

## Representation Theory of Twisted Group Double

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**ABSTRACT.** This text collects useful results concerning the quasi-Hopf algebra  $D^\omega(G)$ . We give a review of issues related to its use in conformal theories and physical mathematics. Existence of such algebras based on 3-cocycles with values in  $R/Z$  which mimic for finite groups Chern-Simons terms of gauge theories, open wide perspectives in the so called "classification program". The modularisation theorem proved for quasi-Hopf algebras by two authors some years ago makes the computation of topological invariants possible. An updated, although partial, bibliography of recent developments is provided.

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## 1 Introduction

Lattice statistical models with boundary conditions or quantum field theories with boundary conditions, have been extensively studied in the last ten years. In an integrable framework and from an algebraic point of view, different boundary conditions may lead to various interesting algebras.

Because of dynamical interactions between excitations, a phenomenon sometimes called “fusion of excitations together” takes place as a consequence of a non trivial geometric transformation. This idea lies at the heart of the celebrated Frobenius-Pasquier-Verlinde formula [1]. Later, R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde applied this to orbifolds.

Such a fusion corresponds, in the mathematical formalism, to a co-product morphism being part of quasi-Hopf (or some quantum algebra) axioms.

Such a geometric transform can be a  $SL_n(Z)$  diffeomorphism of the  $n$ -torus, or a “tour”, along a non-trivial homological cycle. Note that more complicated manifolds are also a very fascinating field of research.

In two-dimensional integrable systems compatibility between the Yang-Baxter equations and the fusion is an interesting general issue.

To our knowledge, R. Dijkgraaf and E. Witten pioneered the use of group 3-cocycles as a discrete analogue of flat connections on 3-manifolds [2]. This viewpoint makes very clear the role of local gauge invariance in building up topological invariants. However, their paper remains in the framework of so-called “topological theories” and cobordism, which can be fruitfully associated with a more algebraic, and algorithmic, approach such as the one initiated by V. G. Drinfeld, M. Jimbo, T. Miwa, N. Reshetikhin, V. Turaev, and many others [3, 4]. This is the viewpoint we will detail here. In some conference proceedings [5], V. Pasquier et al. obtained a nice breakthrough when formulating a twisted double as a quasi-Hopf algebra, called  $D^\omega(G)$ . This is mostly this work that we revisit in this article<sup>1</sup>, including new material and more general approach of classification problems.

In a series of papers D. Altschuler and A. Coste proved that N. Reshetikhin and V. Turaev functorial construction extends to quasi-Hopf algebras and therefore, the invariants from R. Dijkgraaf and E. Witten

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<sup>1</sup>which is a developed version of the preprint IHES/M/99/52

topological theories have been conjectured by them to be exactly recovered from a link-surgery Markov trace [2].

It is clear considering simple lens spaces that this equivalence goes between triangulation and Dehn surgery of the same 3-manifold. The proof of these equalities, as well as consideration of Heegaard splitting presentation, has been provided to our knowledge by S. Piunikhin and followers [6].

## 2 A few heuristic recalls on Drinfeld's Quantum Double

The interest of quasi-triangular Hopf algebras is that they produce naturally solutions of the Yang-Baxter equations in a "universal form", in a first approach, *without spectral parameter*. One can build quasi-triangular Hopf algebras starting from any Hopf algebra introducing two Hopf algebras dual to each other (see for instance [7, 8]). Two Hopf algebras (isomorphic as vector space) are dual to each other if the product and co-product of the first Hopf algebra coincide with the co-product and product of the other one. Let  $e_a$  be the basis vectors of the first Hopf algebra, and  $e^a$  the basis vectors of the second Hopf algebra, we require :

$$\begin{aligned} e_a e_b &= C_{a,b}^c \cdot e_c, & \Delta(e^a) &= C_{a,b}^c \cdot e^b \otimes e^c \\ e^a e^b &= \tilde{C}_c^{a,b} \cdot e^c, & \Delta(e_a) &= \tilde{C}_c^{b,c} \cdot e_b \otimes e_c \end{aligned} \quad (1)$$

where the constants of structure  $C_{a,b}^c$  and  $\tilde{C}_c^{a,b}$  are such that the two Hopf algebras are co-algebras. The antipodes of the two Hopf algebras,  $\gamma$  and  $\rho$  :

$$\gamma(e_a) = \gamma_a^b \cdot e_b, \quad \rho(e^a) = \rho_b^a \cdot e_b$$

are related by a simple matrix inversion :  $\rho = \gamma^{-1}$ .

