Combinatorics of non-Abelian gerbes with connection and curvature

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ABSTRACT. We give a functorial definition of G-gerbes over a simplicial complex when the structure group G is non-Abelian. These combinatorial gerbes are naturally endowed with a connective structure and a curving which define a fibered category equipped with a functorial connection over the space of edge-paths. By computing the curvature of the latter on the faces of an infinitesimal 4-simplex, we recover the cocycle identities satisfied by the curvature of this gerbe. The gauge transformations are defined as natural transformations. The relation with BF-theories suggests that gerbes provide a framework adapted to the geometric formulation of strongly coupled Yang-Mills gauge theories.

Contents

1 Introduction

Fiber bundles with connection are the standard mathematical arena for the gauge theories of condensed matter and particle physics. Indeed, pointlike excitations carry information along paths in space-time and this process is well described by the operation of parallel transport. However, in a strongly coupled regime, the dynamics of non-Abelian gauge fields allows the appearance of extended objects such as loops or membranes and we need a new geometrical setting to describe the transport of information on surfaces and higher dimensional manifolds. It is known that the correct extension of parallel transport to surfaces should be defined in gerbes [9, 6, 7, 14] but the theory is still in development.

In order to develop our intuition of gerbes, we restrict our approach to a discrete setting and use a simplicial complex instead of the Čech - de Rham complex of a covering of a smooth manifold. We extract differential information by considering infinitesimal simplexes. Thus we set a simplicial complex X and a Lie group G which is our structure group. With a discrete topology, we can use any topological group, but when we consider infinitesimal simplexes, we will need a smooth structure group. $\mathfrak{g}$ will denote the Lie algebra of G and $\mathfrak{aut}\,(G)$ the Lie algebra of $\mathrm{Aut}\,(G)$. The word "bundle" will always mean "combinatorial G-bundle with connection" and "gerbes" will in fact mean "combinatorial G-gerbes with 1-connection (connective structure) and 2-connection (curving)". We will use informally infinitesimal simplexes and combinatorial differential forms [5] but without diving into non-standard analysis [18] or synthetic differential geometry [11, 12], where we can handle consistently infinitesimal quantities in a context broader than set theory. We will try to use the notations of [6] where the authors develop the geometry of gerbes in a more general setting, suitable for smooth manifolds or schemes. Our approach is adapted to triangulated spaces and lattice gauge theories, but we prove that it provides the same symmetries (gauge transformations and cocylces) when the simplexes become infinitesimal.

2 Categorical preliminaries

2.1 Basic concepts

We recall here some basic definitions [15] in order to set our notations. A category $\mathbf{C}$ is defined by two kinds of data : a class of objects, Ob ($\mathbf{C}$), and, for any pair of objects x and y, a set of arrows $\mathbf{C}_{xy}$ (or $\mathbf{C}(x,y)$). These sets are equipped with associative composition laws $\mathbf{C}_{xy} \times \mathbf{C}_{yz} \to \mathbf{C}_{xz}$ and identities $1_x \in \mathbf{C}_{xx}$, so that the sets $\mathbf{C}_{xx}$ are monoids. When Ob ($\mathbf{C}$) is a set, $\mathbf{C}$ is called a small category. For any category $\mathbf{C}$, its opposite category, $\mathbf{C}^{\mathrm{op}}$, has the same class of objects but the arrows are reversed. A groupoid is a category in which all the arrows are invertible, and the monoids $\mathbf{C}_{xx}$ are in fact groups.

A weak 2-category $\mathcal{C}$ is defined by three kinds of data : a class of objects, Ob ($\mathcal{C}$), and, for any pair of objects X and Y, a category of 1-arrows $\mathcal{C}_{XY}$. The arrows of $\mathcal{C}_{XY}$ are called the 2-arrows of $\mathcal{C}$. We have maps s_0, t_0, s_1 and t_1, called, respectively, 0-source, 0-target, 1-source and 1-target, and defined as follows. If $X, Y \in$ Ob $(\mathcal{C}), u, v \in$ Ob $(\mathcal{C}_{XY})$ and $\alpha : u \to v$, then $s_0(u) = s_0(v) \equiv X, t_0(u) = t_0(v) \equiv Y, s_1(\alpha) \equiv u$ and $t_1(\alpha) \equiv v$. The composition of these maps thus satisfy the identities $s_0 \circ s_1 = s_0 \circ t_1$ and $t_0 \circ s_1 = t_0 \circ t_1$ The composition of 1-arrows in $\mathcal{C}$ is simply their concatenation, and denoted by $\times$. The 2-arrows can be composed either horizontally, if the 0-target of the first one coincides with the 0-source of the second one, or vertically, if the 1-target of the first one coincides with the 1-source of the second one. When all the 1-arrows and 2-arrows of a 2-category are invertible, it is called a 2-groupoid. The invertibility of a 1-arrow $u : X \to Y$ asserts the existence of another 1-arrow $v : Y \to X$ and of invertible 2-arrows $\alpha : uv \to 1_X$ and $\beta : vu \to 1_Y$. u and v are said to be quasi-inverses of each other.

A category $\mathbf{C}$ is said monoidal when it is endowed with a bifunctor $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$, with associativity isomorphisms $a_{x,y,z} : x(yz) \to (xy)z$ satisfying the pentagon axiom

$$a_{x,y,zw}a_{xy,z,w} = (1_x \times a_{y,z,w})a_{x,yz,w}(a_{x,y,z} \times 1_w)$$

with an identity object I and with left and right identity isomorphisms

$L_x : Ix \to x$ and $R_x : xI \to x$ such that

$$a_{x,I,y}(1_x \times L_x) = (R_x \times 1_y)$$

for any $x, y \in \text{Ob}\ (\mathbf{C})$. If, moreover, for any x of $\mathbf{C}$ there exists an object x^* and invertible arrows $\varepsilon_x : xx^* \to I$ and $\eta_x : x^*x \to I$ then the monoidal category $\mathbf{C}$ is termed compact. A compact, monoidal groupoid is called a gr-category [4].

