Lagrangian formalism for the Dirac equation including charge conjugation

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RÉSUMÉ. La célébration du cinquantième anniversaire de l'article de Yang et Mills amène à en examiner les antécédents, et particulièrement l'équation de Dirac. Il est possible d'écrire toute l'équation de Dirac sans faire intervenir d'aucune manière les nombres complexes, ce qui amène le remplacement des groupes unitaires par des groupes orthogonaux. Puis on réexamine la manière dont on passe de la densité lagrangienne à l'équation d'onde dans le formalisme usuel, ce qui introduit une nouvelle forme de lagrangien, et il est possible de modifier ce lagrangien de telle sorte que l'on obtienne à la fois l'équation d'onde de la particule et de l'antiparticule. Ce nouveau lagrangien implique de nombreuses différences par rapport au formalisme usuel de la théorie de Dirac. On examine en particulier les densités tensorielles sans dérivée et le tenseur d'impulsion-énergie

ABSTRACT. Celebrating the fiftieth anniversary of the Yang-Mills' article, we examine the past history of this paper, particularly the Dirac equation. It is possible to write the Dirac theory without complex numbers, in a totally real frame, as classical physics. This implies to replace unitary groups by orthogonal groups. Next we examine again how the Dirac equation is deduced from the Lagrangian formalism. It introduces another form for the Lagrangian density giving the Dirac equation. It is possible to modify this form to get a Lagrangian density giving both the wave equation of the particle and the wave equation of the antiparticle. This new Lagrangian formalism implies many differences in comparison with the usual Dirac theory. We begin to explore the tensorial densities without derivative, and the momentum-energy deduced from this Lagrangian.

1 - A real frame for the Dirac theory

The celebrated paper of Yang and Mills [1] is a theory of isotopic spin as a local gauge invariance under a SU(2) unitary group. From where comes this model ? Why unitary groups, why SU(2) ? Yang and Mills use a Lagrangian density

$$\mathcal{L} = -\frac{1}{4}\mathbf{f}_{\mu\nu} \cdot \mathbf{f}^*_{\mu\nu} - \overline{\psi}\gamma_\mu(\partial_\mu - i\epsilon\boldsymbol{\tau} \cdot \mathbf{b}_\mu)\psi - m\overline{\psi}\psi \quad (1)$$

ψ and $\overline{\psi}$ terms come from the Dirac theory. Yang and Mills assume that protons and neutrons, particles with spin $\frac{1}{2}$ have waves following a Dirac equation.

Before the discovery of quantum mechanics, physicists knew waves, acoustic waves, electromagnetic waves. All those waves are described by real numbers or vectors above the real number field. Fourier analysis described those waves as series of cosine and sine. It was the same when Louis de Broglie discovered matter's waves, he wrote the wave as $\sin 2\pi\nu(t - \frac{x\beta}{2})$. So the new wave mechanics was very far from unitary groups. One century ago, each physical entity was described by numbers, vectors or tensors above the real field, and never above the complex field. Complex numbers were introduced only by Schrödinger. And even in his first papers, when Schrödinger writes

$$\psi_n = e^{-\frac{x^2}{2}} H_n(x) e^{2\pi i \nu_n t} \quad (2)$$

he notes in the margin that i equals $\sqrt{-1}$ and that we must take only the real part of the right member. Complex numbers appear actually with his fourth paper [2] with the wave equation

$$\frac{\partial\psi}{\partial t} = \pm\frac{2\pi i}{h}E\psi \quad (3)$$

Writing this paper, Schrödinger recognized the intrinsically complex nature of the wave, and was very surprised. Today complex numbers are not discussed, and the gap between gravitation and quantum physics remains.

