative *z*-direction. The Lorentz force, acting on the particle, is

$$\frac{d\vec{p}_M}{dt} = q\left(\vec{v}\times\vec{B}\right) = q\vec{v}\times\left(\nabla\times\vec{A}\right) = q\nabla\left(\vec{v}\vec{A}\right) - q\left(\vec{v}\nabla\right)\vec{A}.$$

For stationary current in the solenoid, $\frac{\partial\vec{A}}{\partial t} = 0$, and $\frac{d\vec{A}}{dt} = \left(\vec{v}\nabla\right)\vec{A}$. Then, taking $\vec{p}_A = q\vec{A}$, we get

$$\frac{d\vec{p}_M}{dt} = -\frac{d\vec{p}_A}{dt} + q\nabla\left(\vec{v}\vec{A}\right). \tag{28}$$

We see that the mechanical force (the total time derivative of the momentum of particle $\vec{p}_M$) is not equal to $-\frac{d\vec{p}_A}{dt}$, but includes the term $q\nabla\left(\vec{v}\vec{A}\right)$. However, it still does not contradict Eq. (27), because we did not include the mechanical momentum of solenoid $\vec{p}_{MS}$ and did not consider the force, acting on the solenoid on behalf of particle. One can show that this force is equal to $-q\nabla\left(\vec{v}\vec{A}\right)$ (see, Appendix A), and

$$\frac{d\vec{p}_{MS}}{dt} = -q\nabla\left(\vec{v}\vec{A}\right). \tag{29}$$

Summing up Eqs. (28), (29), we obtain

$$\frac{d\vec{p}_M}{dt} + \frac{d\vec{p}_{MS}}{dt} = -\frac{d\vec{p}_A}{dt},$$

in accordance with Eq. (27).

Let us consider another example: a charged particle *q* orbits around a tall solenoid S at the constant angular frequency ω (Appendix B, Fig. 4). In this problem the net force, acting on the particle, is equal to zero, while its "momentum" $\vec{p}_A = q\vec{A}$ changes with time. Moreover, this value defines the potential momentum of the whole system "charged particle + solenoid". Then one follows from Eq. (27) that the total time derivative $-\frac{d\vec{p}_A}{dt}$ should be equal to the force, acting on the solenoid on behalf of particle. This result is confirmed by the particular calculations, presented in Appendix B. Simultaneously one can see that this exercise creates a problem with respect to the energy conservation law: the particle can rotate around the solenoid infinitely long (it we negate by its radiation), while the solenoid receives a force which can make work. It seems that the problems of such a kind are not related to Eq. (27); they are common for classical EM theory. In particular, ref. [11] also reveals the problem with formulation of the energy conservation law, considering a system of three charged particles on the basis of Darwin Lagrangian [12].

We also have to stress that Eq. (27) is valid for inertial reference frames only, although a motion of particles under observation can be arbitrary (with small accelerations, allowing neglect of the EM radiation). It means, in particular, that a consideration of interaction between two particles from a non-inertial reference frame, attached to one of them, does not give Eq. (27) and leads to a seeming paradox with the momentum conservation law [13].

The revealed physical meaning of the "potential" momentum was masked in familiar textbooks, which began a consideration of electrodynamics from a motion of charged particle in some abstract external EM field. By such a way it is impossible to find that the total time derivative of the momentum $\vec{p}_A = q\vec{A}$ for a given particle contains a part of force, acting on the sources of this external field.

At the same time, we cannot omit a question about the possible physical meaning of the momentum $\varepsilon_0\left(\vec{E}\times\vec{B}\right)$ for the non-radiating EM field, which is considered in next section.

4 About an energy flux in a non-radiating EM field and the momentum density $\vec{p}_{EM} = \varepsilon_0\left(\vec{E}\times\vec{B}\right)$.

We already mentioned above that the Poynting expression for energy flux density $\vec{S} = \varepsilon_0 c^2\left(\vec{E}\times\vec{B}\right)$ is traditionally applied to both EM radiation and non-radiative EM fields, although for the latter case such energy fluxes were never detected experimentally. There is another problem with the definition $\vec{S} = \varepsilon_0 c^2\left(\vec{E}\times\vec{B}\right)$ for a non-radiating EM field, which is revealed through its application to a single charged particle, moving at the constant velocity $\vec{v}$ in a laboratory frame.