2.2 Torsors and bitorsors

Let G be a fixed Lie group. The topological spaces endowed with a left (resp. right), continuous, free and transitive action of G are called left (resp. right) G-torsors. $\ell, r, \lambda, \rho, \cdots$ will denote generically such actions. A morphism $f : (\ell, X) \to (\ell', X')$ between two left G-torsors is a continuous G-equivariant map : $\ell_g f = f \ell'_g$. These morphisms are in fact homeomorphisms so that the left (resp. right) G-torsors and their morphisms form a groupoid $\mathcal{T}_L(G)$ (resp. $\mathcal{T}_R(G)$). With $\mathcal{T}_L(G)$ and $\mathcal{T}_R(G)$, we can build a group-like category $\mathcal{B}_G$ in the following way [3]. If we want to define a product of two torsors, a one sided action of G isn't enough. However, the left action on $(\ell, Y) \in \text{Ob}\ (\mathcal{T}_L(G))$ can be contracted with the right action on $(X, r) \in \text{Ob}\ (\mathcal{T}_R(G))$ to give a space $X \times_G Y$ (contracted product) defined by

$$\begin{aligned} X \times_G Y &= X \times Y / \sim \\ (x, \ell_g(y)) &\sim (r_g(x), y) \end{aligned}$$

For $X \times_G Y$ to be in the same category as X and Y, they must all carry a left action and a right action of G. A manifold endowed with such actions which commute is called a G-bitorsor. A morphism $f : (\ell, X, r) \to (\ell', X', r')$ between two bitorsors is a continuous map which commutes with both actions on X and X'. Since $\ell_g(x)$, the image of $x \in X$ under the left action of $g \in G$, can be reached by the right action of a unique $h \in G$, the correspondance $g \to h = \varphi(g)$ defines a map $\varphi : G \to G$, which is actually an automorphism of G. Conversely, any $\varphi \in \text{Aut}\,(G)$ determines a unique bitorsor from a left G-torsor (ℓ, X), by defining the right action of $g \in G$ as follows :

$$r_g(x) = \ell_{\varphi^{-1}(g)}(x) \qquad \forall\ x \in X,\ \forall\ g \in G$$

$\mathcal{B}_G$ will denote the gr-category of G-bitorsors and $\mathcal{T}_L(\mathcal{B}_G)$ the (weak) 2-groupoid of $\mathcal{B}_G$-torsors. A $\mathcal{B}_G$-torsor is a category $\mathcal{C}$ on which $\mathcal{B}_G$ acts by equivalences, that is to say each G-bitorsor β is mapped to a functor $F_\beta : \mathcal{C} \to \mathcal{C}$ which admits a quasi-inverse. The simplest $\mathcal{B}_G$-torsors are $\mathcal{T}_L(G)$ and $\mathcal{T}_R(G)$, where the action of $(\lambda, \beta, \rho) \in \mathcal{B}_G$ on $(\ell, T) \in \mathcal{T}_L(G)$ twists ℓ to give the left G-torsor $B \times_G T \simeq (\lambda, T)$. The importance of bitorsors in non-Abelian cohomology comes from the fact that they compare different torsors and can thus act homogeneously as automorphisms on categories which are equivalent to $\mathcal{T}_L(G)$ or $\mathcal{T}_R(G)$.

3 Bundles as representations of a groupoid of paths

3.1 Combinatorial Bundles

If X is a simplicial complex, we can define the groupoid $\Pi_1(X)$ as the category whose objects are its vertices and whose arrows are its edge-paths, i.e. families $(x_0, x_1, \cdots, x_n)$ of successive neighbour vertices ($n \geq 1$). At this level, we can identify the paths which are homotopic in the 1-skeleton. In other words, if $y \neq x$ then

$$(\cdots, x, y, y, z, \cdots) \sim (\cdots, x, y, z, \cdots)$$
$$(\cdots, x, y, x, z, \cdots) \sim (\cdots, x, z, \cdots)$$

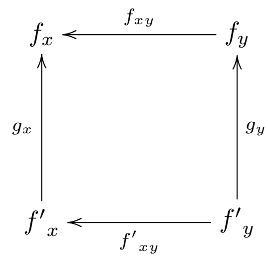
The insertion or deletion of identical successive vertices is a discrete analogue of a local diffeomorphism and the invariance of physical quantities under backtracking is the "zigzag symmetry" invoked by A. Polyakov [16] in the search of a string theory adapted to the description of the confined phase of Yang-Mills theory. Both symmetries must be represented in any quantum theory of loops.

A left bundle with connection over X is defined as a presheaf of left G-torsors on $\Pi_1(X)$, that is to say a contravariant functor $f : \Pi_1(X) \to \mathcal{T}_L(G)$ which represents the groupoid of paths into the groupoid of left G-torsors. The fibers are the objects $f_x, f_y, f_z, \cdots$ associated to the vertices $x, y, z, \cdots$. The connection is determined by the images of the edges, $f_{xy} : f_y \to f_x$, which are morphisms of left G-torsors. Its curvature, c , is the 2-form defined by the holonomies along elementary loops :

$$c = \sum_{[xyzx]} [xyzx] c_{xyzx}$$
$$c_{xyzx} = f_{xy} f_{yz} f_{zx} : f_x \to f_x$$

Each c_{xyzx} is an automorphism of G-torsor, i.e. an element of G, and c is a G-valued 2-form, since $c_{xzyx} = c_{xyzx}^{-1}$, and $c_{xyzx} = 1 \in G$ when $[xyzx]$ is degenerate.

Two bundles f, f' are declared isomorphic when there exists an invertible natural transformation $g : f' \to f$. In components, we have an arrow $g_x : {f'}_x \to f_x$ in $\mathcal{T}_L(G)$ for each x, such that $f_{xy} g_y = g_x {f'}_{xy}$, i.e. the following square commutes



f and f' have therefore the same curvature. When $f_x = {f'}_x$, $\forall x \in X^0$, g_x and g_y give elements of G and we recover the usual expression of gauge transformations.

3.2 The Bianchi Identity

We recall here the homotopical interpretation of the Bianchi identity satisfied by the connection 1-form, A, and its curvature 2-form, F, in order to extend it to the case of gerbes. Let w, x, y and z be four pairwise neighbour points, i.e. an infinitesimal tetrahedron, with w as base point. The homotopy $(wxyw)(wyzw)(wzxw) \sim (wx)(xyzx)(xw)$, which takes place in the 1-skeleton of X, implies the multiplicative Bianchi identity

$$\boxed{c_{wxyw} c_{wyzw} c_{wzxw} = f_{wx} c_{xyzx} f_{xw}} \tag{1}$$

If we choose an origin in each fiber, f_{xy} is mapped to an element of G and, near the identity, we can write $f_{xy} = 1 + A_{xy}$, A being a $\mathfrak{g}$-valued 1-form, i.e. an antisymmetric map ($A_{yx} = -A_{xy}$) from the set of oriented edges of X to $\mathfrak{g}$. We can also write this 1-form as a linear combination of the oriented edges, $A = \sum_{(xy)}(xy)A_{xy}$. The curvature of A is the $\mathfrak{g}$-valued 2-form $F = \sum_{[xyzx]}[xyzx]F_{xyzx}$ defined by

$$\begin{aligned} F_{xyzx} =& c_{xyzx} - 1 \\ =& (1 + A_{xy})(1 + A_{yz})(1 + A_{zx}) - 1 \\ =& (A_{xy} + A_{yz} + A_{zx}) + (A_{xy}A_{yz} + A_{yz}A_{zx} + A_{xy}A_{zx}) \\ F =& dA + \frac{1}{2}[A, A] \end{aligned}$$