One introduces a normal ordering when assembling together the two Hopf algebras (think for instance that one Hopf algebra corresponds to creation operators, and the other to annihilation operators). The coproduct of this double Hopf algebra can be defined as follows :

$$\Delta(e_a e^b) = \tilde{C}_a^{c,d} \cdot C_{e,f}^b \cdot e_c e^f \otimes e_d e^e \quad (2)$$

The transposed coproduct in the double is just :

$$\Delta'(e_a e^b) = \tilde{C}_a^{c,d} \cdot C_{e,f}^b \cdot e_d e^e \otimes e_c e^f \quad (3)$$

The coproduct and the transposed coproduct are intertwined by the following  $R$ -matrix :

$$R \Delta(x) = \Delta'(x) R, \quad \text{where :} \quad R = \sum_a e_a \otimes \mathbf{1} \otimes \mathbf{1} \otimes e^a$$

The antipode in the double is an antihomomorphism that we will not write here. The prescription to get the normal ordering is that the product and coproduct are compatible. Drinfeld's normal ordering is :

$$: e^a e_b : = C_{f,d}^e \cdot C_{e,g}^a \cdot \tilde{C}_b^{h,i} \cdot \tilde{C}_i^{c,f} \cdot \rho_h^g \cdot e_c e^d$$

where we use the Einstein's conventions for summation over up and down indices (contraction).

### 3 Identification of the representations of $D^\omega(G)$

In two very dense papers V.G. Drinfeld [3] has developed the notion of quasi-triangular quasi-Hopf algebra, which seems particularly fruitful in Geometry and Physics [2, 4]. Examples of such a structure can be obtained as follows:

Let  $G$  be a finite group with unit  $e$  and order  $|G|$ . Let  $\omega$  be a normalized 3-cocycle with values in  $U(1)$ , i.e.  $\omega$  is an application from  $G^3$  to  $U(1)$  satisfying for any  $(g_1, g_2, g_3, g_4) \in G^4$  :

$$\omega(g_1, g_2, g_3) \omega(g_1, g_2 g_3, g_4) \omega(g_2, g_3, g_4) = \omega(g_1 g_2, g_3, g_4) \omega(g_1, g_2, g_3 g_4) \tag{4}$$

$$\omega(g_1, g_2, e) = \omega(g_1, e, g_2) = \omega(e, g_1, g_2) \tag{5}$$

Where  $e$  is the unit element of  $G$ . In this multiplicative notation  $\omega$  is the exponential of an additive cocycle with values in  $i R/2\pi Z$ . These equations imply a number of identities which are derived in the appendix.

From these data one defines [5] a finite dimensional algebra over the complex numbers as the linear span of the  $|G|^2$  generators  $(g \lfloor_x)_{(g,x) \in G^2}$  satisfying the relations  $g_1 \lfloor_x \cdot g_2 \lfloor_y = \delta_{g_1, x g_2 x^{-1}} \theta_{g_1}(x, y)$

where  $\theta_g(x, y) = \omega(g, x, y) \omega(x, y, (xy)^{-1} gxy) \omega(x, x^{-1} g x, y)^{-1}$ . (6)

$D^\omega(G)$  is associative by virtue of (20) and has unit element  $1_{D^\omega(G)} = \sum_{g \in G} g \lfloor_e$ . It can be equipped with a ribbon quasi-Hopf quasi-triangular structure described in details in [5, 9] allowing one to compute topological invariants of links and three manifolds by the method explained in [4, 9]. Here we wish to present a study of the representations of  $D^\omega(G)$  which are the objects "colouring" the links in this method.