Next Born gave an interpretation of the wave's modulus as giving the probability density. So we get a norm and a Hermitian scalar product associated to this norm

$$\| \psi \|^2 = \iiint |\psi(\vec{x}, t)|^2 dv \quad (4)$$

$$< \psi|\psi' > = \iiint \psi^* \psi' dv$$

At the same time Uhlenbeck and Goudsmit made the hypothesis of a proper rotation of the electron, and next Pauli gave a wave equation for a wave with two components (spin up and spin down) : $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$. The Hermitian scalar product becomes

$$< \psi|\psi' >= \iiint (\psi_1^*\psi_1' + \psi_2^*\psi_2')dv = \iiint \psi^\dagger\psi' dv \qquad (5)$$

And The SU(2) unitary group comes when you search for a transformation of ψ under a spatial rotation : If $u = (u_x, u_y, u_z)$ is a unitary vector of the rotation axis and α is the rotation angle, then ψ is transformed into

$$\psi' = R\psi \ ; \quad R = \begin{pmatrix}\cos(\frac{\alpha}{2}) - iu_z\sin(\frac{\alpha}{2}) & (-iu_x - u_y)\sin(\frac{\alpha}{2})\\ (-iu_x + u_y)\sin(\frac{\alpha}{2}) & \cos(\frac{\alpha}{2}) + iu_z\sin(\frac{\alpha}{2})\end{pmatrix} \qquad (6)$$

The R matrix is unitary, and $\det R = 1$, so R is an element of the SU(2) group. To see the gap with classical physics, remember that R is not unique, R and $-R$ giving the same spatial rotation. In fact the spatial rotation group SO(3) and the unitary group SU(2) have the same Lie algebra, and SU(2) is the covering group of SO(3). But these two groups are not isomorphic. Nevertheless it's not SO(3) but SU(2) which is used in the Pauli's theory !

To get his wave equation Dirac knew the Pauli equation and wanted a linear and first order wave equation in the relativistic case, to conserve the probabilistic interpretation of nonrelativistic quantum mechanics. The Dirac spinor has now four components

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} \qquad (7)$$

So the Hermitian scalar product and the associated norm are

$$\parallel \psi \parallel^2 = \iiint (\psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_3^*\psi_3 + \psi_4^*\psi_4)dv = \iiint \psi^\dagger\psi dv \quad (8)$$
$$< \psi|\psi' > = \iiint \psi^\dagger\psi' dv$$

The Dirac equation needs four complex matrices γ^μ and we use now

$$\begin{aligned}
\gamma_0=\gamma^0=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\ ;\ \gamma_j=-\gamma^j=\begin{pmatrix}0 & -\sigma_j\\ \sigma_j & 0\end{pmatrix}\\
I=\sigma_0=\sigma^0=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\ ;\ \sigma_1=-\sigma^1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\
\sigma_2=-\sigma^2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\ ;\ \sigma_3=-\sigma^3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\
\gamma_{ij}=\gamma_i\gamma_j\ ;\ \gamma_5=-i\gamma_{0123}=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}
\end{aligned} \tag{9}$$

Contrarily to habits of early quantum mechanics, we don't use now the imaginary time of Dirac or Yang-Mills papers, but instead we use a true +, -, -, - signature for the spacetime metric. The Dirac equation reads

$$\begin{aligned}
[\gamma^\mu(\partial_\mu+iqA_\mu)+im]\psi=0 \quad (10)\\
q=\frac{e}{\hbar c}\ ;\ m=\frac{m_0c}{\hbar}
\end{aligned}$$

where e is the negative charge of the electron and A_μ are the covariant components ot the spacetime vector A, exterior electromagnetic potential.

The link between Dirac spinors and unitarity is not as strong as with the Pauli wave equation : Firstly if matrices of Dirac and Yang-Mills paper are hermitic, it's only because the spacetime metric is twisted with an imaginary time. If we use an appropriate spacetime metric as in (9)-(10), only the γ_0 is hermitic. Secondly the covering group of the Lorentz restricted group of spacetime rotations is $SL(2,\mathbb{C})$. SU(2) is a subgroup of $SL(2,\mathbb{C})$, but only a subgroup. Rotations involving time are not unitary.