Expanding each member of the equation (1) to the second order, we obtain

$$\begin{aligned} c_{wxyw}c_{wyzw}c_{wzxw} =& 1 + F_{wxyw} + F_{wyzw} + F_{wzxw} \\ f_{wx}c_{xyzx}f_{xw} =& 1 + A_{wx} + A_{xw} + F_{xyzx} + A_{wx}F_{xyzx} + A_{xw}F_{xyzx} \\ =& 1 + F_{xyzx} + [A_{wx}, F_{xyzx}] \end{aligned}$$

and this implies the Bianchi identity

$$F_{xyzx} - F_{wxyw} - F_{wyzw} - F_{wzxw} + [A_{wx}, F_{xyzx}] = 0$$

usually written in terms of differential forms :

$$\boxed{dF + [A, F] = 0}$$

4 Gerbes as representations of a groupoid of surfaces

4.1 Combinatorial gerbes

In this section, we use the 2-skeleton of X to define a groupoid $\mathcal{P}(X)$ whose objects are *all* the edge-paths of X and whose arrows are generated by the homotopies along oriented 2-cells. With the concatenation of paths, noted $\times$, $\mathcal{P}(X)$ is a monoidal groupoid. Two paths with the same boundary points, say $\gamma = (x_0, x_1, \cdots, x_n)$ and $\gamma' = (x_0, x_1, \cdots, x_p, y, x_{p+1}, \cdots, x_n)$, form an edge of $\mathcal{P}(X)$ if they differ only by the insertion of a vertex y which is a common neighbour to x_p and x_{p+1}. If the vertex inserted is already present near the insertion point, i.e. if $y = x_p$ or $y = x_{p+1}$, then γ' is homotopic to γ in X^1, but we won't identify these two paths. The arrows of $\mathcal{P}(X)$ are then obtained by composition of these edges. Thus, an arrow from γ_0 to γ_n is a family of paths $(\gamma_0, \cdots, \gamma_n)$ such that, for all $i \in \{0, \cdots, n-1\}$, (γ_i, γ_{i+1}) be an edge of $\mathcal{P}(X)$. We can identify the paths of $\mathcal{P}(X)$ (also called 2-paths) with different speeds or with backtrackings as follows. If γ is a neighbour of γ_i, then

$$(\cdots, \gamma_{i-1}, \gamma_i, \gamma_i, \gamma_{i+1}, \cdots) \sim (\cdots, \gamma_{i-1}, \gamma_i, \gamma_{i+1}, \cdots)$$
$$(\cdots, \gamma_{i-1}, \gamma_i, \gamma, \gamma_i, \gamma_{i+1}, \cdots) \sim (\cdots, \gamma_{i-1}, \gamma_i, \gamma_{i+1}, \cdots)$$

The generating edges of $\mathcal{P}(X)$, which are the shortest homotopies between the shortest paths, will be written with brackets :

$$[xzy] = (xy, xzy) = \text{insertion of } z \text{ between } x \text{ and } y$$
$$[xzy]^* = (xzy, xy) = \text{deletion of } z \text{ between } x \text{ and } y$$
$$[xzyx] = (xx, xzyx) = \text{insertion of } (zy) \text{ between } x \text{ and } x$$
$$[xzyx]^* = (xzyx, xx) = \text{deletion of } (zy) \text{ between } x \text{ and } x$$

Remark : When we say that the inverse of $[xzy] = (xy, xzy)$ is $[xzy]^* = (xzy, xy)$, we mean that their vertical composition $(xy, xzy) \circ (xzy, xy)$ is the boundary of a degenerate 3-cell and not strictly equal to $(xy, xy) = 1_{(xy)}$.

We now define G-gerbes as contravariant, monoidal functors

$$\Phi : \Pi_1(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{T}_L(\mathcal{B}_G))$$

which represent the groupoid $\Pi_1(\mathcal{P}(X))$, into the groupoid $\mathcal{P}(\mathcal{T}_L(\mathcal{B}_G))$ whose objects are the paths in the 2-groupoid of left $\mathcal{B}_G$-torsors, $\mathcal{T}_L(\mathcal{B}_G)$. The choice of $\mathcal{T}_L(\mathcal{B}_G)$ as a target 2-category is a natural one, because its 2-arrows are mapped to elements of G, once we have chosen an equivalence between its objects and the gauge gr-category $\mathcal{B}_G$. This definition hides much combinatorial geometry. For example, we will explore in the following section the cocycle relations satisfied by the curvature of Φ. The fiber of Φ at x is the category Φ_x associated to the trivial edge (xx), where $\Phi_{xx} = (\Phi_x \xrightarrow{\text{Id}} \Phi_x)$. Since each edge (xy) defines 1-connection functor $\Phi_{xy} : \Phi_y \to \Phi_x$, we can define the (functor valued) curvature 2-form $C = \sum_{[xyzx]}[xyzx]C_{xyzx}$ by

$$C_{xyzx} = \Phi_{xy}\Phi_{yz}\Phi_{zx}$$

It is important not to confuse C_{xyzx}, which is given by the usual composition of functors, and the concatenation $\Phi_{xyzx} = \Phi_{xy} \times \Phi_{yz} \times \Phi_{zx}$, which is a fibered category over the path $(xyzx)$. The functor C_{xyzx}, associated to the 2-cell $[xyzx]$, is an equivalence of the $\mathcal{B}_G$-torsor Φ_x, therefore it is naturally equivalent to the functorial action of some bitorsor $\beta_{xyzx} \in \text{Ob}\,(\mathcal{B}_G)$ satisfying

$$\begin{aligned} \beta_{xyzx}\beta_{xzyx} &= 1_{\mathcal{B}_G} \\ \Phi_{xy}\beta_{yzxy} &= \beta_{xyzx}\Phi_{xy} \end{aligned}$$

We will keep the same notation for this bitorsor, its functorial action on Φ_x, and the corresponding element of $\text{Aut}\,(G)$. But Φ provides us also with natural equivalences which represent in $\mathcal{T}_L(\mathcal{B}_G)$ the corresponding elementary homotopies. More precisely, the 2-arrows

$$\begin{aligned} \Phi_{[xzy]} &: \Phi_{xz} \times \Phi_{zy} \longrightarrow \Phi_{xy} \\ \Phi_{[xzy]^*} &: \Phi_{xy} \longrightarrow \Phi_{xz} \times \Phi_{zy} \\ \Phi_{[xyzx]} &: \Phi_{xy} \times \Phi_{yz} \times \Phi_{zx} \longrightarrow \Phi_{xx} \\ \Phi_{[xyzx]^*} &: \Phi_{xx} \longrightarrow \Phi_{xy} \times \Phi_{yz} \times \Phi_{zx} \end{aligned}$$

induce natural transformations between the compositions of 1-connection functors

$$\begin{aligned} K_{[xzy]} &: \Phi_{xz}\Phi_{zy}\beta_{yzxy} \longrightarrow \Phi_{xy} \\ K_{[xzy]^*} &: \Phi_{xy} \longrightarrow \Phi_{xz}\Phi_{zy}\beta_{yzxy} \\ K_{[xzyx]} &: C_{xzyx}\beta_{xyzx} \longrightarrow 1_{\Phi_x} \\ K_{[xzyx]^*} &: 1_{\Phi_x} \longrightarrow C_{xzyx}\beta_{xyzx} \end{aligned}$$

such that, for any $a \in \mathrm{Ob}(\Phi_y)$, we have

$$\begin{aligned} K_{[\dots]^*} &= (K_{[\dots]})^{-1} \\ K_{[xzy]}(a) &= \Phi_{xy}(K_{[yxzy]}(a)) \\ &= K_{[xzyx]}(\Phi_{xy}(a)) \end{aligned}$$