**Proposition 1:** Any finite dimensional representation of  $D^\omega(G)$  is a direct sum of irreducible ones.

Proof: Let  $(\pi, V)$  be a representation with a non trivial stable subspace  $W$ . Let  $P$  be any projector onto  $W$  ( $P^2 = P, ImP = W, KerP \oplus W = V$ ). Define

$$P_o = \frac{1}{|G|} \sum_{(g,x) \in G^2} \frac{1}{\theta_{g^{-1}}(x, x^{-1})} \pi(x^{-1}g^{-1}x \lfloor_{x^{-1}}) P \pi(g^{-1} \lfloor_x)$$

This formula implies  $ImP_o \subset ImP$ . Furthermore, because of (20) derived in the appendix below,  $\theta_{x^{-1}g^{-1}x}(x^{-1}, x) = \theta_{g^{-1}}(x, x^{-1})$  so that  $P_o|_{ImP} = Id_{ImP}$ , implying  $ImP = ImP_o$ . One sees that  $KerP_o$  is such that  $V = W \oplus KerP_o$ , allowing a proof of prop. 1 by induction on  $dimV$ .

We shall use the following group theoretical setup:  $\{C_A\}_{A=1, \dots, P}$  the set of conjugacy classes of  $G$ ,  $|C_A|$  the number of elements in  $C_A$  and  $\Gamma = \{g_A\}_{A=1, \dots, P}$  a system of representatives of these classes. For any  $A$ ,  $p_A$  will be the order of  $g_A$ ,  $N_A = \{h \in G/hg_Ah^{-1} = g_A\}$  the centralizer of  $g_A$ ,  $|N_A|$  its number of elements equal to  $|G|/|C_A|$ ,  $\chi_A = \{x_{A,j}\}_{j=1, \dots, |C_A|}$  a system of representatives of  $G/N_A$  ( $e$  being one of the  $x_{A,j}$ ),  $\langle g_A \rangle = \{e, g_A, \dots, g_A^{p_A-1}\}$  the cyclic subgroup generated by  $g_A$ ,  $\mathcal{H}_A = \{h_{A,a}\}_{a=1, \dots, |N_A|/p_A}$  a system of representatives of  $N_A/\langle g_A \rangle$ .

Equivalently, for any  $g \in G$  there exist a unique couple  $(A, j)$  such that  $g = x_{A,j}g_Ax_{A,j}^{-1}$  and  $p_A = inf\{n/g^n = e\}$ .  $A$  being fixed, any  $x \in G$  can be written uniquely  $x = x_{A,j}h$ ,  $x_{A,j} \in \chi_A$ ,  $h \in N_A$  and any  $h \in N_A$  can be written uniquely

$h = g_A^k h_{A,a} = h_{A,a}g_A^k$ ,  $k \in \{0, \dots, p_A - 1\}$ ,  $h_{A,a} \in \mathcal{H}_A$ . When working within a given class  $C_A$  we will denote  $x_{A,j}$  and  $h_{A,a}$  simply by  $x_j$  and  $h_a$ .

**Proposition 2:** the left regular representation of  $D^\omega(G)$  is the direct sum of  $|G|$  representations  $(W_g)_{g \in G} : D^\omega(G) = \bigoplus_{g \in G} W_g$ , where

$$W_g = \text{Span}\{k \underset{x}{\lfloor} / x^{-1} kx = g\}; \dim(W_g) = |G|.$$

If  $g$  and  $g'$  are conjugate,  $W_g$  and  $W_{g'}$  are equivalent representations.

Proof:  $W_g$  is the image of  $D^\omega(G)$  by the projection  $a \longrightarrow a \cdot \varepsilon_g$  where the elements  $\varepsilon_g = g \underset{e}{\lfloor}$  form a decomposition of  $1_{D^\omega(G)}$  into a sum of orthogonal idempotents.