Actually it is possible to write all the Dirac theory with only real numbers, which is nearer to classical physics. For this translation, all you have to do is to use real Clifford algebras and separate the real and imaginary parts of the four ψ_i. For instance, if we let

$$\begin{aligned}
\psi_1=a_1+ia_4\ ;\ \psi_2=-a_3-ia_2\\
\psi_3=a_8+ia_5\ ;\ \psi_4=a_6+ia_7
\end{aligned} \tag{11}$$

we can associate to each ψ of the Dirac theory an element ϕ with value in the real Clifford algebra of the physical space :

$$\phi = f(\psi) = a_1 + a_2\sigma_{32} + a_3\sigma_{31} + a_4\sigma_{12} + a_5\sigma_{123} + a_6\sigma_1 + a_7\sigma_2 + a_8\sigma_3$$
$$\sigma_{ij} = \sigma_i\sigma_j \tag{12}$$

With

$$\widehat{\phi} = a_1 + a_2\sigma_{32} + a_3\sigma_{31} + a_4\sigma_{12} - a_5\sigma_{123} - a_6\sigma_1 - a_7\sigma_2 - a_8\sigma_3 \tag{13}$$

$$\nabla = \sigma^\mu\partial_\mu \ ; \quad A = \sigma^\mu A_\mu \tag{14}$$

$$\widehat{\nabla} = \partial_0 + \vec{\partial} \ ; \quad \vec{\partial} = \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3$$
$$\widehat{A} = A^0 - \vec{A} \ ; \quad \vec{A} = A^1\sigma_1 + A^2\sigma_2 + A^3\sigma_3 \tag{15}$$

it is possible to establish an equivalence [3] between the Dirac equation (10) and

$$\nabla\widehat{\phi} + qA\widehat{\phi}\sigma_{12} + m\phi\sigma_{12} = 0 \tag{16}$$

There is a link between ϕ and $\widehat{\phi}$ and the Weyl spinors ξ and η defined by

$$U\psi = \begin{pmatrix}\xi\\ \eta\end{pmatrix} \quad ; \quad U = U^{-1} = \frac{1}{\sqrt{2}}(\gamma_0 + \gamma_5)$$
$$\xi = \frac{1}{\sqrt{2}}\begin{pmatrix}\psi_1 + \psi_3\\ \psi_2 + \psi_4\end{pmatrix} \quad ; \quad \eta = \frac{1}{\sqrt{2}}\begin{pmatrix}\psi_1 - \psi_3\\ \psi_2 - \psi_4\end{pmatrix} \tag{17}$$

We get with the Pauli matrices (9) which form a matrix representation of the space algebra

$$\phi = \sqrt{2}(\xi \quad \sigma_{13}\eta^*) \ ; \ \widehat{\phi} = \sqrt{2}(\eta \quad \sigma_{13}\xi^*) \tag{18}$$

We have precedently explained [3] how the preceding relations are Lorentz-invariant.

As f applying ψ to ϕ is an isomorphism, experimental results coming from (10) are also coming from (16), as the very precise calculation of the energy levels of the H atom.

With the real formalism, the norm (8) becomes

$$\|\phi\|^2 = \|\Psi\|^2 = \iiint J^0 dv = \iiint (\sum_{i=1}^{i=8} {a_i}^2) dv \tag{19}$$

Now, there is no reason to associate to this norm an Hermitian scalar product, unknown in real vector fields. The preceding norm is naturally associated to the Euclidean scalar product

$$\phi \cdot \phi' = \iiint (\sum_{i=1}^{i=8} a_i a'_i) dv \tag{20}$$

It is possible to associate to the real formalism a matrix formalism, associating to each ϕ of the space algebra the unicolumn matrix

$$\Phi = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{pmatrix} \tag{21}$$