Similarly, Φ induces natural transformations

$$\begin{aligned} \bar{K}_{[xzy]} &: \beta_{xyzx}\Phi_{xz}\Phi_{zy} \longrightarrow \Phi_{xy} \\ \bar{K}_{[xzy]^*} &: \Phi_{xy} \longrightarrow \beta_{xyzx}\Phi_{xz}\Phi_{zy} \\ \bar{K}_{[xzyx]} &: \beta_{xyzx}C_{xzyx} \longrightarrow 1_{\Phi_x} \\ \bar{K}_{[xzyx]^*} &: 1_{\Phi_x} \longrightarrow \beta_{xyzx}C_{xzyx} \end{aligned}$$

satisfying identical conditions. (Thus it seems possible to break the chiral symmetry by giving to the variables K and $\bar{K}$ different dynamics.) For example, $K_{[xzy]}$ represents the elementary homotopy from (xy) to (xzy) as a 2-arrow in the reversed direction (Φ being contravariant) and whose component at a is the arrow

$$K_{[xzy]}(a) : \Phi_{xz}\Phi_{zy}\beta_{yzxy}(a) \longrightarrow \Phi_{xy}(a)$$

The functoriality of Φ implies the naturality of $K_{[xzy]}$, i.e. for each arrow $u : a \to b$ in Φ_y we have a commutative diagram

$$\begin{array}{ccc} \Phi_{xz}\Phi_{zy}\beta_{yzxy}(a) & \xrightarrow{K_{[xzy]}(a)} & \Phi_{xy}(a) \\ \Big\downarrow{\scriptstyle \Phi_{xz}\Phi_{zy}\beta_{yzxy}(u)} & & \Big\downarrow{\scriptstyle \Phi_{xy}(u)} \\ \Phi_{xz}\Phi_{zy}\beta_{yzxy}(b) & \xrightarrow[K_{[xzy]}(b)]{} & \Phi_{xy}(b) \end{array}$$

i.e. the relation

$$\boxed{\Phi_{xy}(u) \circ K_{[xzy]}(a) = K_{[xzy]}(b) \circ \Phi_{xz}\Phi_{zy}\beta_{yzxy}(u)} \qquad (2)$$

We have to consider the paths with and without backtrackings as different because the degenerate 2-cell $[xyx]$ is mapped to a 2-arrow

$$\Phi_{[xyx]} : \Phi_{xy} \times \Phi_{yx} \to \Phi_{xx}$$

which is a part of the combinatorial data contained in Φ and amounts to the choice of a quasi-inverse of the 1-connection (see [5], §(7.4)). Indeed, if we choose $\beta_{xyyx} = 1_{\Phi_x}$, then $K_{[xyx]}$ is precisely the arrow we need to invert Φ_{xy}, since

$$K_{[xyx]} : \Phi_{xy}\Phi_{yx}\beta_{xyyx} = \Phi_{xy}\Phi_{yx} \to 1_{\Phi_x}$$

A local gauge fixing is obtained by choosing an object α_x in Φ_x for each vertex x of an infinitesimal tetrahedron. Then, each object a of Φ_x defines (the isomorphy class of) a bitorsor b_{xa} by

$$b_{xa}(\alpha_x) \simeq a$$

This allows us to define the $\mathfrak{aut}\,(G)$-valued 1-form $\mu = \sum_{(xy)}(xy)\mu_{xy}$ by

$$b_{y,\Phi_{xy}(\alpha_x)} = 1_{\mathrm{Aut}\,(G)} + \mu_{xy}$$

and the $\mathfrak{g}$-valued 2-form $B = \sum_{[xyzx]}[xyzx]B_{xyzx}$ by

$$K_{[xyzx]} = 1_G + B_{xyzx}$$

Moreover, the expansion of β near the identity provides the "fake curvature" [5], which is the $\mathfrak{aut}\,(G)$-valued 2-form ν defined by

$$\nu = \sum_{[xyzx]} [xyzx]\nu_{xyzx}$$
$$\beta_{xyzx} = 1_{\mathrm{Aut}\,(G)} + \nu_{xyzx}$$

The naturality condition (2) induces the following identity in $\mathrm{Aut}\,(G)$:

$$(\mathrm{Ad}_{1+B_{yxzy}})(1+\mu_{xy}) = (1+\mu_{xz})(1+\mu_{zy})(1+\nu_{yzxy})$$

Expanding it to the second order, we obtain an identity in $\mathfrak{aut}(G)$:

$$\mu_{xz} + \mu_{zy} + \mu_{yx} + \mu_{xz}\mu_{zy} = \nu_{yxzy} + [B_{yxzy}, \cdots]$$

the limit of which is

$$\boxed{\nu = d\mu + \mu^2 - \mathrm{ad}_B} \tag{3}$$

where $\mathrm{ad}_B = [B, \cdots]$. This identity can also be taken as the definition of ν.

4.2 The functorial connection of Φ

In this section, we are going to define more precisely the functorial connection of Φ. The fiber of Φ above $\gamma = (x_0, \cdots, x_n)$, is the category Φ_γ whose objects are the families of objects connected by arrows as follows :

$$\begin{aligned} \mathrm{Ob}\ (\Phi_\gamma) = \{ &(a, u) = (a_0, u_{01}, a_1, \cdots, a_{n-1}, u_{n-1,n}, a_n) \mid \\ &a_i \in \mathrm{Ob}\ (\Phi_{x_i}) \text{ and } u_{j,j+1} : \Phi_{x_j x_{j+1}}(a_{j+1}) \to a_j \} \end{aligned}$$

The arrows of Φ_γ are the homotopies between such families :

$$\Phi_\gamma\big((a,u);(b,v)\big) = \{(\alpha_i : a_i \to b_i)_{0 \le i \le n} \mid \alpha_i \circ u_i = v_i \circ \Phi_{x_i, x_{i+1}}(\alpha_i)\}$$