In such a case the term $\vec{j}\vec{E}$ in Eq. (5) describes a self-action of the non-radiating inertially moving particle with its own electromagnetic field. Standard renormalization procedure implies that this term should be dropped. However, a simple cancellation of the term $\vec{j}\vec{E}$ leads to another physical difficulty. Namely, in the rest frame of a charged particle we can write (*du*/*dt*)=0. In the laboratory frame this equality transforms to

$$\frac{\partial u}{\partial t}+(\vec{v}\nabla)u=0\text{, or}$$

$$\frac{\partial u}{\partial t}+\nabla(\vec{v}u)=0\,. \qquad (30)$$

Now let us show that Eqs. (5) and (30) are mathematically equivalent to each other, if we take into account that for the EM field of a charged particle

$$\vec{B}=\frac{\vec{v}\times\vec{E}}{c^2}\,. \qquad (31)$$

Indeed,

$$\begin{aligned}\nabla\vec{S} &= \varepsilon_0 c^2\nabla\left(\vec{E}\times\vec{B}\right)=\varepsilon_0 c^2\left[\vec{B}\left(\nabla\times\vec{E}\right)-\vec{E}\left(\nabla\times\vec{B}\right)\right]=\\ &=-\varepsilon_0 c^2\left[\vec{B}\frac{\partial\vec{B}}{\partial t}+\frac{\vec{E}\left(\nabla\times\left(\vec{v}\times\vec{E}\right)\right)}{c^2}\right]=\\ &=-\varepsilon_0 c^2\left[\vec{B}\frac{\left(\vec{v}\times\partial\vec{E}/\partial t\right)}{c^2}+\frac{\vec{E}\vec{v}\left(\nabla\vec{E}\right)}{c^2}-\frac{\vec{E}\left(\vec{v}\nabla\right)\vec{E}}{c^2}\right]=\\ &=\varepsilon_0\vec{B}\left(\vec{v}\times\partial\vec{E}/\partial t\right)-\vec{E}\vec{j}+\varepsilon_0\vec{E}\left(\vec{v}\nabla\right)\vec{E}\end{aligned}\,. \qquad (32)$$

Here we used the vector identity $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\vec{b}\left(\vec{a}\vec{c}\right)-\vec{c}\left(\vec{a}\vec{b}\right)$, as well as the equality

$$\vec{v}\left(\nabla\vec{E}\right)=\vec{v}\frac{\rho}{\varepsilon_0}=\frac{\vec{j}}{\varepsilon_0},$$

where   is the charge density. Further, using Eq. (4), we can write

$$\vec{B}\left(\vec{v}\times\partial\vec{E}/\partial t\right)=\vec{B}\left[\vec{v}\times\left(c^2\left(\nabla\times\vec{B}\right)-\frac{\vec{j}}{\varepsilon_0}\right)\right]=-c^2\vec{B}\left(\vec{v}\nabla\right)\vec{B} \tag{33}$$

(under the transformation of Eq. (33) we again use the identity $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\vec{b}\left(\vec{a}\vec{c}\right)-\vec{c}\left(\vec{a}\vec{b}\right)$, and take into account that the vectors $\vec{v}$ and $\vec{B}$ are orthogonal to each other, so that $\vec{v}\vec{B}=0$. From Eqs. (33) and (32) we derive

$$\nabla\vec{S}=\varepsilon_0\vec{E}\left(\vec{v}\nabla\right)\vec{E}+\varepsilon_0c^2\vec{B}\left(\vec{v}\nabla\right)\vec{B}-\vec{E}\vec{j}\,. \tag{34}$$

Substituting $\nabla\vec{S}$ from Eq. (34) into Eq. (5), we obtain:

$$\frac{\partial u}{\partial t}+\varepsilon_0\vec{E}\left(\vec{v}\nabla\right)\vec{E}+\varepsilon_0c^2\vec{B}\left(\vec{v}\nabla\right)\vec{B}=0\,. \tag{35}$$

In turn, one can see that

$$\varepsilon_0\vec{E}\left(\vec{v}\nabla\right)\vec{E}+\varepsilon_0c^2\vec{B}\left(\vec{v}\nabla\right)\vec{B}=\nabla\left[\vec{v}\left(\frac{\varepsilon_0E^2}{2}+\frac{\varepsilon_0c^2B^2}{2}\right)\right]=\nabla\left(\vec{v}u\right),$$

and Eq. (35) transforms to Eq. (30). We can rewrite Eq. (30) as

$$\frac{\partial u}{\partial t}+\nabla\vec{S}_U=0\,, \tag{36}$$

where

$$\vec{S}_U=\vec{v}u \tag{37}$$

is known as Umov's vector [14].