Denote a given  $g_o \in G$   $g_o = x_o g_A x_o^{-1}$  where  $x_o$  is one of the  $x_{A,j}$  ( $x_o = e$  if  $g_o = g_A$ ); then a basis of  $W_{g_o}$  is  $\left( x_j g_A x_j^{-1} \underset{x_j h x_o^{-1}}{\lfloor} \right)_{(h, x_j) \in N_A \times \mathcal{X}_A}$ . An intertwiner  $\Phi_{g_A}^{g_o}$  from  $W_{g_A}$  to  $W_{g_o}$  is

$$\Phi_{g_A}^{g_o} \left( x_j g_A x_j^{-1} \underset{x_j h}{\lfloor} \right) = \theta_{x_j g_A x_j^{-1}}(x_j h, x_o^{-1}) x_j g_A x_j^{-1} \underset{x_j h x_o^{-1}}{\lfloor}$$

The commutativity of  $\Phi_{g_A}^{g_o}$  with left multiplication by  $g_1 \underset{x_1}{\lfloor}$  results from identity (20) below with  $(g, x, y, z) = (g_1, x_1, x_j, x_o^{-1})$ .

**Proposition 3:** Each  $W_{g_o}$  ( $g_o = x_o g_A x_o^{-1}$ ) splits into  $p_A$  subrepresentations  $(W_{g_o, \lambda_{A,i}})_{i=1, \dots, p_A}$  which are eigenspaces for the action of the central element  $v^{-1} = \sum_{k \in G} k \underset{k}{\lfloor}$  with eigenvalues  $\lambda_{A,i}$  (denoted in short  $\lambda_i$ ) which are the  $p_A$ th-roots of  $\omega_A = \prod_{n=0}^{p_A-1} \omega(g_A, g_A^n, g_A)$ . For  $(g, y) \in G^2$  such that  $y^{-1} g y = g_o$ , set

$$\psi_{\lambda_i, g, y} = \sum_{k=0}^{p_A-1} \lambda_i^{p_A-1-k} \prod_{n=0}^{k-1} \omega(g, g^n y, y^{-1} g y) g \underset{g^k y}{\lfloor}. \tag{7}$$

Then  $v^{-1} \cdot \psi_{\lambda_i, g, y} = \lambda_i \psi_{\lambda_i, g, y}$  (8)

and  $\psi_{\lambda_i, g, g y} = \frac{\lambda_i}{\omega(g, y, y^{-1} g y)} \psi_{\lambda_i, g, y}$ , (9)

so that a basis of  $W_{g_o, \lambda_i}$  is  $(\psi_{\lambda_i, x_j g_A x_j^{-1}, x_j h_a x_o^{-1}})_{(x_j, h_a) \in \mathcal{X}_A \times \mathcal{H}_A}$ .

Proof: For any  $g \underset{y}{\lfloor} \in W_{g_o}$ , and for any  $k \in \{0, \dots, p_A - 1\}$ , set :  $\phi_k = \prod_{n=0}^{k-1} \omega(g, g^n y, y^{-1} g y) g \underset{g^k y}{\lfloor}$ .

The characteristic polynomial of the action of  $v^{-1}$  on the space spanned by the  $\phi_k$ 's is then (cf (23)) :

$$\det(\lambda - \rho_{reg}(v^{-1})) = \lambda^{p_A} - \prod_{n=0}^{p_A-1} \omega(g, g^n y, y^{-1} g y) = \lambda^{p_A} - \omega_A,$$

leading to eigenvectors  $\psi_{\lambda_i, g, y}$  which, altogether, are a generating system of  $W_{g_o, \lambda_i}$ . Each eigenspace  $W_{g_o, \lambda_i}$  is stable under the action of  $D^\omega(G)$  because  $v^{-1}$  is central and the  $\lambda_i$ 's are distinct.

**Definition :**  $A$  being fixed let  $(\pi, V)$  be a projective representation of  $N_A$  on a vector space  $V$ , with cocycle  $\theta_{g_A}$ , one has in  $End(V)$  :

$$\pi(h_3)\pi(h_2) = \theta_{g_A}(h_3, h_2)\pi(h_3 h_2). \quad (10)$$

Then, following [5], we define the "DPR-induced" of  $(\pi, V)$  as the representation  $(\rho_\pi, C[\chi_A] \otimes V)$  of  $D^\omega(G)$  given by:

$$\rho_\pi(g \lfloor \_ ) |x_j \rangle \otimes |v \rangle = \delta_{g, x_k g_A x_k^{-1}} \frac{\theta_g(x, x_j)}{\theta_g(x_k, h_2)} |x_k \rangle \otimes \pi(h_2) |v \rangle \quad (11)$$

where  $|v \rangle \in V$  and  $(x_k, h_2) \in \chi_A \times N_A$  are defined by  $x x_j = x_k h_2$ .