If (γ, γ') is an oriented edge of $\mathcal{P}(X)$, and if γ and γ' differ by the 2-cell $[xzy]$, we define $\Phi_{\gamma\gamma'} : \Phi_{\gamma'} \to \Phi_\gamma$ by the following action on the objects of $\Phi_{\gamma'}$:

$$\begin{aligned} &\Phi_{\gamma\gamma'}(\cdots, a_x, u_{xz}, a_z, u_{zy}, a_y, \cdots) = \\ &(\cdots, a_x, u_{xz} \circ \Phi_{xz}(u_{zy}) \circ K_{[xzy]^*}(a_y), \beta_{yxzy}(a_y), \beta_{yxzy}(\cdots)) \end{aligned}$$

where the dots on the left of a_x denote the unchanged entries and all the entries on the right of a_y are twisted by β_{yzxy} pulled back to the corresponding vertex. The functor $\Phi_{\gamma'\gamma} : \Phi_\gamma \to \Phi_{\gamma'}$ acts as follows :

$$\begin{aligned} &\Phi_{\gamma'\gamma}(\cdots, b_x, v_{xy}, b_y, \cdots) = \\ &(\cdots, b_x, v_{xy} \circ K_{[xzy]}(b'_y), \Phi_{zy}\beta_{yzxy}(b_y), \Phi_{zy}\beta_{yzxy}(1_{b_y}), \beta_{yzxy}(b_y), \beta_{yzxy}(\cdots)) \end{aligned}$$

The initial entries v_{xy} and b_y are deleted and the entries on the right of b_y are twisted by β_{yxzy} pulled back to the corresponding vertex. One can check easily that $\Phi_{\gamma'\gamma}$ and $\Phi_{\gamma\gamma'}$ are quasi-inverses one of another. In the definition of the 3-curvature of Φ, given below, we will also need the expression of the connection functor $\Phi_{(xx,xyzx)}$ which maps the category of sections of Φ_{xyzx} into that of sections of Φ_{xx}. The image of $(a_{0x}, u_{xy}, a_y, u_{yz}, a_z, u_{zx}, a_{1x}) \in \text{Ob }(\Phi_{xyzx})$ via $\Phi_{(xx,xyzx)}$ is

$$\begin{aligned}&\Phi_{(xx,xyzx)}(a_{0x}, u_{xy}, a_y, u_{yz}, a_z, u_{zx}, a_{1x}) = \\ &\big(a_{0x}, u_{xy} \circ \Phi_{xy}(u_{yz}) \circ \Phi_{xy}\Phi_{yz}(u_{zx}) \circ K_{[xyzx]^*}(a_{1x}), \beta_{xyzx}(a_{1x})\big)\end{aligned}$$

Conversely, $\Phi_{(xyzx,xx)}$ maps (b_{0x}, v_{xx}, b_{1x}) to

$$\begin{aligned}&\Phi_{(xyzx,xx)}(b_{0x}, v_{xx}, b_{1x}) = \\ &\big(b_{0x}, v_{xx} \circ K_{[xyzx]}(b_{1x}), \Phi_{yz}\Phi_{zx}\beta_{xzyx}(b_{1x}), \Phi_{yz}\Phi_{zx}\beta_{xzyx}(1_{b_{1x}}), \\ &\Phi_{zx}\beta_{xzyx}(b_{1x}), \Phi_{zx}\beta_{xzyx}(1_{b_{1x}}), \beta_{xzyx}(b_{1x})\big)\end{aligned}$$

When we move the path $\gamma = (x_0, \cdots, x_n)$ at an end point, to the neighbour path $\gamma' = (x_0, \cdots, x_{n-1}, y)$, y being a neighbour of x_n and of x_{n-1}, the sweeping functor $\Phi_{\gamma'\gamma}$ acts on $(\cdots, a_{n-1}, u_{n-1}, a_n)$ as follows :

$$\begin{aligned}&\Phi_{\gamma'\gamma}(\cdots, a_{n-1}, u_{n-1}, a_n) = \\ &\big(\cdots, a_{n-1}, u_{n-1} \circ K_{[x_{n-1}yx_n]^*}(a_n), \Phi_{yx_n}\beta_{x_nyx_{n-1}x_n}(a_n)\big)\end{aligned}$$

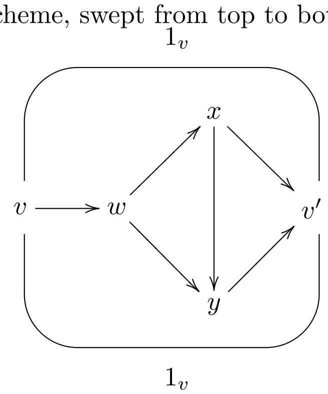
The objects of $\bar{\Phi}$ are defined like those of Φ but with reversed arrows :

$$\begin{aligned}\text{Ob}(\bar{\Phi}) = \{&(a, u) = (a_0, u_{01}, a_1, \cdots, a_{n-1}, u_{n-1,n}, a_n) \mid \\ &a_i \in \text{Ob }(\Phi_{x_i}) \text{ and } u_{j,j+1} : a_j \to \Phi_{x_jx_{j+1}}(a_{j+1})\}\end{aligned}$$

The elements of $\bar{\Phi}((a, u); (b, v))$ are defined as homotopies between the objects (a, u) and (b, v) and the connection $\bar{\Phi}_{\gamma\gamma'}$ is defined by formulas similar to those for Φ but using $\bar{K} : \beta C \to 1$ and $\bar{K}^{-1} : 1 \to \beta C$ instead of K and K^{-1}.

We define the 3-curvature of Φ as the 3-form Ω which, to each oriented 3-cell $\sigma = \langle wxyzw \rangle$ such that $w = s_0(\sigma) = t_0(\sigma)$ and $(ww) = s_1(\sigma) =$

$t_1(\sigma)$, associates the 2-arrow $\Omega_\sigma : \Phi_{ww} \to \Phi_{ww}$ obtained by composition of the sweeping functors corresponding to the four oriented faces of this 3-cell. Let's now compute Ω_σ when the boundary of σ is represented by the following pasting scheme, swept from top to bottom :



Let $(b \overset{\alpha}{\longleftarrow} a) \in \mathrm{Ob}(\Phi_{vv'})$ and let's twist a by the 1-arrows $C\beta$ associated to the boundaries of the 2-cells swept successively. This defines a sequence of objects $a_i \in \mathrm{Ob}(\Phi_v)$

$$\begin{aligned}
a_1 &= C_{vwyv}\beta_{vywv}(a)\\
a_2 &= \Phi_{vy}C_{ywxy}\beta_{yxwy}\Phi_{yv}(a_1)\\
a_3 &= C_{vxyv}\beta_{vyxv}(a_2)\\
a_4 &= C_{vwxv}\beta_{vxwv}(a_3)
\end{aligned}$$