With this unicolumn matrix the scalar product is

$$\phi \cdot \phi' = \iiint \Phi^t \Phi' dv \tag{22}$$

and the Dirac equation reads now

$$[\Gamma^\mu(\partial_\mu + qA_\mu P_3) + mP_3]\Phi = 0 \tag{23}$$

where Γ_μ and P_j are the real 8×8 matrices

$$\Gamma^0 = \Gamma_0 = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix} \; ; \; \Gamma^1 = -\Gamma_1 = \begin{pmatrix} 0 & \gamma_{013} \\ \gamma_{013} & 0 \end{pmatrix} \tag{24}$$

$$\Gamma^2 = -\Gamma_2 = \begin{pmatrix} 0 & -\gamma_{03} \\ \gamma_{03} & 0 \end{pmatrix} \; ; \; \Gamma^3 = -\Gamma_3 = \begin{pmatrix} 0 & -\gamma_{01} \\ \gamma_{01} & 0 \end{pmatrix} \tag{25}$$

$$P_1 = \begin{pmatrix} \gamma_{013} & 0 \\ 0 & \gamma_{013} \end{pmatrix} ; \; P_2 = \begin{pmatrix} -\gamma_{05} & 0 \\ 0 & \gamma_{05} \end{pmatrix} ; \; P_3 = \begin{pmatrix} -\gamma_{135} & 0 \\ 0 & \gamma_{135} \end{pmatrix} \tag{26}$$

These matrices verify

$$\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu = 2g^{\mu\nu} I_8 \; ; \;\; \Gamma^\mu P_j = P_j \Gamma^\mu \tag{27}$$

$\overline{\psi} = \psi^\dagger \gamma_0$ of the usual frame is now replaced by

$$\overline{\Phi} = \Phi^t \Gamma_0 \tag{28}$$

And as $(\Gamma^\mu)^t\Gamma_0 = \Gamma_0\Gamma^\mu$ and $P_j^t = -P_j$, transposing (23) we get an equivalent equation

$$\partial_\mu\overline{\Phi}\Gamma^\mu - qA_\mu\overline{\Phi}P_3\Gamma^\mu - m\overline{\Phi}P_3 = 0 \tag{29}$$

In the complex frame the 16 matrices made of the four γ_μ and their products form a basis of the 16-dimensionnal matrix field. It is closed. In the real frame the 16 matrices made of the four Γ_μ and their products are not able to form a basis of the 64-dimensionnal matrix field. But this matrix field has a basis made of the 16 matrices I_8, Γ^μ, $\Gamma^{\mu\nu}, \mu < \nu$, $\Gamma^{\mu\nu\rho}, \mu < \nu < \rho$, Γ^{0123}, and of the 48 matrices P_j, $\Gamma^\mu P_j$, $\Gamma^{\mu\nu}P_j, \mu < \nu$, $\Gamma^{\mu\nu\rho}P_j, \mu < \nu < \rho$, $\Gamma^{0123}P_j$, j=1, 2, 3. These 64 matrices split into two parts : $\frac{8\times 7}{2} = 28$ verify $M^t = -M = M^{-1}$ and form a basis of the Lie algebra of SO(8), $\frac{8\times 9}{2} = 36$ verify $M^t = M = M^{-1}$ and allow to build 36 tensorial densities $\Phi^t M\Phi$. These 36 tensorial densities are the components of

$$\Omega_1 = \overline{\Phi}\Phi \tag{30}$$

$$J^\mu = \overline{\Phi}\Gamma^\mu\Phi \tag{31}$$

$$S^{\mu\nu}{}_{(j)} = \overline{\Phi}\Gamma^{\mu\nu}P_{(j)}\Phi \tag{32}$$

$$K^{\mu\nu\rho}{}_{(j)} = \overline{\Phi}\Gamma^{\mu\nu\rho}P_{(j)}\Phi \tag{33}$$

$$\Omega_2 = \overline{\Phi}\Gamma^{0123}\Phi \tag{34}$$

In the complex frame, the Dirac theory only knows the scalar Ω_1, the conservative current vector J^μ and the tensors $S^{\mu\nu}{}_{(3)}$ and $K^{\mu\nu\rho}{}_{(3)}$ and the pseudoscalar Ω_2. Most of the tensors that we may build are unknown, the Dirac theory in the complex frame is actually incomplete.