Thus, for a charged particle, moving at a constant velocity $\vec{v}$, we derived two mathematically equivalent forms of the energy balance equations:

$$\frac{\partial u}{\partial t}+\nabla\vec{S}+\vec{j}\vec{E}=0,$$

$$\frac{\partial u}{\partial t}+\nabla\vec{S}_U=0.$$

One sees from there that we cannot omit the term $\vec{j}\vec{E}$ in Eq. (5), because by such a way we destroy an equivalence of Eqs. (5) and (30). From this point of view Eq. (30) seems more attractive, because it does not contain this term of self-action. Physically this means that the electromagnetic field moves uniformly in space together with its charged particle source at the velocity $\vec{v}$. On the other hand, the EM field "knows" only two velocities: 0 and *c*. Hence, a representation of the EM field, moving at $v\neq c$ cannot describe a real physical situation. As a result, we reveal that Eqs. (5) and (30), being equivalent mathematically, are both physically unsatisfactory in description of the energy flux of a single charged particle. In these conditions, one can propose the following method of overcoming this difficulty: both expressions for the energy flux density, $\vec{S}$ and $\vec{S}_U$ are relevant, but the first of them describes "differential energy fluxes" $\vec{S}$, propagating at the velocity *c*, which compose an "integral energy flux" $\vec{S}_U$, propagating with an effective velocity equal to the velocity of the source particle $\vec{v}$. The idea to distinguish the "differential" and "integral" energy fluxes in a non-radiative EM field was already discussed in scientific literature [15], and an equivalence of Eqs. (5), (30) makes it more attractive. In addition, there is another attractive feature of this idea, if we assume that the experimental observations in the non-radiating EM fields deal with the integral energy fluxes. Then the momentum density of the EM field, produced by a single charged particle moving at the constant velocity $\vec{v}$, is

$$\vec{p}_{EM}=\frac{\vec{S}_u}{c^2}=\vec{v}\frac{u}{c^2}=m_{EM}\vec{v} \qquad (38)$$

where m_{EM} is the density of EM mass. One sees that the definition (38), expressed via the Umov vector, provides the equality

$$u = m_{EM}c^2 \tag{39}$$

for the energy and mass densities in accordance with the familiar Einstein's expression. Integrating Eq. (39) over the while space, we get

$$U = M_{EM}c^2$$

(U is the total EM energy, M_{EM} is the total EM mass). Thus, the expression of energy flux density through the vector of Umov formally eliminates the familiar problem "4/3" [16] for a moving electron.

Now consider a closed system of charged particles with the momentary velocities $\vec{v}_l$ at the considered instant (l=1...N, N being the number of particles). We assume that there are no external mechanical forces, and the accelerations of particles are small enough to neglect their EM radiation. Generalizing the calculations (31)-(35) to this system, we again obtain Eq. (36), where

$$\vec{S}_U = \varepsilon_0 \sum_l \vec{v}_l \frac{\left(\vec{E}_\Sigma \vec{E}_l\right)}{2} + \varepsilon_0 c^2 \sum_l \vec{v}_l \frac{\left(\vec{B}_\Sigma \vec{B}_l\right)}{2}. \tag{40}$$

Here $\vec{E}_\Sigma = \sum_l \vec{E}_l$, $\vec{B}_\Sigma = \sum_l \vec{B}_l$ are the resultant electric and magnetic fields created by the charged particles. Thus, the "integral" energy flux density of a system of charged particles is described by the obtained Eq. (40). If all particles move uniformly at the constant velocity $\vec{v}$ at the considered instant, Eq. (40) transforms to

$$\vec{S}_U = \vec{v}\varepsilon_0 \left(\frac{\vec{E}_\Sigma^{\,2}}{2} + \frac{\vec{B}_\Sigma^{\,2}}{2} \right).$$

Here the resultant fields again move uniformly with the system of charged source particles.

At the same time, we always have to remember that a real EM field cannot move at the velocity 0<v<c, such a motion is relevant for an "integral" energy flux, which is composed from "differential" fluxes, propagating at the speed of light.

Next we analyze the tensor for EM energy for the EM radiation and the non-radiative EM field. It is known that the motional equation for an EM field with the Lagrangian density $-\frac{\varepsilon_0}{4}F_{ik}F^{ik}$ (F^{ik} is the tensor of EM field) gives [8]

$$T^{ik} = -\varepsilon_0 \frac{\partial A^l}{\partial x^l} F_l^k + \frac{\varepsilon_0}{4} g^{ik} F_{lm} F^{lm} . \tag{41}$$

where A^l is the four-potential, and **g** is the metric tensor. A physically meaningful tensor of EM energy should be symmetrical. Using the gauge arbitrariness in its choice,

$$T^{ik} \to T^{ik} + \frac{\partial \psi^{ikl}}{\partial x^l} \text{ (where } \psi^{ikl} = -\psi^{ilk} \text{)}, \tag{42}$$

the tensor (41) can be transformed to the symmetric form

$$T^{ik} = \varepsilon_0 \left(-F^{il} F_l^k + \frac{1}{4} g^{ik} F_{lm} F^{lm} \right). \tag{43}$$