The K arrows corresponding to these 2-cells are

$$\begin{aligned}
K_{[vwyv]^*}(a) &: a \longrightarrow a_1\\
\Phi_{vy}\big(K_{[ywxy]^*}(\Phi_{yv}(a_1))\big) &: a_1 \longrightarrow a_2\\
K_{[vxyv]}(a_2) &: a_2 \longrightarrow a_3\\
K_{[vwxv]}(a_3) &: a_3 \longrightarrow a_4
\end{aligned}$$

The composition of these four arrows defines the 3-curvature arrow $\Omega_\sigma(a)$:

$$\boxed{\Omega_\sigma(a) = K_{[vwxv]}(a_3) \circ K_{[vxyv]}(a_2) \circ \Phi_{vy}\big(K_{[ywxy]^*}(\Phi_{yv}(a_1))\big) \circ K_{[vwyv]^*}(a)}$$

Choosing a local section of Φ, we can expand the 1-connection as $\Phi_{xy} = 1_{\text{Aut}}(G) + \mu_{xy}$ and the 2-connection as $K = 1_G + B$. Since $(1_{\mathcal{B}_G} - C)$ is an infinitesimal of second order, the expansion of $\Omega_\sigma(a) = 1_G + \omega_{vwxyv}$ to the third order gives

$$1 + \omega_{vwxyv} = [1 + B_{vwxv}][1 + B_{vxyv}]\big[(1 + \mu_{vy}) \cdot (1 - B_{ywxy})\big][1 - B_{vwyv}]$$

$$\omega_{vywxv} = B_{ywxy} - B_{vywv} - B_{vyxv} - B_{vxwv} + \mu_{vy} \cdot B_{ywxy}$$

And the expression of the curvature 3-form,

$$\omega = \sum_{\langle wxyzw \rangle} \langle wxyzw \rangle \omega_{wxyzw}$$

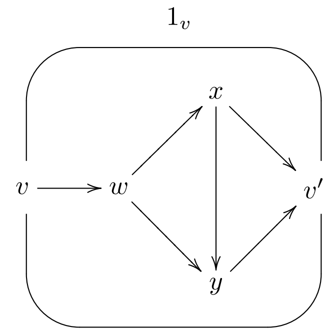
in terms of differential forms becomes :

$$\boxed{\omega = dB + \mu \cdot B} \tag{4}$$

4.3 Cocycle identities

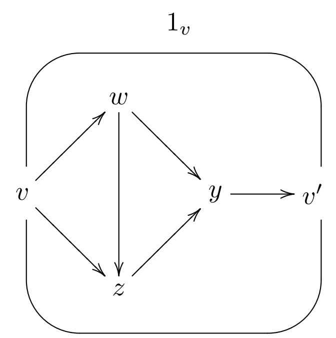
In the proof of the Bianchi identity, we have used a homotopy in the 1-skeleton of an infinitesimal tetrahedron. By analogy, in order to obtain the 3-cocycle identities satisfied by Φ, we can use a homotopy in the 2-skeleton of an oriented 4-simplex of X. So let v, w, x, y and z be five pairwise neighbour vertices of X. Let's choose v as base point, let's double it into v and v' and let's sweep the boundary of the four tetrahedra containing v, according to the following pasting schemes, from top to bottom :

First face (F_1) :



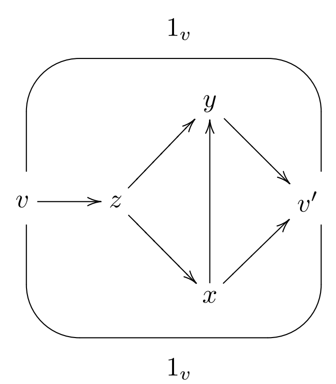
$$F_1 = \Phi_{[vwxv]} \circ (1_{\Phi_{vwx}} \times \Phi_{[xyv]}) \circ (1_{\Phi_{vw}} \times \Phi_{[wxy]^*} \times 1_{\Phi_{yv}}) \circ \Phi_{[vwyv]^*}$$

Second face (F_2) :



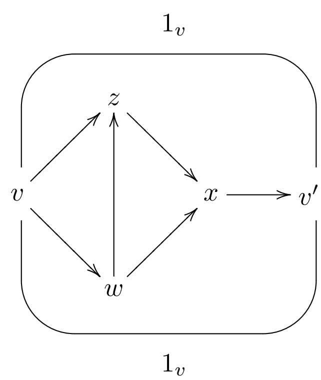
$$F_2 = \Phi_{[vwyv]^*} \circ (1_{\Phi_{vw}} \times \Phi_{[wzy]} \times 1_{\Phi_{yv}}) \circ (\Phi_{[vwz]^*} \times 1_{\Phi_{[zyv]}}) \circ \Phi_{[vzyv]^*}$$

Third face (F_3) :



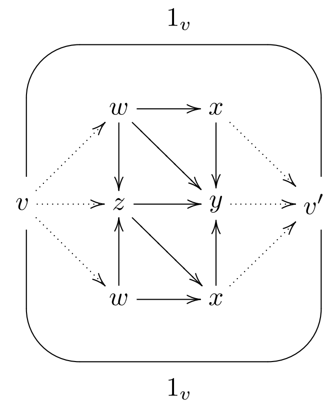
$$F_3 = \Phi_{[vzyv]} \circ (1_{\Phi_{vz}} \times \Phi_{[zxy]} \times 1_{\Phi_{yv}}) \circ (1_{\Phi_{vzx}} \times \Phi_{[xyv]^*}) \circ \Phi_{[vzxv]^*}$$

Fourth face (F_4) :



$$F_4 = \Phi_{[vzxv]} \circ (\Phi_{[vwz]} \times 1_{\Phi_{zxv}}) \circ (1_{\Phi_{vw}} \times \Phi_{[wzx]^*} \times 1_{\Phi_{xv}}) \circ \Phi_{[vwxv]^*}$$

The composition of these four 2-arrows, which are loops in loop space, amounts to sweeping the boundary of the tetrahedron opposite to v (fifth face, F_5) :



$$
\begin{aligned}
F_5 = &\Phi_{[vwxv]} \circ (1_{\Phi_{vwx}} \times \Phi_{[xyv]}) \circ (1_{\Phi_{vw}} \times \Phi_{[wxy]^*} \times 1_{\Phi_{yv}}) \circ \\
&(1_{\Phi_{vw}} \times \Phi_{[wzy]} \times 1_{\Phi_{yv}}) \circ (\Phi_{[vwz]^*} \times 1_{\Phi_{zyv}}) \circ (\Phi_{[vwz]} \times \Phi_{[zxy]} \times 1_{\Phi_{yv}}) \circ \\
&(1_{\Phi_{vw}} \times \Phi_{[wzx]^*} \times \Phi_{[xyv]^*}) \circ \Phi_{[vwxv]^*}
\end{aligned}
$$