**Proposition 4:** For any  $g_o \in C_A$ ,  $(\rho_{reg}, W_{g_o})$  is equivalent to the representation DPR-induced from  $(\pi_{g_o}, C[N_A])$  given by :

$$\pi_{g_o}(h_2) |h_1 \rangle = \theta_{g_A}(h_2, h x_o^{-1}) |h_2 h \rangle$$

Proof: An explicit intertwiner  $\Theta_{g_o} : W_{g_o} \longrightarrow C[\chi_A] \otimes C[N_A]$  is:

$$\Theta_{g_o}(x_j g x_j^{-1} \lfloor \_ ) = \frac{1}{\theta_{x_j g x_j^{-1}}(x_j, h x_o^{-1})} |x_j \rangle \otimes |h \rangle. \quad (12)$$

Also note that  $\pi_{g_o}$  is equivalent to  $\pi_{g_A}$  with intertwiner  $\Psi : (\pi_{g_A}, C[N_A]) \longrightarrow (\pi_{g_o}, C[N_A])$ ,  $\Psi |h \rangle = \theta_{g_A}(h, x_o^{-1}) |h \rangle$ .

Having these equivalences of representations we can now focus on the subrepresentations of the  $W_{g_A}$ 's :

**Proposition 5:**  $(\rho_{reg}, W_{g_A, \lambda_i})$  is equivalent to the representation DPR-induced from  $(\pi^{\lambda_i}, C[\mathcal{H}_A])$  where :

$$\pi^{\lambda_i}(h_2)|h_a \rangle = \frac{\lambda_i^l}{\prod_{n=0}^{l-1} \omega(g_A, g_A^n, g_A)} \frac{\theta_{g_A}(h_2, h_a)}{\theta_{g_A}(h_b, g_A^l)} |h_b \rangle \quad (13)$$

$(l, h_b) \in \{0, \dots, p_A - 1\} \times \mathcal{H}_A$  being defined by  $h_2 h_a = g_A^l h_b$ .

Proof: The fact that  $\pi^{\lambda_i}$  satisfies (10) results from (24).

$$\Omega(\psi_{\lambda_i, x_j g_A x_j^{-1}, x_j h_a}) = \theta_{x_j g_A x_j^{-1}}(x_j, h_a) \Theta_{g_A}(\psi_{\lambda_i, x_j g_A x_j^{-1}, x_j h_a}) \quad (14)$$

is an intertwiner  $W_{g_A, \lambda_i} \rightarrow C[\chi_A] \otimes C[\mathcal{H}_A]$ , as a rather tedious computation shows.

**Proposition 6:**  $(\pi_{g_A}, C[N_{g_A}])$  is the direct sum of  $p_A$   $\theta_{g_A}$ -projective representations  $N_{A, \lambda_i}$  equivalent to the  $(\pi^{\lambda_i}, C[\mathcal{H}_A])$ ,  $C[N_{g_A}] = \bigoplus_{i=0}^{p_A-1} N_{A, \lambda_i}$ .

Proof:

$$\Pi|h_a \rangle = \sum_{m=0}^{p_A-1} \theta_{g_A}(h_a, g_A^m) \frac{\prod_{n=0}^{m-1} \omega(g_A, g_A^n, g_A)}{\lambda_i^m} |g_A^m h_a \rangle \quad (15)$$

is an intertwiner  $(\pi^{\lambda_i}, C[\mathcal{H}_A]) \rightarrow (\pi_{g_A}, C[N_{g_A}])$  whose image we call  $N_{A, \lambda_i}$ .

$$\mathcal{P}|g_A^k h_a \rangle = \frac{\lambda_i^k}{\prod_{n=0}^{k-1} \omega(g_A, g_A^n, g_A) \theta_{g_A}(h_a, g_A^k)} |h_a \rangle \quad (16)$$

is an intertwiner  $(\pi_{g_A}, C[N_{g_A}]) \rightarrow (\pi^{\lambda_i}, C[\mathcal{H}_A])$ .  $\Pi \circ \mathcal{P}$  gives an explicit expression for the projector  $C[N_{g_A}] \rightarrow N_{A, \lambda_i}$ .