Eq. (43) represents the conventional expression for the tensor of EM field. However, one should stress that the transformation of Eq. (41) into Eq. (43) uses the equality [8]

$$\frac{\partial F_i^k}{\partial x_i} = 0 , \tag{44}$$

which represents the Maxwell equation for a source-free EM field, i.e., EM radiation. Therefore, the tensor (43) is applicable for the EM radiation. Thus, for a non-radiating EM field we have to apply another transformation of Eq. (41) to symmetric form than was made for EM radiation. One of methods to solve this problem is to test different functions ψ^{ikl} in (42) to obtain a physically meaningful result. We can avoid such a complex way, proceeding from Eq. (40) for the integral energy flux density in a non-radiative EM field, which gives the components $T_n^{0\alpha} = T_n^{\alpha 0}$ (=1...3, the subscript *n* signifies a non-radiating EM field). To simplify our analysis, we further consider free of charge space volume, where

$$\frac{\partial T^{ik}}{\partial x^i} = 0 \,. \tag{45}$$

Then, one can find that a form of the tensor of EM energy, satisfying both conditions (40) and (45), is

$$T_n^{ik} = \sum_l \frac{dx^i{}_{(l)}}{dt}\frac{dx^k{}_{(l)}}{dt}\left[\frac{\varepsilon_0\left(\vec{E}_\Sigma, \vec{E}_l\right)}{2} + \frac{\varepsilon_0 c^2\left(\vec{B}_\Sigma, \vec{B}_l\right)}{2}\right], \tag{46}$$

if we assume that the velocities of all charged particles have constant values.

At the same time, it is clear that Eq. (46) cannot determine the tensor of EM energy of a non-radiating EM field in the general case, where the particles move at velocities variable with time. Indeed, in this case a fraction of EM energy is dissipated as EM radiation, and the equality (45) is violated. Simple addition of the tensor of EM radiation (43) to *lhs* of Eq. (46) does not recover the equality (45) under variable velocities. This difficulty reflects a known fact that Umov's vector can be defined only for constant velocities of sources of EM field. In these conditions, perhaps, we have to recognize that a physically valid expression for the tensor of EM energy is given by Eq. (43), which describes the EM radiation and "differential" fluxes in the non-radiative EM field. Then a conception about an "integral" energy flux loses its physical meaning, remaining convenient for mathematics. In such a case the above-mentioned problem "4/3" is resolved by the conventional way (via introducing the "Poincaré stresses"). Perhaps, a tensor of EM energy for a mixture of EM radiation and non-radiating EM field, involving the "integral" energy fluxes, can be found at least in principle. Maybe, one of possible ways to solve this problem is to analyse the global energy tensor, including EM field and a matter, and to apply the Bessonov's method to derive the energy conservation law [2]. In case of success, it would allow consideration of the "integral" fluxes as physically real. Another way is the re-definition of energy density, proposed by Chubykalo [17] in the discussion about zero-energy solutions of Maxwell's equations [18]. However, as was mentioned in [19], in this case the energy of EM field, integrated over the whole space, should be infinite. Further, recent experimental observations by Chubykalo et al. [20] allow to express some doubts in the validity of conventionally accepted expression for the energy density: the Umov vector, being constructed with such an energy density, does not describe an electromagnetic energy flux. Another hypothesis, which could be related with conceptions of "integral" and "differential" energy fluxes, is about a massive photon [21]. Recent derivation of the generalized Maxwell equations by Dvoeglazov [22] in his comment on the paper of Gersten [23] allows to generalize the equations for such the photons. Experimental aspects to test this hypothesis were analyzed in [24]. One can mention that the most recent developments of technique of nuclear resonant radiation experiments (in particular, the proposed back-scattering mirrors for X-ray and Mössbauer radiation [25], which can be used in Michelson-Morley type experiments with gamma-resonant radiation) open a way for experimental test of the hypothesis on massive photon.

However, even without resolution of this complex problems, we can conclude that both "differential" and "integral" energy fluxes in a non-radiating EM field are attached to their sources, and the momentum of such a field, defined via the energy flux, represents an intrinsic attribute of the charged particles, like their EM mass. This result allows one to make two conclusions:

-the momentum $\vec{p}_{EM} = \varepsilon_0 \left(\vec{E} \times \vec{B} \right)$, associated with "differential" energy fluxes in a non-radiating EM field, can be transformed to a mechanical momentum only under direct interactions of charged particles, and such a transformation is not responsible for the violation of Newton's third law in EM interaction. The violation occurs under variation of EM fields alone in the points of location of particles, when we apply Eq. (27)). Remember, that Eq. (27), obtained via the Lagrangian (17), includes a mechanical mass of the particles. Hence, it is naturally to propose that the momentum $\vec{p}_{EM} = \varepsilon_0 \left(\vec{E} \times \vec{B} \right)$ describes the electromagnetic momentum of charged particles, associated with their EM masses;

-the vector product of $\vec{E}$ and $\vec{B}$, when the electric and magnetic fields are taken from different non-radiating sources, has no physical meaning, and it does not define the energy flux and momentum density of the non-radiating EM field.