The functors defined by these five pasting schemes map $\Phi_{vv'}$ to itself. If we choose a section of Φ above this 4-simplex, i.e. an object of reference in each fiber, these functors are reduced to the action of an element of G on the arrows of Φ_v and the identity of these 2-arrows becomes an equality in G. By expanding the product $F_1F_2F_3F_4$ of the first four faces to the fourth order, and equating it to F_5, we obtain a combinatorial representation of the cocycle identity satisfied by ω. The proof using differential forms is much easier :

$$
\begin{aligned}
d\omega + \mu \cdot \omega &= d(dB + \mu \cdot B) + \mu \cdot (dB + \mu \cdot B) \\
&= d\mu \cdot B - \mu \cdot dB + \mu \cdot dB + \mu^2 \cdot B \\
&= (d\mu + \mu^2) \cdot B
\end{aligned}
$$

Since B a $\mathfrak{g}$-valued 2-form, we have $[B, B] = 0$ and we obtain :

$$
\boxed{d\omega + \mu \cdot \omega = \nu \cdot B} \tag{5}
$$

This identity is the higher dimensional analogue of the Bianchi identity given in [6]. Similar identities appeared already in [1]. The 2-curvature

ν also satisfies a cocycle identity, since (3) implies

$$\begin{aligned} d\nu =& [d\mu, \mu] - \mathrm{ad}_{dB} \\ [\mu, \nu] =& [\mu, d\mu] + [\mu, \mu^2] - [\mu, \mathrm{ad}_B] \end{aligned}$$

where the bracket is in $\mathfrak{aut}(G)$. Since

$$\begin{aligned} [\mu, d\mu] =& -[d\mu, \mu] \\ [\mu, \mathrm{ad}_B] =& \mathrm{ad}_{\mu \cdot B} \\ [\mu, \mu^2] =& 0 \quad \text{(Jacobi)} \end{aligned}$$

we obtain

$$\boxed{d\nu + [\mu, \nu] = -\mathrm{ad}_\omega} \tag{6}$$

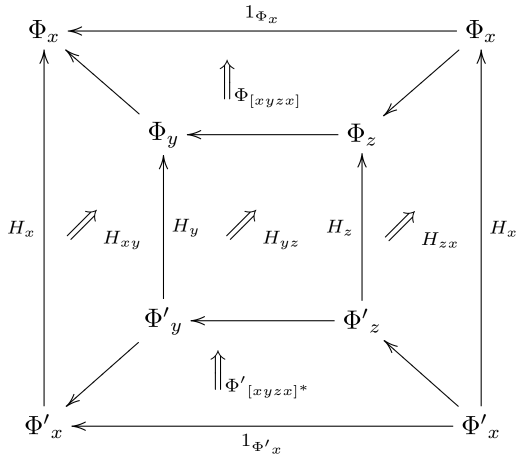
This is the relation (11.1.1) of [6]. We will see below how, in four dimensions, (5) and (6) imply the topological symmetry of the action of the pure BF-theory :

$$S(\mu, B) = \int_M \mathrm{tr}_{\mathrm{ad}}(\nu \, \mathrm{ad}_B)$$

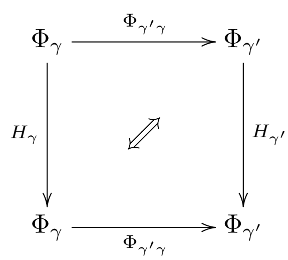
The section $(a, u) \in \mathrm{Ob}(\Phi_\gamma)$, transported by the successive sweeping functors $\Phi_{\gamma\gamma'}$, plays a role similar to the fermion transported by the gauge field in Yang-Mills theory.

4.4 Gauge transformations

As in the case of bundles, we need to know when two gerbes define the same gauge invariant quantities and what the automorphisms of a given gerbe are. Our functorial approach suggests to define the gauge transformations as equivalences of monoidal cofunctors. Two gerbes Φ and Φ' are declared isomorphic when there exists an invertible natural equivalence of monoidal cofunctors, $H : \Phi' \to \Phi$, i.e. for each vertex x a 1-arrow in $\mathcal{T}_L(\mathcal{B}_G)$, $H_x : {\Phi'}_x \to \Phi_x$, and, for each edge xy, a 2-arrow $H_{xy} : H_x {\Phi'}_{xy} \to \Phi_{xy} H_y$ in $\mathcal{T}_L(\mathcal{B}_G)$, such that the following composition of 2-arrows be invertible



When $\Phi = \Phi'$, the gauge transformation H is a self-equivalence of Φ : if, for each path $\gamma \in \mathrm{Ob}\,(\mathcal{P}(X))$, $H_\gamma : \Phi_\gamma \to \Phi_\gamma$ denotes the action of H on Φ_γ, then for each edge (γ, γ') of $\mathcal{P}(X)$, there exists an invertible natural transformation from $H_{\gamma'}\Phi_{\gamma'\gamma}$ to $\Phi_{\gamma'\gamma}H_\gamma$ represented by the following diagram



In order to obtain the local expression of the gauge transformed 2-connection, let's pick $(b \overset{\alpha}{\longleftarrow} a) \in \mathrm{Ob}\,(\Phi_{(xx)})$ and let's compare the action of $\Phi_{(xyzx,xx)}H_{(xx)}$

$$\begin{aligned}\Phi_{(xyzx,xx)}H_{(xx)}(b \xleftarrow{\alpha} a) = \big(&H_x(b), H_x(\alpha)\circ K_{[xyzx]}(H_x(a)),\\ &\Phi_{yz}\Phi_{zx}\beta_{xzyx}H_x(a), \Phi_{yz}\Phi_{zx}\beta_{xzyx}H_x(1_a),\\ &\Phi_{zx}\beta_{xzyx}H_x(a), \Phi_{zx}\beta_{xzyx}H_x(1_a), \beta_{xzyx}H_x(a)\big)\end{aligned}$$

with the action of $H_{(xyzx)}\Phi_{(xyzx,xx)}$ on the same initial section :

$$\begin{aligned}&H_{(xyzx)}\Phi_{(xyzx,xx)}(b \xleftarrow{\alpha} a) =\\ &\qquad\big(H_x(b), H_x(\alpha)\circ H_x(K_{[xyzx]}(a))\circ H_{xy}^{-1}(\Phi_{yz}\Phi_{zx}\beta_{xzyx}(a)),\\ &\qquad H_y\Phi_{yz}\Phi_{zx}\beta_{xzyx}(a), H_{yz}^{-1}(\Phi_{zx}\beta_{xzyx}(a)),\\ &\qquad H_z\Phi_{zx}\beta_{xzyx}(a), H_{zx}^{-1}(\beta_{xzyx}(a)), H_x\beta_{xzyx}(a)\big)\end{aligned}$$