The Dirac equation (23) is invariant under the orthogonal gauge group SO(8) : If A is any orthogonal 8×8 matrix, the transformation

$$\Phi \mapsto \Phi' = A\Phi \quad ; \quad A^{-1} = A^t \tag{35}$$

$$\Gamma^\mu \mapsto \Gamma'^\mu = A\Gamma^\mu A^{-1} \tag{36}$$

$$P_3 \mapsto P_3' = AP_3A^{-1} \tag{37}$$

gives, if $\partial_\mu A = 0$

$$[\Gamma'^\mu(\partial_\mu + qA_\mu P_3') + mP_3']\Phi' = A[\Gamma^\mu(\partial_\mu + qA_\mu P_3) + mP_3]\Phi = 0 \tag{38}$$

So SO(8) is the natural gauge group of the Dirac equation.

2 - Lagrangian density

The Yang-Mills Lagrangian (1) comes directly from the Dirac Lagrangian : the Dirac equation (10) may be deduced from the Lagrangian density

$$\mathcal{L} = \overline{\psi}[\gamma^\mu(-i\partial_\mu + qA_\mu) + m]\psi \tag{39}$$

Often it is said that variation with respect to $\overline{\psi}$ gives

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} = 0 \tag{40}$$

but it is false, because any variation of ψ gives a variation of $\overline{\psi}$.

Now, if we use the real form (23) for the Dirac equation, the Lagrangian density (39) reads

$$\mathcal{L} = \overline{\Phi}[\Gamma^\mu(-P_3\partial_\mu + qA_\mu) + m]\Phi \tag{41}$$

And a correct calculation of the variation of the Lagrangian density with respect to variation of Φ gives (23). But if the bad calculation (40) allows to get a good result, it is because the Lagrangian density of the Dirac theory may be changed into

$$\mathcal{L} = \overline{\psi}'[\gamma^\mu(-i\partial_\mu + qA_\mu) + m]\psi = \overline{\Phi}'[\Gamma^\mu(-P_3\partial_\mu + qA_\mu) + m]\Phi \tag{42}$$

where ψ' or Φ' is another wave which may be varied independently of ψ or Φ, and gives the Dirac equation, because

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi}')} = 0 \tag{43}$$

is correct. But now we get another wave equation, by the variation of the Lagrangian density with respect to the variation of ψ or Φ, which is

$$[\gamma^\mu(\partial_\mu + iqA_\mu) + im]\psi' = 0 \tag{44}$$

And so we get again a Dirac equation for the ψ' wave, with the same charge and the same mass.

What could be, physically, this ψ' wave ? It is evident, now that we know the positron and charge conjugation, that the conjugated wave has always been present in the Dirac theory. If we follow Ziino [4] we could interpret (44) as

$$[\gamma^\mu(\partial_\mu + i(-q)(-A_\mu)) + im]\psi' = 0 \tag{45}$$

and the ψ' should be the positron's wave, with a charge conjugation changing the sign of every charge, and consequently changing the sign of the electromagnetic potential. Nevertheless it is difficult to think that it is correct, because there is only one potential term in the Lagrangian density, and gauge invariance implies the same variation for ψ and ψ'.

So we should like to get a Lagrangian density giving not twice the same equation for ψ and ψ', but together the wave equation of the particle and the antiparticle. To build such a Lagrangian density, we need to consider the spacetime algebra.