Looking at Figs. 1-2, we see that in the problems of subsections 2.1-2.2 we tried to calculate the momentum of the EM field from the magnetic field of one source (conductive loop (2.1) and tall solenoid (2.2)) and the electric field from another source (charged particles). That is why it is not surprising that we obtained a seeming violation of the momentum conservation law. The second conclusion, above, also resolves a number of paradoxes, dealing with strange pictures of the energy fluxes in non-radiating EM fields [1], and explains the failure to experimentally detect energy fluxes of such a kind.

5 Conclusions

1. A transformation of the momentum of a non-radiating EM field into mechanical momentum, which is responsible for violation of Newton's third law in electromagnetic interactions, occurs in accordance with Eq. (27), where $\vec{P}_A = \sum q_i \vec{A}_i$ is termed "potential" momentum of a system of charged particles.

2. The momentum density $\vec{p}_{EM} = \varepsilon_0 \left(\vec{E} \times \vec{B}\right)$, associated with the "differential" energy flux in a non-radiating EM field (here the vectors $\vec{E}$ and $\vec{B}$ are taken from the same source of EM field), represents an intrinsic attribute of the source charged particle, like its EM mass. It can be transformed into mechanical momentum only under direct interaction (e.g., collisions) of the charged particles.

3. The "momentum density" $\vec{p}_{EM} = \varepsilon_0 \left(\vec{E} \times \vec{B}\right)$, where the fields $\vec{E}$ and $\vec{B}$ are taken from different sources of EM field, has no physical meaning, and it does not define the momentum of the non-radiating EM field.

Acknowledgements

The author warmly thanks Thomas E. Phipps for helpful discussions.

Appendix A. Calculation of the force, acting on a solenoid due to a charged particle, moving inside the solenoid

Let at the initial instant t=0 the charged particle q moves inside a tall solenoid at the velocity $\vec{v}$, lying in the xy-plane (Fig. 3). The radius of the solenoid is equal to r, the distance between the particle and the axis of solenoid is $R<r$ at t=0. The axis of solenoid is collinear to the axis z. One requires to determine the force, experienced by the solenoid, carrying the current i.

In the non-relativistic limit a moving charged particle creates the magnetic field

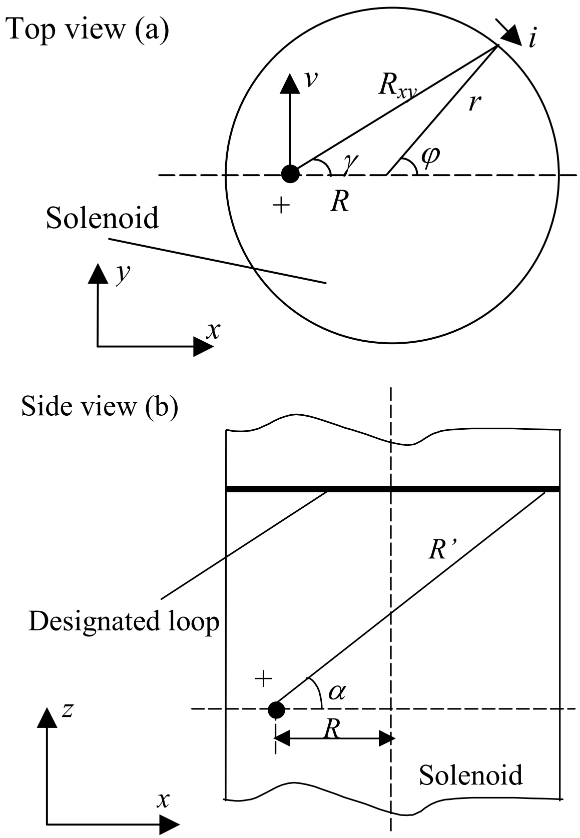

Fig. 3. The charged particle $+q$ moves inside the solenoid

$$\vec{B} = \frac{q\vec{v}\times\vec{n}}{4\pi\varepsilon_0 c^2 R'^2}, \tag{A1}$$

where $\vec{n}$ is the unit vector, joining the point-like charge and designated space point R'. This field induces a magnetic force, acting on each element dl of solenoid with the current $\vec{i}$:

$$d\vec{F} = \left(\vec{i} \times \vec{B}\right) dl \ . \tag{A2}$$

Without a lose of generality, we can choose the axis x to be orthogonal to $\vec{v}$ at t=0. Then the component B_y=0 in Eq. (A1). Also taking into account, that i_z=0 in solenoid, we obtain the components of force for a single loop of the solenoid as

$$dF_{lx} = i_y B_z dl = -iB_z r \cos\varphi d\varphi \ , \tag{A3}$$

$$dF_{ly} = -i_x B_z dl = -iB_z r \sin\varphi d\varphi \ , \tag{A4}$$

$$dF_{lz} = -i_y B_x dl = iB_x r \cos\varphi d\varphi \ . \tag{A5}$$

where φ is the circumferential angle (Fig. 3, a). Firstly, let us calculate the component of total force along the axis x. Eq. (A1) gives:

$$B_z = -\frac{qvn_x}{4\pi\varepsilon_0 c^2 R'^2} \ . \tag{A6}$$

Substituting Eq. (A6) into Eq. (A3), we obtain

$$dF_{lx} = \frac{qvn_x ir \cos\varphi d\varphi}{4\pi\varepsilon_0 c^2 R'^2} \ . \tag{A7}$$