Equating the compositions of these arrows, we obtain :

$$\begin{aligned}K_{[xyzx]}(H_x(a)) = &H_x(K_{[xyzx]}(a))\circ H_{xy}^{-1}(\Phi_{yz}\Phi_{zx}\beta_{xzyx}(a))\\ &\circ\Phi_{xy}H_{yz}^{-1}(\Phi_{zx}\beta_{xzyx}(a))\circ\Phi_{xy}\Phi_{yz}H_{zx}^{-1}(\beta_{xzyx}(a))\end{aligned}$$

In a local chart, we have

$$\begin{aligned}K_{[xyzx]}(a) &= 1_G + B_{xyzx}\\ K_{[xyzx]}(H_x(a)) &= 1_G + {B'}_{xyzx}\\ \Phi_{xy} &= 1_{\mathrm{Aut}\,(G)} + \mu_{xy}\\ {\Phi'}_{xy} &= 1_{\mathrm{Aut}\,(G)} + {\mu'}_{xy}\\ H_x &= 1_{\mathrm{Aut}\,(G)} + \xi_x\\ H_{xy} &= 1_G + \eta_{xy}\end{aligned}$$

where $\xi_x \in \mathfrak{aut}\,(G)$ and $\eta_{xy} \in \mathfrak{g}$. The expansion of the previous identity up to first order in (ξ, η) provides us with the expression of the infinitesimal gauge transformations :

$$\begin{aligned}1 + {B'}_{xyzx} &= \big[(1-\xi_x)\cdot(1+B_{xyzx})\big][1-\eta_{xy}]\big[(1+\mu_{xy})\cdot(1-\eta_{yz})\big][1-\eta_{zx}]\\ {B'}_{xyzx} &= B_{xyzx} - \xi_x\cdot B_{xyzx} - (\eta_{xy}+\eta_{yz}+\eta_{zx}+\mu_{xy}\cdot\eta_{yz})\\ &\quad +(\eta_{xy}\eta_{xz}+\eta_{xy}\eta_{yz}+\eta_{yz}\eta_{zx})\end{aligned}$$

Using the differential forms $\xi = \sum_x x\xi_x$ and $\eta = \sum_{(xy)}(xy)\eta_{xy}$, and keeping only the terms linear in (ξ, η), the η^2 term disappears and we obtain

$$\boxed{B' = B - \xi \cdot B - (d\eta + \mu \cdot \eta)} \tag{7}$$

(The formula (11.3.6) of [6] reduces to the above relation when λ and π vanish. These differential forms correspond to the face which closes the big outer square in our cubic diagram, which we have taken as the identity functor on Φ_{xx}.)

The infinitesimal effect of H on the 1-form μ is obtained by expanding up to first order in (ξ, η) the equivalence between the 1-connection functors :

$$\begin{aligned} {\Phi'}_{xy} &\simeq H_x^{-1}\Phi_{xy}H_y \\ 1 + {\mu'}_{xy} &= (1 - \xi_x)(1 + \mu_{xy})(1 + \xi_y) \\ {\mu'}_{xy} &= \mu_{xy} + \xi_y - \xi_x + \mu_{xy}\xi_y - \xi_x\mu_{xy} \end{aligned}$$

In terms of differential forms :

$$\boxed{\mu' = \mu + d\xi + [\mu, \xi]} \tag{8}$$

Introducing the notation $\nabla = d + \mu$, we have

$$\begin{aligned} \nabla^2 &= d\mu + \mu^2 \\ \nabla\xi &= d\xi + [\mu, \xi] \\ \nabla B &= dB + \mu \cdot B = \omega \\ \nabla\eta &= d\eta + \mu \cdot \eta \end{aligned}$$

and similarly with ∇' and μ'. The effect of H on the 2-curvature 2-form ν is given by

$$\begin{aligned} \nu' &= d\mu' + (\mu')^2 + [B, \cdots] \\ &= d(\mu + \nabla\xi) + (\mu + \nabla\xi)^2 + [B - \xi \cdot B - \nabla\eta, \cdots] \\ &= \big(d\mu + \mu^2 + [B, \cdots]\big) + d\nabla\xi + \mu\,\nabla\xi + (\nabla\xi)\,\mu - [\xi \cdot B + \nabla\eta, \cdots] \end{aligned}$$

$$\boxed{\nu' = \nu + \nabla^2\xi - [\xi \cdot B + \nabla\eta, \cdots]} \tag{9}$$

The first order gauge variation of the 3-curvature 3-form ω is given by

$$\begin{aligned}\omega' &= dB' + \mu' \cdot B' \\ &= dB - d(\xi \cdot B) - d(\nabla\eta) + (\mu + \nabla\xi) \cdot (B - \xi \cdot B - \nabla\eta) \\ &= \omega - \nabla(\xi \cdot B) + (\nabla\xi) \cdot B - \nabla^2\eta\end{aligned}$$

$$\boxed{\omega' = \omega - \xi \cdot \omega - \nabla^2\eta} \tag{10}$$

We deduce from these relations the variation of the full BF action :

$$\begin{aligned}S' - S &= -\int_M \mathrm{tr}_{\mathrm{ad}}\big(\nabla(\xi \cdot \omega) + \nu \cdot \nabla\eta\big) \\ &= -\int_{\partial M} \mathrm{tr}_{\mathrm{ad}}\big(\xi \cdot \omega + \frac{1}{2}\,\mu \cdot \nabla\eta\big)\end{aligned}$$

Our approach does not suppose the existence of a globally defined 2-form B. Thus, all integral formulas depend on the choice of a Čech 1-cocycle taking its values in Aut (G) and adapted to an open hypercover of M.

5 Perspectives

As suggested in (§4.4), the geometry of gerbes is adapted to the kinematics of topological field theories of type BF. Indeed, if we compare the gauge transformations given in (§4.3) with those of the BF-theory ([8], [13]) we can make the following correspondence :

$$\text{gerbes} \longrightarrow BF - \text{theory}$$

$$\begin{aligned}\mu &\longrightarrow [A, \cdots] \\ \nu &\longrightarrow [F_A, \cdots] - [B, \cdots]\end{aligned}$$

This suggests us that the right framework for BF type field theories is the space of gerbes and not the narrower theory of (principal or associated) bundles. We can also interpret directly the Faraday 2-form F as a

2-connection whose 3-curvature ω vanishes, due to the Bianchi identity. The G-bundle on which F is defined can be interpreted as a local section of a flat gerbe, and Yang-Mills theory appears to be defined more naturally as the dynamics of a functorial connection whose 2-curvature vanishes.

Conclusion : We have developped a simplicial approach to non-Abelian gerbes and proved, using a 2-groupoid combinatorics, that it leads to the same cohomology classes as the differential approach. The specificity of our viewpoint resides in the usage of the space of edge-paths. The importance of the space of paths is suggested by gauge field theories where physical quantities are naturally associated to loops. Conversely, the developement of the theory of gerbes may open the door to the resolution of long standing problems of physics like the mathematical proof of confinement of quarks in QCD or the construction of integrable models in dimension higher than two [1].

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