3 - Spacetime algebra and 16×16 matrices

Spacetime algebra is the best frame to study the Dirac equation, and this was used by Hestenes [5], Boudet [6] and many physicists. Each element Ψ of this algebra may be written

$$\begin{aligned}\Psi =& a_1 + a_2\gamma_{23} + a_3\gamma_{13} + a_4\gamma_{21} + a_5\gamma_{0123} + a_6\gamma_{10} + a_7\gamma_{20} + a_8\gamma_{30}\\ &+ a_9\gamma_0 + a_{10}\gamma_{023} + a_{11}\gamma_{013} + a_{12}\gamma_{021} + a_{13}\gamma_{132}\\ &+ a_{14}\gamma_1 + a_{15}\gamma_2 + a_{16}\gamma_3 \end{aligned} \tag{46}$$

and we will associate to each Ψ a real unicolumn

$$X = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{16} \end{pmatrix} \tag{47}$$

We can associate to each ϕ of (12) a Ψ with value into the spacetime algebra :

$$\Psi = a_1 + a_2\gamma_{23} + a_3\gamma_{13} + a_4\gamma_{21} + a_5\gamma_{0123} + a_6\gamma_{10} + a_7\gamma_{20} + a_8\gamma_{30} \tag{48}$$

which is made also of the eight a_i of (11). Ψ is concerned only with the even subalgebra of the spacetime algebra, and the Dirac equation in this frame reads

$$\partial\Psi\gamma_{21} = m\Psi\gamma_0 + qA\Psi \tag{49}$$

$$\partial = \gamma^\mu\partial_\mu \quad ; \quad A = \gamma^\mu A_\mu \tag{50}$$

As the spacetime algebra is 16-dimensionnal, and as the Dirac equation is linear, we often have to use linear applications of spacetime algebra into spacetime algebra. These linear applications form a 256-dimensionnal vector space, isomorphic to the space of the 16×16 real matrices. Among these linear applications are the 16 left multiplications

$$l:\Psi\mapsto\gamma\Psi\ ;\quad \gamma=1,\ \gamma_\mu,\ \gamma_{\mu\nu},\ \gamma_{\mu\nu\rho},\ \gamma_{0123} \tag{51}$$

and the 16 right multiplications

$$r:\Psi\mapsto\Psi\gamma\ ;\quad \gamma=1,\ \gamma_\mu,\ \gamma_{\mu\nu},\ \gamma_{\mu\nu\rho},\ \gamma_{0123} \tag{52}$$

We note L_μ, $L_{\mu\nu}$,... the matrices of the left multiplication by γ_μ, $\gamma_{\mu\nu}$, and so on, and R_μ, $R_{\mu\nu}$,... the matrices of the right multiplication, using the unicolumn X. To compute these 256 matrices we only need to know

$$L_\mu=\begin{pmatrix}0 & \Gamma_\mu\\ \Gamma_\mu & 0\end{pmatrix}\ ;\quad \mu=0,\ 1,\ 2,\ 3 \tag{53}$$

$$R_0=\begin{pmatrix}0 & I_8\\ I_8 & 0\end{pmatrix}\ ;\quad R_j=\begin{pmatrix}0 & \Gamma_{0123}P_j\\ -\Gamma_{0123}P_j & 0\end{pmatrix}\ ;\quad j=1,\ 2,\ 3 \tag{54}$$

$$L_{\mu\nu}=L_\mu L_\nu\quad ;\quad R_{\mu\nu}=R_\nu R_\mu \tag{55}$$

A complete calculation of these 256 matrices gives two results : the 256 matrices M=LR=RL form a basis of the vector space of all the 16×16 real matrices. So each linear application of the spacetime algebra into itself is a linear combination of products of right and left multiplications. And these 256 matrices split into two parts : $\frac{16\times 15}{2}=120$ verify $M^t=-M=M^{-1}$ and form a basis of the Lie algebra of SO(16), $\frac{16\times 17}{2}=136$ verify $M^t=M=M^{-1}$ and allow to build 136 tensorial densities X^tMX