One can see from Fig. 3, that

$$R'^2 = R_{xy}{}^2 + z^2 \ , \qquad R_{xy}{}^2 = R^2 + 2Rr\cos\varphi + r^2 \ , \qquad \cos\alpha = \frac{R_{xy}}{R'} \ ,$$

$$\cos\gamma = \frac{R + r\cos\varphi}{R_{xy}} \ , \ n_x = \cos\alpha\cos\gamma = \frac{R + r\cos\varphi}{\sqrt{R^2 + 2Rr\cos\varphi + r^2 + z^2}} \ . \tag{A8}$$

Substituting the values of (A8) into (A7), one gets:

$$dF_x = \frac{qvir\left(R + r\cos\varphi\right)\cos\varphi d\varphi}{4\pi\varepsilon_0 c^2\left(R^2 + 2Rr\cos\varphi + r^2 + z^2\right)^{3/2}} . \qquad \text{(A9)}$$

From there the force, acting on a single loop of solenoid along the axis *x*, is

$$dF_{lx} = \frac{qvir}{4\pi\varepsilon_0 c^2}\int_0^{2\pi}\frac{\left(R + r\cos\varphi\right)\cos\varphi d\varphi}{\left(R^2 + 2Rr\cos\varphi + r^2 + z^2\right)^{3/2}} = \frac{qvi}{4\pi\varepsilon_0 c^2}\int_0^{2\pi}\frac{\left(\frac{R}{r} + \cos\varphi\right)\cos\varphi d\varphi}{\left(1 + \frac{2R}{r}\cos\varphi + \frac{R^2}{r^2} + \frac{z^2}{r^2}\right)^{3/2}} .$$

The fragment of solenoid with the length *dz* contains *ndz* loops. Hence, the force, acting on this fragment is

$$dF_{lx} = \frac{qvindz}{4\pi\varepsilon_0 c^2}\int_0^{2\pi}\frac{\left(\frac{R}{r} + \cos\varphi\right)\cos\varphi d\varphi}{\left(1 + \frac{2R}{r}\cos\varphi + \frac{R^2}{r^2} + \frac{z^2}{r^2}\right)^{3/2}} .$$

From there the total force along the axis *x*, acting on the solenoid due to the moving particle, is

$$F_x = \frac{qvin}{4\pi\varepsilon_0 c^2}\int_{-\infty}^{\infty}\int_0^{2\pi}\frac{\left(\frac{R}{r} + \cos\varphi\right)\cos\varphi d\varphi dz}{\left(1 + \frac{2R}{r}\cos\varphi + \frac{R^2}{r^2} + \frac{z^2}{r^2}\right)^{3/2}} .$$

Taking into account that $\frac{in}{\varepsilon_0 c^2} = B$, we obtain

$$F_x = \frac{qvB}{4\pi}\int_{-\infty}^{\infty}\int_0^{2\pi}\frac{\left(\frac{R}{r} + \cos\varphi\right)\cos\varphi d\varphi dz}{\left(1 + \frac{2R}{r}\cos\varphi + \frac{R^2}{r^2} + \frac{z^2}{r^2}\right)^{3/2}} .$$

Integration over z gives:

$$F_x = \frac{qvB}{4\pi}\int_0^{2\pi}\left(\frac{R}{r}+\cos\varphi\right)\cos\varphi d\varphi\int_{-\infty}^{\infty}\frac{dz}{\left(1+\frac{2R}{r}\cos\varphi+\frac{R^2}{r^2}+\frac{z^2}{r^2}\right)^{3/2}} =$$

$$= \frac{qvB}{2\pi}\int_0^{2\pi}\frac{\left(\frac{R}{r}+\cos\varphi\right)}{\left(1+\frac{2R}{r}\cos\varphi+\frac{R^2}{r^2}\right)}\cos\varphi d\varphi\,. \qquad \text{(A10)}$$

The remaining integral over φ is equal to

$$\int_{-\infty}^{\infty}\frac{\left(1+\frac{r}{R}\cos\varphi\right)}{1+\frac{2r}{R}\cos\varphi+\frac{r^2}{R^2}}\cos\varphi d\varphi = \pi\,. \qquad \text{(A11)}$$

Substituting Eq. (A11) into Eq. (A10), we obtain

$$F_x = \frac{qvB}{2}\,. \qquad \text{(A12)}$$

Taking into account that inside the solenoid the vector potential is equal to $A = BR/2$ and circulated in the clock-wise direction, one sees that Eq. (A12) gives the same force, as

$$F_x = -\left[q\nabla\left(\vec{v}\vec{A}\right)\right]_x$$

(see, Eq. (29)). In a similar way one can show that the components F_y and F_z, computed from Eqs. (A4) and (A5), coincide with corresponding components of the force

$$\vec{F} = -q\nabla\left(\vec{v}\vec{A}\right). \qquad \text{(A13)}$$

Thus, the moving charged particle creates the force (A13), exerted on the solenoid.