We now reach our main point:

**Proposition 7 :** The irreducible representations of  $D^\omega(G)$  are (up to equivalence) characterized by a couple  $(A, \alpha)$  where  $A$  is the label of a conjugacy class in  $G$  and  $(\alpha, V_\alpha)$  is an irreducible projective representation of  $N_{g_A}$  with 2-cocycle  $\theta_{g_A}$  which can be realized as a subrepresentation of  $(\pi_{g_A}, C[N_{g_A}])$ . Furthermore  $\alpha(g_A) = \lambda_i Id_{V_\alpha}$ , for one  $\lambda_i$ , so that  $(\alpha, V_\alpha)$  is equivalent to a subrepresentation of  $N_{A, \lambda_i} (\sim \pi^{\lambda_i})$ . Then

the corresponding representation of  $D^\omega(G)$  is equivalent to the DPR-induced  $(\rho_\alpha, V_{A\alpha} = C[\chi_A] \otimes V_\alpha)$ .

Proof: This is clear once one has noticed that:

$$\pi^{\lambda_i}(g_A) = \lambda_i \frac{\theta_{g_A}(g_A, h_\alpha)}{\theta_{g_A}(h_\alpha, g_A)} Id_{C[\mathcal{H}_A]} = \lambda_i Id_{C[\mathcal{H}_A]}$$

(so that  $\pi^{\lambda_i}$  is  $\theta_{g_A}$ -projective, and  $g_A^{p_A} = e$  implies  $\lambda_i^{p_A} = \omega_A$ ) and

$$dim D^\omega(G) = \sum_A |C_A| dim W_{g_A} = \sum_A |C_A| |\chi_A| dim C[N_{g_A}] \tag{17}$$

$$= \sum_A |\chi_A|^2 \sum_\alpha dim^2(V_\alpha) = \sum_{A\alpha} dim^2(V_{A\alpha}) . \tag{18}$$

From (11) one gets:

**Proposition 8 :** The characters of the representations of  $D^\omega(G)$  are:

$$\chi_{A,\alpha}(g \lfloor_x) = \delta_{g \in C_A} \delta_{x \in N_g} \frac{\theta_g(x, x_j)}{\theta_g(x_j, h)} \chi_\alpha(h) \tag{19}$$

where  $N_g$  is the centralizer of  $g$ , and  $(x_j, h) \in \chi_A \times N_{g_A}$  are defined by  $g = x_j g_A x_j^{-1}$ ,  $x = x_j h x_j^{-1}$ .

**Summary of the results.**

These eight propositions, which concatenated in Spinoza’s style some partly available results, first proved the semi simplicity of the twisted double algebra. They also described the irreducible representations from the regular representation and used an interesting generalization of character theory for quasi-Hopf algebras.

**4 Perspectives**

These results allow the computation of the Verlinde  $S$  matrix using the ribbon functor applied to the coloured Hopf link. The complete formulae, involving a number of phases (i.e. exponentials of the additive 3-cocycle) are presented in [10]. These phases are absolutely non trivial in the general case of any non abelian group and in the so-called "non cohomologically trivial" situation.

It is worth noticing that such rather complicated objects appear, if one wants to classify conformal theories. Classifying such theories with a small number of primary fields will use classification of *finite groups* (and computing their 3-cocycles) with a small number of conjugacy classes, a task performed, up thirteen classes, by [11]. It has been noticed, at least since E. Landau that there are only a finite number of finite groups with a given number of classes (but how many ? may be physicists inspired by recent classifications or statistical approaches could give interesting comments to this question). Then, there seems to be many more modular invariant partition functions than in the affine case.

These 3-cocycles have been considered in the context of string and membranes studies using names such that "orbifolds with finite torsion". One idea is that any CFT can live on a string worldsheet, and our algebra encodes twisted boundary conditions, in the sense of G. t'Hooft and C.P. Korthals-Altes, which is known to relate to non-commutative geometry.