We associate now to each couple $(\psi,\ \psi')$ or $(\Phi,\ \Phi')$ made of the wave of an electron and a positron one Ψ and one X by

$$X=\begin{pmatrix}\Phi\\ \Phi'\end{pmatrix} \tag{56}$$

So the wave Ψ_e of the electron is the even part of Ψ and the wave Ψ_p of the positron is the odd part of Ψ. The Dirac equations

$$\partial\Psi_e\gamma_{21}=m\Psi_e\gamma_0+qA\Psi_e \tag{57}$$

$$\partial\Psi_p\gamma_{21}=m\Psi_p\gamma_0-qA\Psi_p \tag{58}$$

are the even and odd parts of a unique equation

$$\partial\Psi=m\Psi\gamma_{012}+qA\gamma_{0123}\Psi\gamma_{03} \tag{59}$$

Which is equivalent to

$$Ł^{\mu}\partial_{\mu}X = qA_{\mu}L^{\mu}L_{0123}R_{03}X + mR_{012}X \tag{60}$$

or to the system

$$\Gamma^{\mu}\partial_{\mu}\Phi + qA_{\mu}\Gamma^{\mu}P_3\Phi + mP_3\Phi = 0 \tag{61}$$
$$\Gamma^{\mu}\partial_{\mu}\Phi' - qA_{\mu}\Gamma^{\mu}P_3\Phi' + mP_3\Phi' = 0 \tag{62}$$

To get a Lagrangian density for this system we let

$$\overline{X} = X^t L_0 = (\overline{\Phi}' \quad \overline{\Phi}) \tag{63}$$

and we propose the Lagrangian density

$$\mathcal{L} = \overline{X}R_2L^{\mu}\partial_{\mu}X - qA_{\mu}\overline{X}L^{\mu}L_{0123}R_{032}X + m\overline{X}R_{01}X \tag{64}$$

which gives

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}X)} = \overline{X}R_2L^{\mu}R_{032}X + m\overline{X}R_{01}X \tag{65}$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial X} &= (L_0R_2L^{\mu}\partial_{\mu}X)^t - qA_{\mu}(L_0L^{\mu}L_{0123}R_{032}X)^t \\ &\quad - qA_{\mu}\overline{X}L^{\mu}L_{0123}R_{032} + m(L_0R_{01}X)^t + m\overline{X}R_{01} \\ &= -\partial_{\mu}\overline{X}R_2L^{\mu} - 2qA_{\mu}\overline{X}L^{\mu}L_{0123}R_{032} + 2m\overline{X}R_{01}\end{aligned} \tag{66}$$

So the Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial X} = \partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}X)}\right) \tag{67}$$

gives the transposed equation

$$\partial_{\mu}\overline{X}L^{\mu} = -qA_{\mu}\overline{X}L^{\mu}L_{0123}R_{03} - m\overline{X}R_{012} \tag{68}$$

and is equivalent to (60).

From (60) and (68), it is easy to get the conservation of the vector current

$$j_+^{\mu} = \overline{X}L^{\mu}X = \overline{\Phi}\Gamma^{\mu}\Phi + \overline{\Phi}'\Gamma^{\mu}\Phi' \tag{69}$$
$$\partial_{\mu}j_+^{\mu} = 0 \tag{70}$$

but also the conservation of

$$j^\mu_- = \overline{X}L^\mu L_{0123}R_{0123}X = -\overline{\Phi}\Gamma^\mu\Phi + \overline{\Phi}'\Gamma^\mu\Phi' \quad (71)$$

$$\partial_\mu j^\mu_- = 0 \quad (72)$$

And so we have a separate conservation of the probability current of the electron and of the probability current of the positron.