Appendix B. Calculation of the force, acting on a solenoid due to a charged particle, rotating around the solenoid

Let a charged particle q rotates in the xy-plane around a tall solenoid S at the constant angular frequency ω (Fig. 4). The radius of solenoid is equal to r, the distance between the particle and axis of solenoid is $R>r$. The axis of solenoid is collinear to the axis z. Under calculation of the force, acting on the solenoid due to the charged particle, we assume that at t=0 the axis x is orthogonal to orbital velocity of particle. Then, using designations of Fig. 3, we again obtain Eq. (A9). However, now $R>r$, and we derive for a single loop

$$dF_{lx}=\frac{qvir}{4\pi\varepsilon_0 c^2}\int_0^{2\pi}\frac{(R+r\cos\varphi)\cos\varphi\, d\varphi}{\left(R^2+2Rr\cos\varphi+r^2+z^2\right)^{3/2}}=\frac{qvir}{4\pi\varepsilon_0 c^2 R}\int_0^{2\pi}\frac{\left(1+\frac{r}{R}\cos\varphi\right)\cos\varphi\, d\varphi}{\left(1+\frac{2r}{R}\cos\varphi+\frac{r^2}{R^2}+\frac{z^2}{R^2}\right)^{3/2}}.$$

The fragment of solenoid with the length dz contains ndz loops. Hence, the force, acting on the fragment with the length dz is

$$dF_x=\frac{qvirndz}{4\pi\varepsilon_0 c^2 R}\int_0^{2\pi}\frac{\left(1+\frac{r}{R}\cos\varphi\right)\cos\varphi d\varphi}{\left(1+\frac{2r}{R}\cos\varphi+\frac{r^2}{R^2}+\frac{z^2}{R^2}\right)^{3/2}}.$$

From there the total force acting on the solenoid along the axis x is

$$F_x=\frac{qvirn}{4\pi\varepsilon_0 c^2 R}\int_{-\infty}^{\infty}\int_0^{2\pi}\frac{\left(1+\frac{r}{R}\cos\varphi\right)\cos\varphi d\varphi dz}{\left(1+\frac{2r}{R}\cos\varphi+\frac{r^2}{R^2}+\frac{z^2}{R^2}\right)^{3/2}}.$$