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## Appendix : Group cohomology identities

$\theta_g$  defined above satisfies:

$$\theta_g(x, y) \theta_g(xy, z) = \theta_g(x, yz) \theta_{x^{-1}gx}(y, z) \quad (20)$$

Proof: For instance simplify successively (20) with the help of (1) written with  $(g_i)_{i=1, \dots, 4} = (x, y, z, (xyz)^{-1}gxyz)$ ;  $(x, y, (xy)^{-1}gxy, z)$ ;  $(g, x, y, z)$  and finally  $(x, x^{-1}gx, y, z)$ .

$$\text{For any } g, y \quad \prod_{n=0}^{k-1} \omega(g, g^n y, y^{-1}gy) = \theta_g(g^k, y) \prod_{n=0}^{k-1} \omega(g, g^n, g) \quad (21)$$

Proof: Let us form the product of the  $k$  identities obtained from (1) with  $(g_i)_{i=1, \dots, 4} = (g, g^n, y, y^{-1}gy)$  for  $n = 0, \dots, k-1$ , and divide it by a similar product for  $(g_i) = (g, g^n, y, y^{-1}gy)$  :

$$\prod_{n=0}^{k-1} \omega(g, g^n y, y^{-1}gy) = \theta_g(y, y^{-1}g^k y) \prod_{n=0}^{k-1} \omega(y^{-1}gy, y^{-1}g^n y, y^{-1}gy) \quad (22)$$

Proof: Let us build similar products of identities for

$$(g_i) = (g, y, y^{-1}g^ny, y^{-1}gy), (y^{-1}, gy, y^{-1}g^ny, y^{-1}gy), (y^{-1}, y, y^{-1}g^ny, y^{-1}gy)$$

and simplify a factor involving five  $\omega$ 's by the identity written with  $(g_i) = (y^{-1}, y, y^{-1}gy, y^{-1}g^ky)$ .

When  $k = p_A$  and  $g \in C_A$  ( $g^{p_A} = e$ ) (21) and (22) insure that

$$\prod_{n=0}^{p_A-1} \omega(g, g^ny, y^{-1}gy) = \prod_{n=0}^{p_A-1} \omega(g, g^n, g) = \prod_{n=0}^{p_A-1} \omega(g_A, g_A^n, g_A) = \omega_A \quad (23)$$

For any positive integers  $k, l$  and any  $g \in G$ :

$$\prod_{n=0}^{k+l-1} \omega(g, g^n, g) = \theta_g(g^l, g^k) \prod_{n=0}^{k-1} \omega(g, g^n, g) \prod_{n=0}^{l-1} \omega(g, g^n, g) \quad (24)$$

$$\text{and this implies the symmetry : } \theta_g(g^k, g^l) = \theta_g(g^l, g^k). \quad (25)$$

Proof: Set

$$Q_k = \frac{\theta_g(g^l, g^k)}{\prod_{n=k}^{k+l-1} \omega(g, g^n, g)},$$

(1) with  $(g_i) = (g, g^l, g^k, g); (g^l, g, g^k, g)$  implies that  $Q_{k+1} = Q_k$ , therefore equal to  $Q_0 = 1/\prod_{n=0}^{l-1} \omega(g, g^n, g)$ .

Note for completeness the corresponding identity with negative powers:

$$\prod_{n=0}^{k+l-1} \omega(g, g^{-n}, g) = \frac{\omega(g, g^{-l}, g)}{\theta_g(g^{-l}, g^{1-k})} \prod_{n=0}^{k-1} \omega(g, g^{-n}, g) \prod_{n=0}^{l-1} \omega(g, g^{-n}, g) \quad (26)$$

and the relation between positive and negative powers :

$$\prod_{n=0}^l \omega(g, g^{-n}, g) = \theta_g(g^{-l}, g^l) \prod_{n=0}^{l-1} \omega(g, g^n, g) \quad (27)$$

Proof: by induction and use of (1) with  $(g_i) = (g, g^{-l-1}, g^{l+1}, g)(g, g^{-l-1}, g, g^{l+1})(g^{-l}, g, g^l, g)(g, g^{-l}, g^l, g)$ .

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