The real matrix playing the role of the i of quantum mechanics is here

$$\widehat{i} = L_{0123}R_{03} = \begin{pmatrix} -P_3 & 0 \\ 0 & P_3 \end{pmatrix} \quad (73)$$

which anticommutes with the four L^μ and with R_{012}. The wave equation (60) and the Lagrangian density (64) are invariant under the local gauge transformation

$$A_\mu \mapsto A_\mu + \frac{1}{q}\partial_\mu a \ ; \quad X \mapsto e^{a\widehat{i}}X \quad (74)$$

If M is one of the 6 matrices I_{16}, R_0, R_{01}, R_{02}, R_{03}, or R_{123}, and if N is one of the 10 matrices R_1, R_2, R_3, R_{21}, R_{32}, R_{13}, R_{021}, R_{032}, R_{013}, R_{0123}, we can split the 136 tensorial densities into

* 6 scalars $\overline{X}MX$
* 6 spacetime vectors $\overline{X}L^\mu MX$
* 10 tensors $\overline{X}L^{\mu\nu}NX$
* 10 tensors $\overline{X}L^{\mu\nu\rho}NX$
* 6 pseudoscalars $\overline{X}L^{0123}MX$

It is curious to remark that the conservative j^μ_- current is in fact one of the ten $\overline{X}L^{\mu\nu\rho}NX$. It implies that the j^μ_- current is not a spacetime vector, but a spacetime pseudovector.

From the Lagrangian density (64), as with the complex frame, we can calculate the momentum-energy tensor linked to the wave. This tensor reads now

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu X)}\partial_\nu X - \delta^\mu_\nu\mathcal{L} \quad (75)$$

But $\mathcal{L} = 0$ when the wave equation is satisfied and

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu X)} = \overline{X}R_2L^\mu \quad (76)$$

So we get

$$T^{\mu}_{\nu} = \overline{X}R_2L^{\mu}\partial_{\nu}X \tag{77}$$

$$= -\overline{\Phi}\Gamma_{0123}P_2\partial_{\nu}\Phi' + \overline{\Phi}'\Gamma_{0123}P_2\partial_{\nu}\Phi \tag{78}$$

And we can begin to see the considerable difference between our Lagrangian density and the classical one : the energy-momentum tensor (78) is entirely made of cross-terms between the Φ wave of the electron and the Φ' wave of the positron. So we have absolutely no energy with an electron alone, we have an energy-momentum tensor linked to the wave only in the case of a couple electron-positron.

Back to Yang-Mills

The Yang-Mills' isospin was the beginning of non-commutative local gauge groups used to understand electromagnetic, weak and strong interactions, and to unify all known interactions. Beyond what is now called the "standard model", many facts remain unexplained, for instance why we have not only electrons, but also muons and tauons.

If we look at the real frame for the Dirac theory, we see immediately that the 3 index play a very particular role, and we have explained in a previous paper [7] that we can build two other wave equations or Lagrangian densities by a circular permutation of the 1, 2, 3 index. Here we can associate to the Dirac equation (59) two other equations

$$\partial\Psi = m\Psi\gamma_{023} + qA\gamma_{0123}\Psi\gamma_{01} \tag{79}$$

$$\partial\Psi = m\Psi\gamma_{031} + qA\gamma_{0123}\Psi\gamma_{02} \tag{80}$$

So we have here naturally three kinds of wave equation, not one, two or four. It seems evidently linked to the existence of three and only three generations of fundamental fermions.

The Lagrangian density (64) is invariant under the global gauge transformation

$$\begin{aligned} X &\mapsto MX \ ; \quad M^t = M^{-1} \\ L^{\mu} &\mapsto ML^{\mu}M^t \\ R_{\mu} &\mapsto MR_{\mu}M^t \end{aligned} \tag{81}$$

Amongst the preceding global gauge transformations are those transforming one of the three wave equations (59), (79), (80) into another, with

$M = e^{aR_j}$, and as it is so easy to transform one equation into another, it is difficult to understand (59), (79) and (80) as describing three different things. Only one of all the global gauge transformations is local in the Dirac equation, the electric gauge (74). But the paper of Yang and Mills explains how, adding appropriate potentials and fields, we can extend a global to a local gauge. And as the SO(16) Lie group of all transformations (81) is big enough to contain the U(1)xSU(2)xSU(3) of the standard model, it is a good candidate to unify electromagnetic, weak and strong interactions of the three families of fundamental fermions.

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