Taking into account that $\frac{in}{\varepsilon_0 c^2}=B$, we obtain

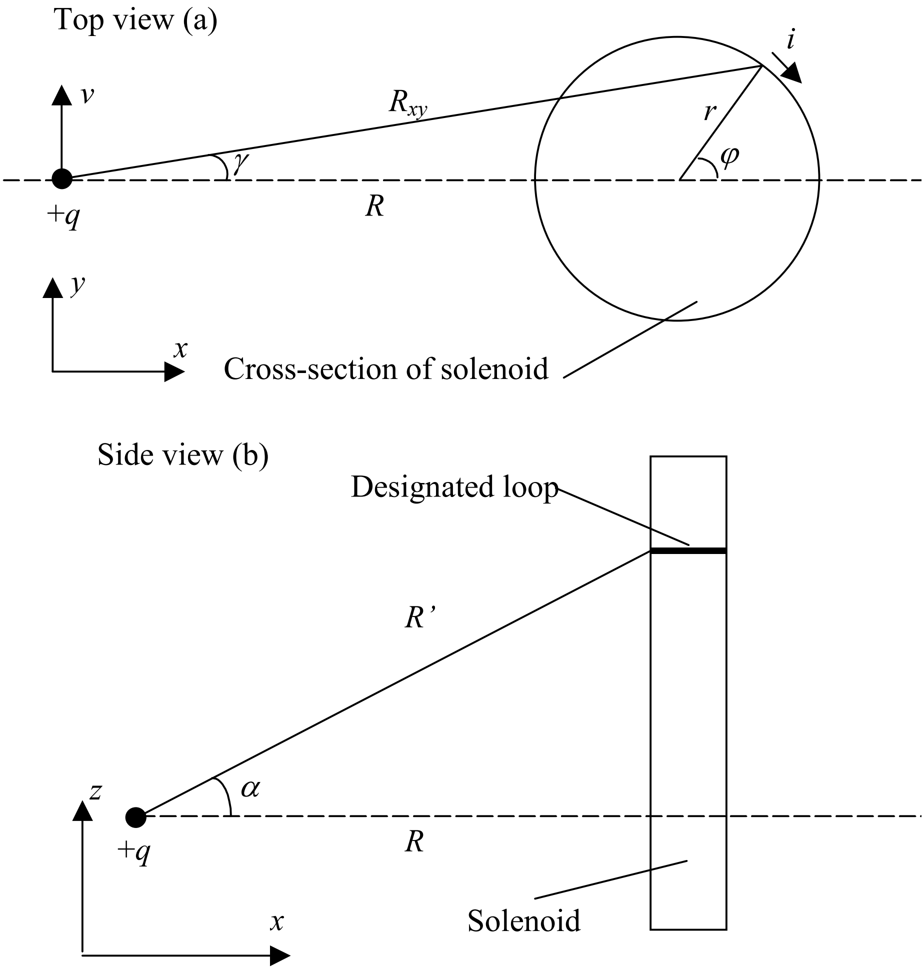

Fig. 4. The charged particle $+q$ orbits around the solenoid

$$F_x = \frac{qvrB}{4\pi R}\int_{-\infty}^{\infty}\int_{0}^{2\pi}\frac{\left(1+\frac{r}{R}\cos\varphi\right)\cos\varphi d\varphi dz}{\left(1+\frac{2r}{R}\cos\varphi+\frac{r^2}{R^2}+\frac{z^2}{R^2}\right)^{3/2}}.$$

This equation can also be expressed via the value of vector potential of solenoid A, using the equality $A = Br^2/R$ (outside the solenoid):

$$F_x = \frac{qvA}{2\pi r}\int_{-\infty}^{\infty}\int_{0}^{2\pi}\frac{\left(1+\frac{r}{R}\cos\varphi\right)\cos\varphi d\varphi dz}{\left(1+\frac{2r}{R}\cos\varphi+\frac{r^2}{R^2}+\frac{z^2}{R^2}\right)^{3/2}}. \tag{B1}$$

Integration over z gives:

$$\begin{aligned} F_x &= \frac{qvAR^2}{2\pi r}\int_{0}^{2\pi}\left(1+\frac{r}{R}\cos\varphi\right)\cos\varphi d\varphi\int_{-\infty}^{\infty}\frac{dz}{\left(R^2+2rR\cos\varphi+r^2+z^2\right)^{3/2}} = \\ &= \frac{qvAR^2}{\pi r}\int_{-\infty}^{\infty}\frac{\left(1+\frac{r}{R}\cos\varphi\right)}{R^2+2rR\cos\varphi+r^2}\cos\varphi d\varphi = \frac{qvA}{\pi r}\int_{-\infty}^{\infty}\frac{\left(1+\frac{r}{R}\cos\varphi\right)}{1+\frac{2r}{R}\cos\varphi+\frac{r^2}{R^2}}\cos\varphi d\varphi \end{aligned}. \tag{B2}$$

The remaining integral over φ is equal to

$$\int_{-\infty}^{\infty}\frac{\left(1+\frac{r}{R}\cos\varphi\right)}{1+\frac{2r}{R}\cos\varphi+\frac{r^2}{R^2}}\cos\varphi d\varphi = -\frac{\pi r}{R}. \tag{B3}$$

Substituting Eq. (B3) into Eq. (B2), we obtain

$$F_x = -\frac{qvA}{R}. \tag{B4}$$

This expression describes the momentary force from the rotating particle with the negative x-coordinate, when the axis x be orthogonal to its orbital velocity. It shows that the force is directed along the line, joining the axis of solenoid and momentary position of the rotating particle. One follows from there that the direction of the force, exerted by the particle on solenoid, rotates together with the particle at the same angular frequency ω. Hence, the projections of this force change with time as

$$F_x = \frac{qvA}{R} = q\omega A \cos \omega t \,. \tag{B5}$$

$$F_y = \frac{qvA}{R} = q\omega A \sin \omega t \,, \tag{B6}$$

and $F_z = 0$.

One can see that Eqs. (B5) and (B6), taken together, can be written in the vector form as

$$\vec{F} = -q\left(\vec{\omega} \times \vec{A}\right). \tag{B7}$$

On the other hand, for the vector field of solenoid, circulated in the clock-wise direction,

$$\frac{d\vec{A}}{dt} = \left(\vec{\omega} \times \vec{A}\right). \tag{B8}$$

Comparison of Eqs. (B7) and (B8) shows that

$$\vec{F} = -q\frac{d\vec{A}}{dt} = -\frac{d\vec{p}_A}{dt} \,. \tag{B9}$$

Thus, we have shown that the force, acting on the solenoid due to a rotating particle, is equal with the opposite sign to the total time derivative of the potential momentum $q\vec{A}$ for the system “solenoid +particle”.

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Manuscrit reçu le 22 janvier 2004, révisé le 4 Mars 2004.